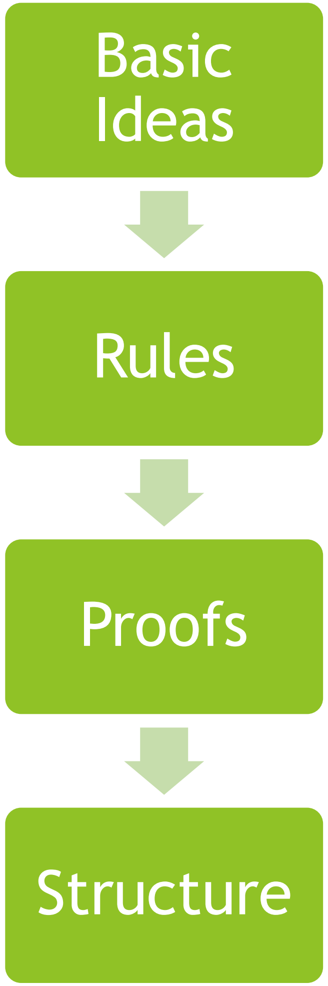
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,{}^{1} \quad \dfrac{\dfrac{}{(A \to B) \land (B \to C)}\,{}^{2}}{A \to B}}{B} \quad \dfrac{\dfrac{}{(A \to B) \land (B \to C)}\,{}^{2}}{B \to C}}{C}}{A \to C}\,{}^{1}}{(A \to B) \land (B \to C) \to (A \to C)}\,{}^{2}$$

