Homework 16 (due Thursday 13 December)

Exercises 3.2.5, 3.2.7, 4.1.9 and 4.1.10 from the syllabus.

Exercise 3.2.5. Prove that there are at most $2^{\aleph_0}$ degrees. [10pts]

Hint: Note that there are only countably many sets in every degree. Hence, there are *at least* $2^{\aleph_0}$ degrees. Conclude that there are *exactly* $2^{\aleph_0}$ degrees.

Exercise 3.2.7.

(a.) Let $\{A_y \mid y \in \omega\}$ be any countable sequence of sets. Define the *infinite join*

$$\oplus_y A_y = \oplus\{A_y \mid y \in \omega\} := \{\langle x, y\rangle \mid x \in A_y, y \in \omega\}$$

Prove that $\deg(\oplus_y A_y)$ is the *uniform* least upper bound for $\{\deg(A_y) \mid y \in \omega\}$ in the sense that if there exists a set C and a computable function f such that $A_y = \Phi^C_{f(y)}$ for all y, then $\oplus_y A_y \leq_T C$. [20pts]

(b.) Prove that this operation is not well-defined on degrees. Namely, define $\{A_y \mid y \in \omega\}$ and $\{B_y \mid y \in \omega\}$ such that $A_y \equiv_T B_y$ but $\oplus_y A_y \not\equiv_T \oplus_y B_y$. [20pts]

Exercise 4.1.9. Prove that $A \in \bigcup_n (\Sigma_n \cup \Pi_n)$ iff A can be obtained from a computable relation by a finite number of applications of projection and complementation. [10pts]

Exercise 4.1.10. Prove that $\mathrm{Ext} \in \Sigma_3$ for

$$\mathrm{Ext} := \{x \mid \varphi_x \text{ is extendible to a total computable function}\}$$

[20 pts]