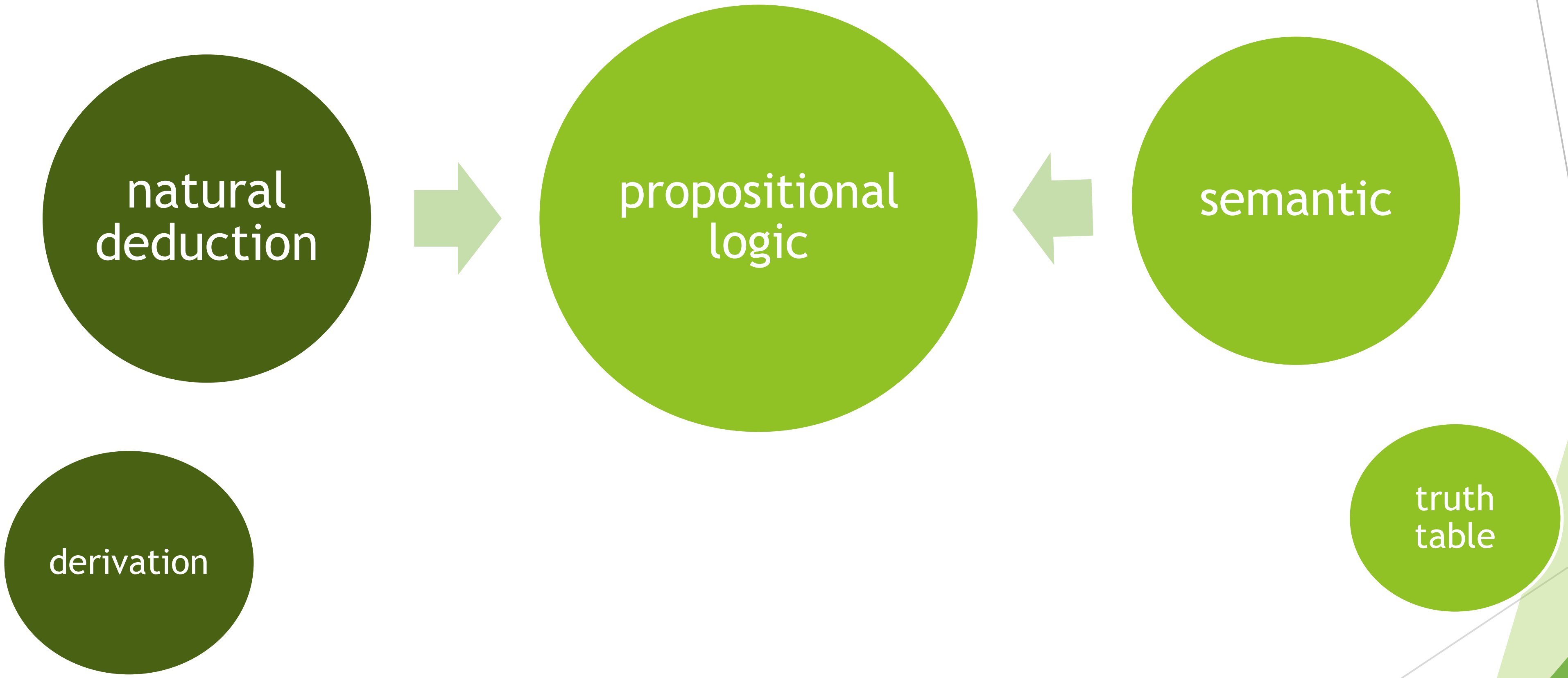
Natural Deduction

Propositional Logic

Outline



Basic ideas


natural deduction
propositional logic
semantic
derivation
truth table

Basic ideas

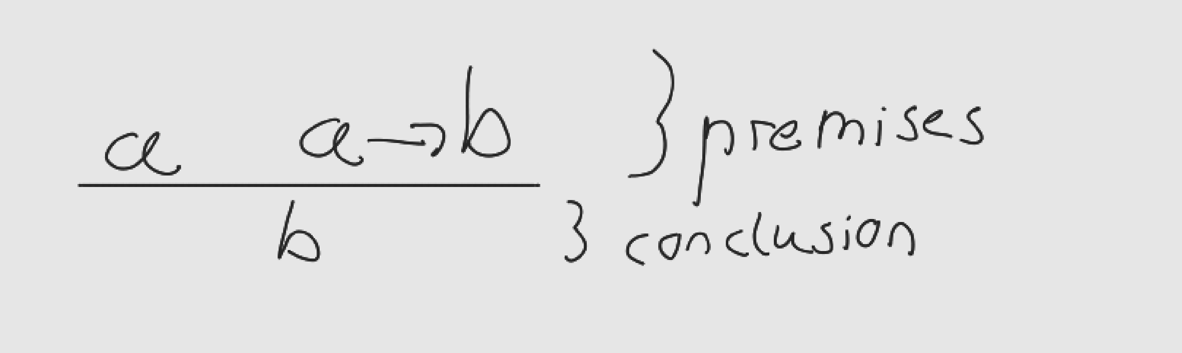
comply the derivation rules

Syntax

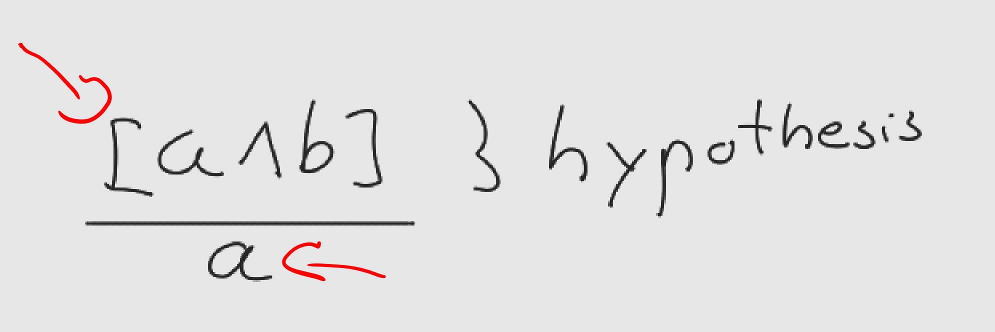
- connectives: $\wedge$, $\rightarrow$, $\bot$
- $\neg a$: $a \rightarrow \bot$
- $a \leftrightarrow b$: $a \rightarrow b \wedge b \rightarrow a$

Syntax

- connectives: ∧, →, ⊥
- ¬a: a →⊥
- a ↔ b: a→b ∧ b→a
- from a, a →b conclude b:



- if a∧b is true, than a is true:



Dont be afraid of other syntax!

$$\dfrac{\dfrac{\dfrac{\overline{A\ \text{true}}^{\,u} \qquad \overline{B\ \text{true}}^{\,w}}{A \wedge B\ \text{true}}\,\wedge_I}{B \supset (A \wedge B)\ \text{true}}\,\supset_{I^w}}{A \supset (B \supset (A \wedge B))\ \text{true}}\,\supset_{I^u}$$

1	$((p \to q) \wedge (\neg r \to \neg q))$	
2	$\quad p$	
3	$\quad ((p \to q) \wedge (\neg r \to \neg q))$	1, Reiteration
4	$\quad (p \to q)$	3, $\wedge$E
5	$\quad q$	2,4, $\to$E
6	$\quad (\neg r \to \neg q))$	3, $\wedge$E
7	$\quad\quad \neg r$	
8	$\quad\quad (\neg r \to \neg q))$	6, Reiteration
9	$\quad\quad \neg q$	7,8, $\to$E
10	$\quad\quad q$	5, Reiteration
11	$\quad \neg\neg r$	7–10, $\neg$I
12	$\quad r$	11, $\neg\neg$E
13	$(p \to r)$	2–12, $\to$I
14	$(((p \to q) \wedge (\neg r \to \neg q)) \to (p \to r))$	1–13, $\to$I

Derivation rules

- basic rules: express intuitive meaning of connectives

Derivation rules

- basic rules: express intuitive meaning of connectives

Elimination	Introduction
eliminate connectives	Introduce connectives

Introduction rules

- live

$$\frac{\begin{array}{c}[\neg a]\\ \vdots \\ \bot\end{array}}{a}\ \text{RAA}$$

↖ Conclusion

Elimination rules

- live

Overview

INTRODUCTION RULES ELIMINATION RULES

$$(\wedge I)\ \frac{\varphi \quad \psi}{\varphi \wedge \psi}\wedge I \qquad (\wedge E)\ \frac{\varphi \wedge \psi}{\varphi}\wedge E \quad \frac{\varphi \wedge \psi}{\psi}\wedge E$$

$$(\to I)\ \begin{array}{c}[\varphi]\\ \vdots \\ \dfrac{\psi}{\varphi \to \psi}\to I\end{array} \qquad (\to E)\ \frac{\varphi \quad \varphi \to \psi}{\psi}\to E$$

We have two rules for $\bot$, both of which eliminate $\bot$, but introduce a formula.

$$(\bot)\ \frac{\bot}{\varphi}\bot \qquad (\mathrm{RAA})\ \begin{array}{c}[\neg\varphi]\\ \vdots \\ \dfrac{\bot}{\varphi}\mathrm{RAA}\end{array}$$

First example

- 1. a →b ↔ ¬(a ∧ ¬b)

INTRODUCTION RULES ELIMINATION RULES

$$(\wedge I)\ \frac{\varphi \quad \psi}{\varphi \wedge \psi}\wedge I \qquad (\wedge E)\ \frac{\varphi \wedge \psi}{\varphi}\wedge E \quad \frac{\varphi \wedge \psi}{\psi}\wedge E$$

$$(\to I)\ \begin{array}{c}[\varphi]\\ \vdots\\ \psi\end{array} \over \varphi \to \psi$$ $\to I$

$$(\to E)\ \frac{\varphi \quad \varphi \to \psi}{\psi} \to E$$

We have two rules for ⊥, both of which eliminate ⊥, but introduce a formula.

$$(\bot)\ \frac{\bot}{\varphi}\bot \qquad (\mathrm{RAA})\ \begin{array}{c}[\neg\varphi]\\ \vdots\\ \dfrac{\bot}{\varphi}\,\mathrm{RAA}\end{array}$$

Proof strategy

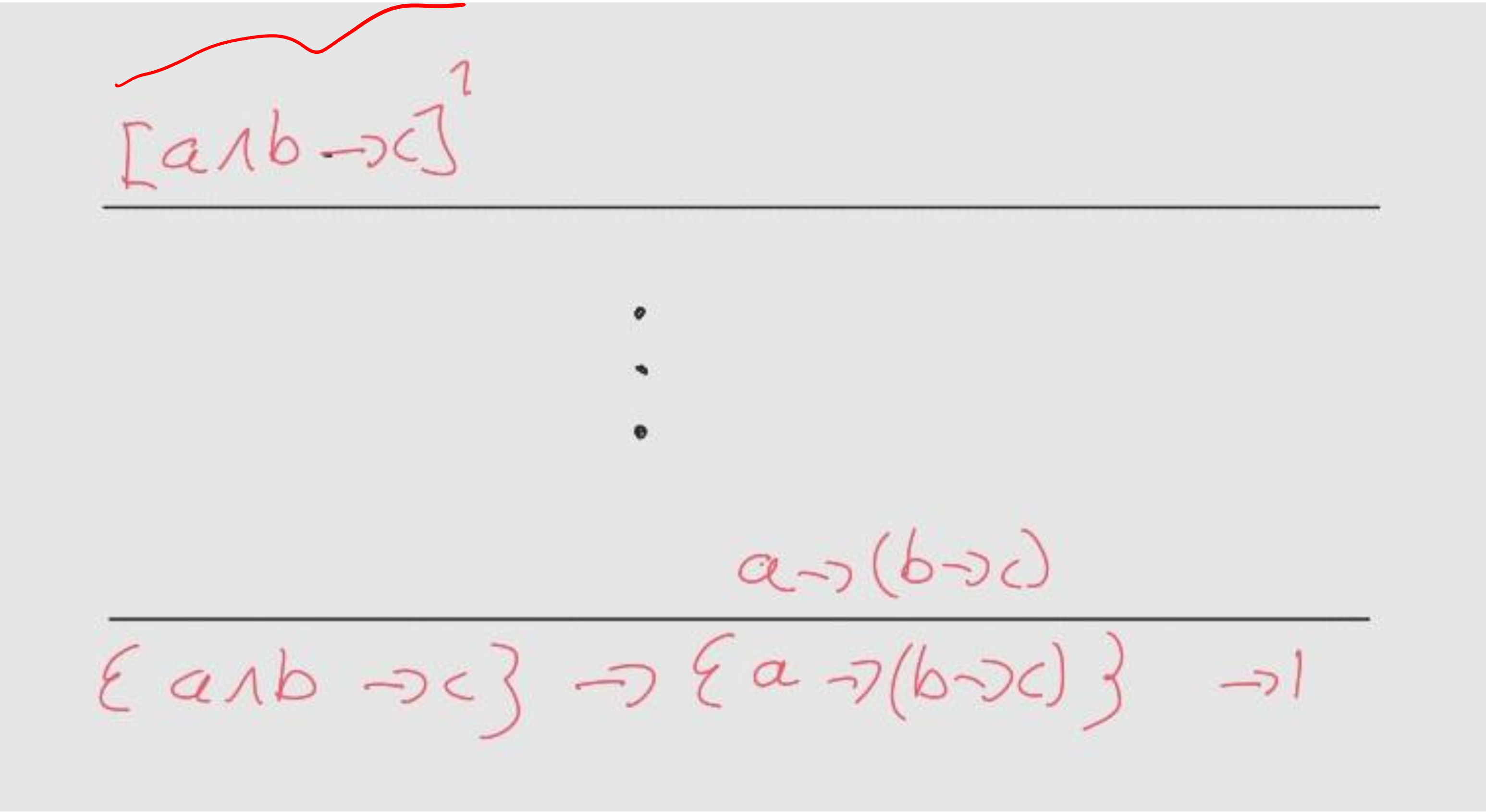
$\{a \wedge b \rightarrow c\} \rightarrow \{a \rightarrow (b \rightarrow c)\}$

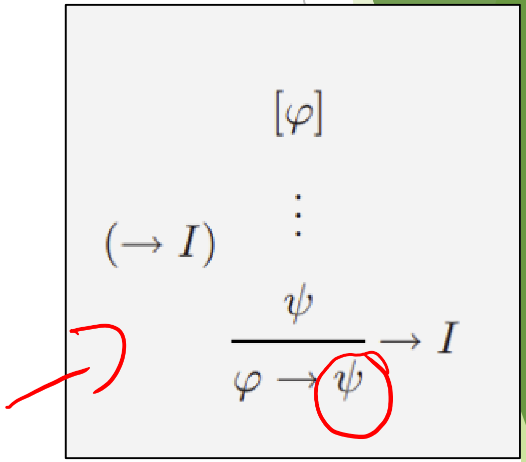
$(\rightarrow I)$

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \rightarrow \psi} \rightarrow I$$

Proof strategy

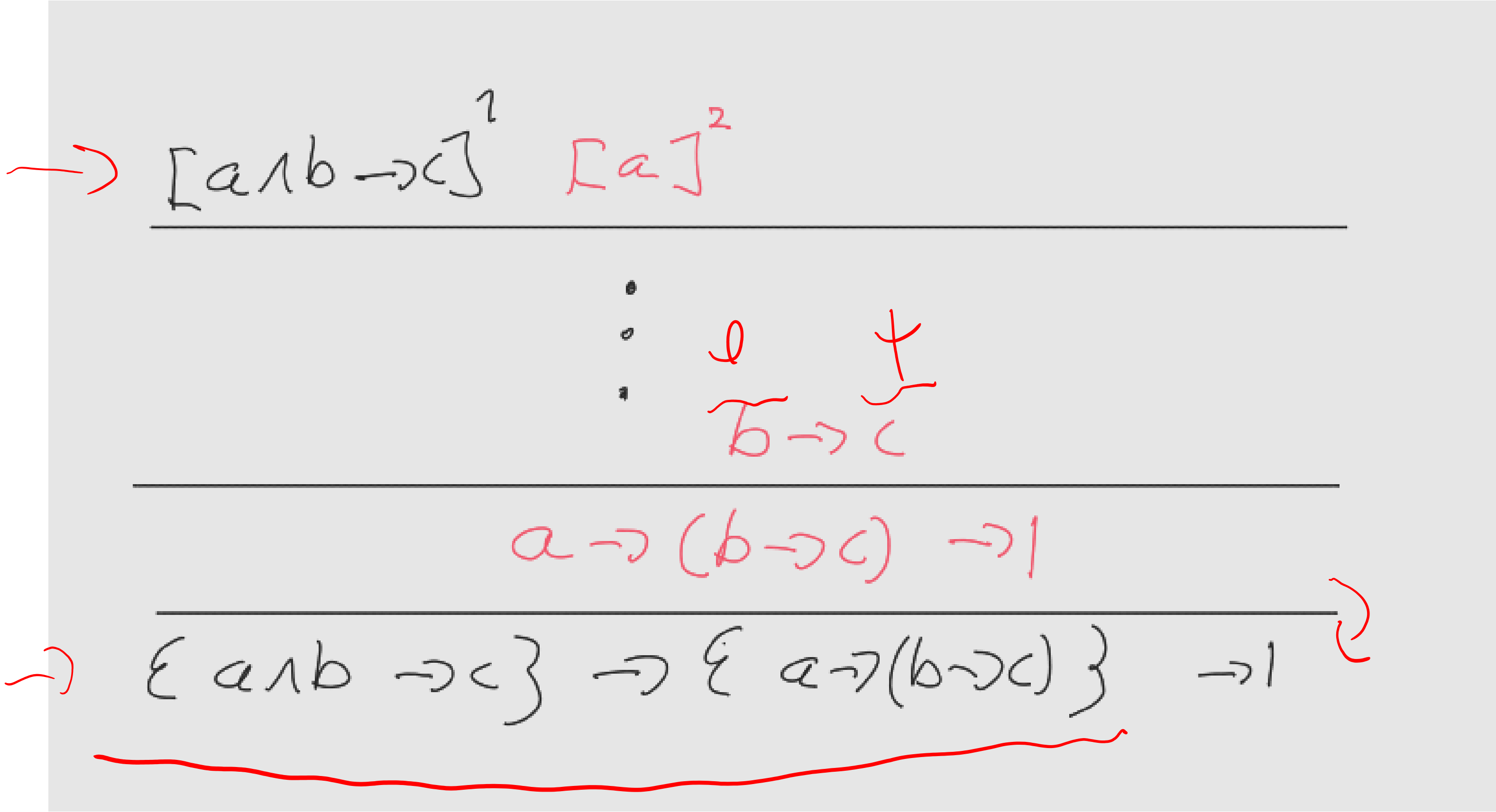
{a ∧ b →c} → {a →(b →c)}

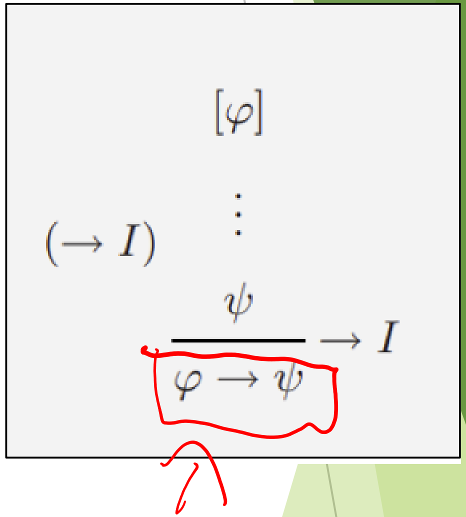




Proof strategy

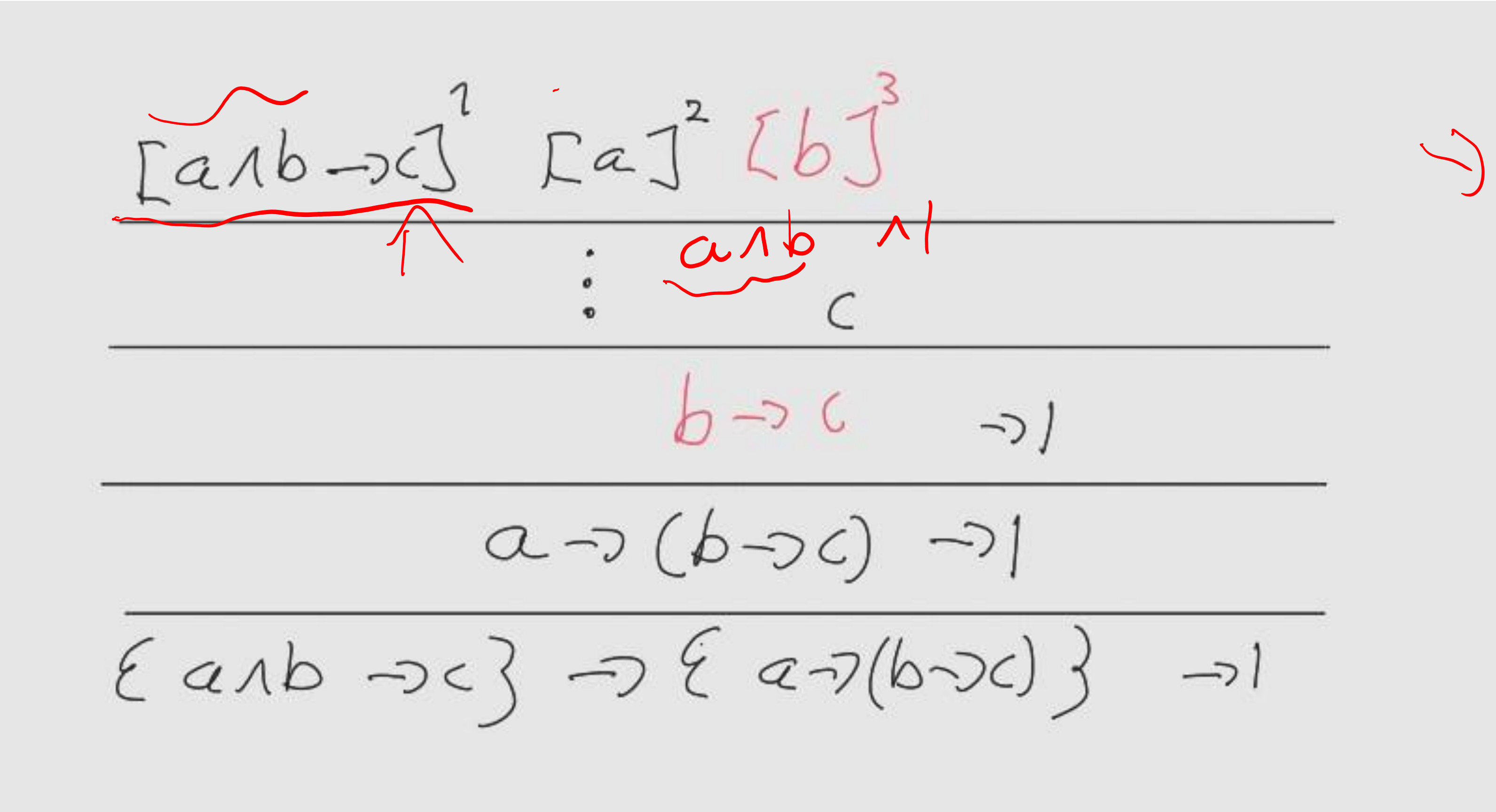
{a ∧ b →c} → {a →(b →c)}

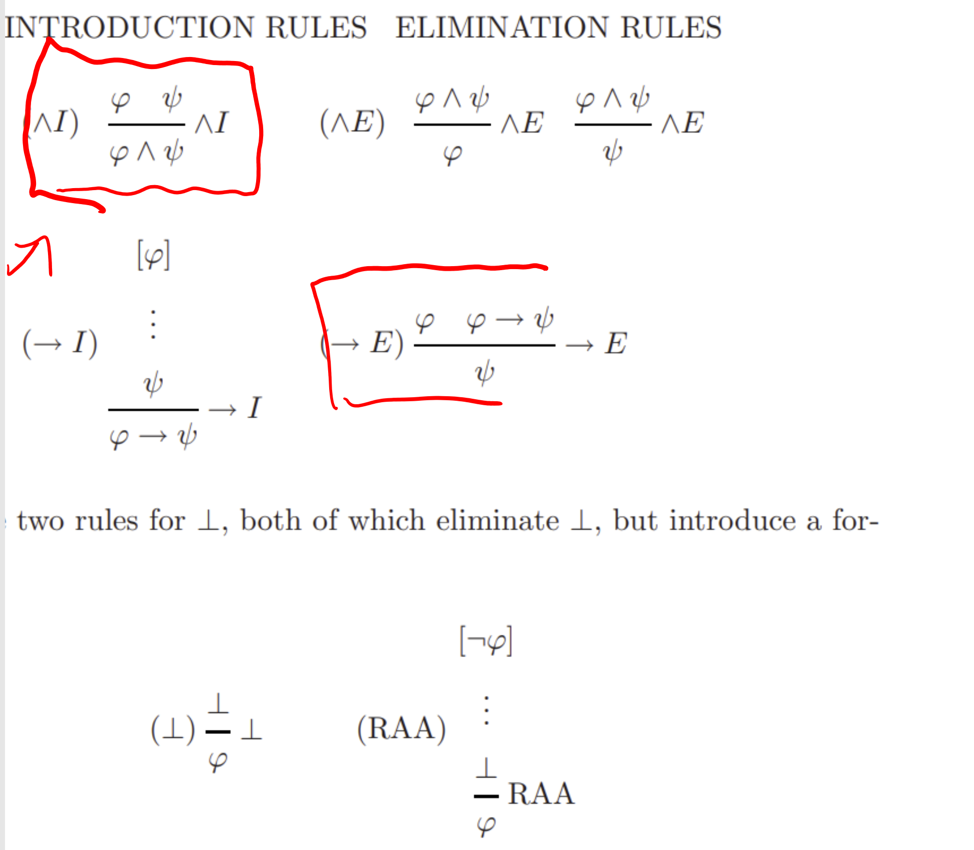




Proof strategy

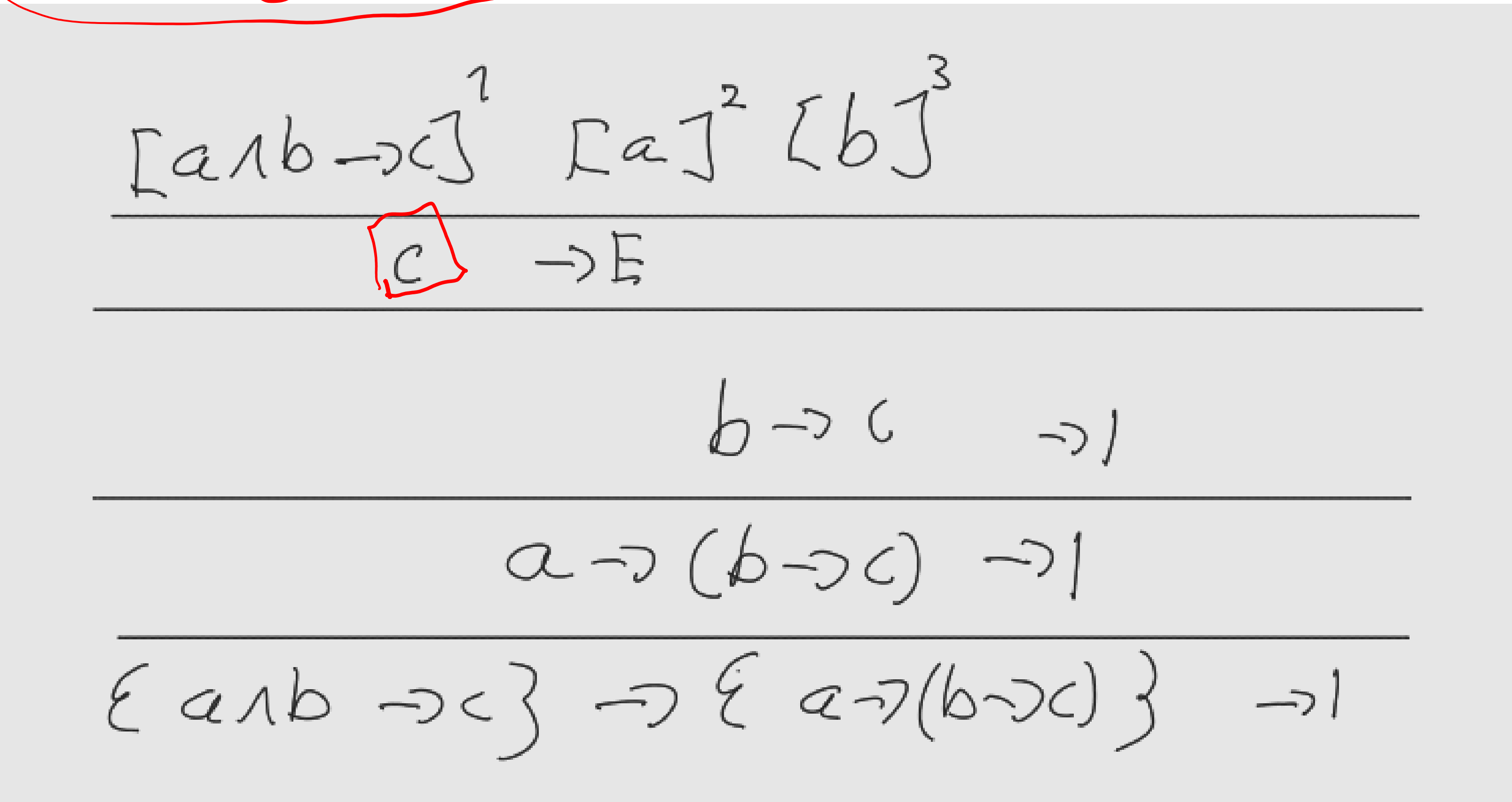
{a ∧ b →c} → {a →(b →c)}

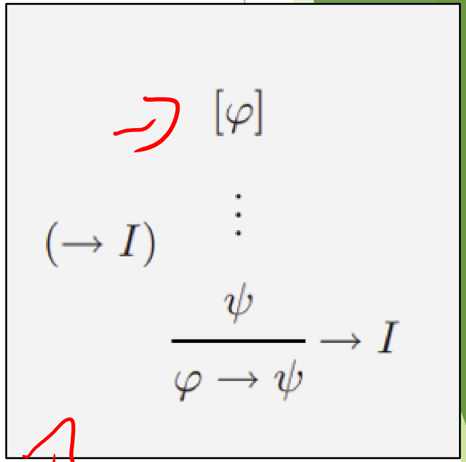




Proof strategy

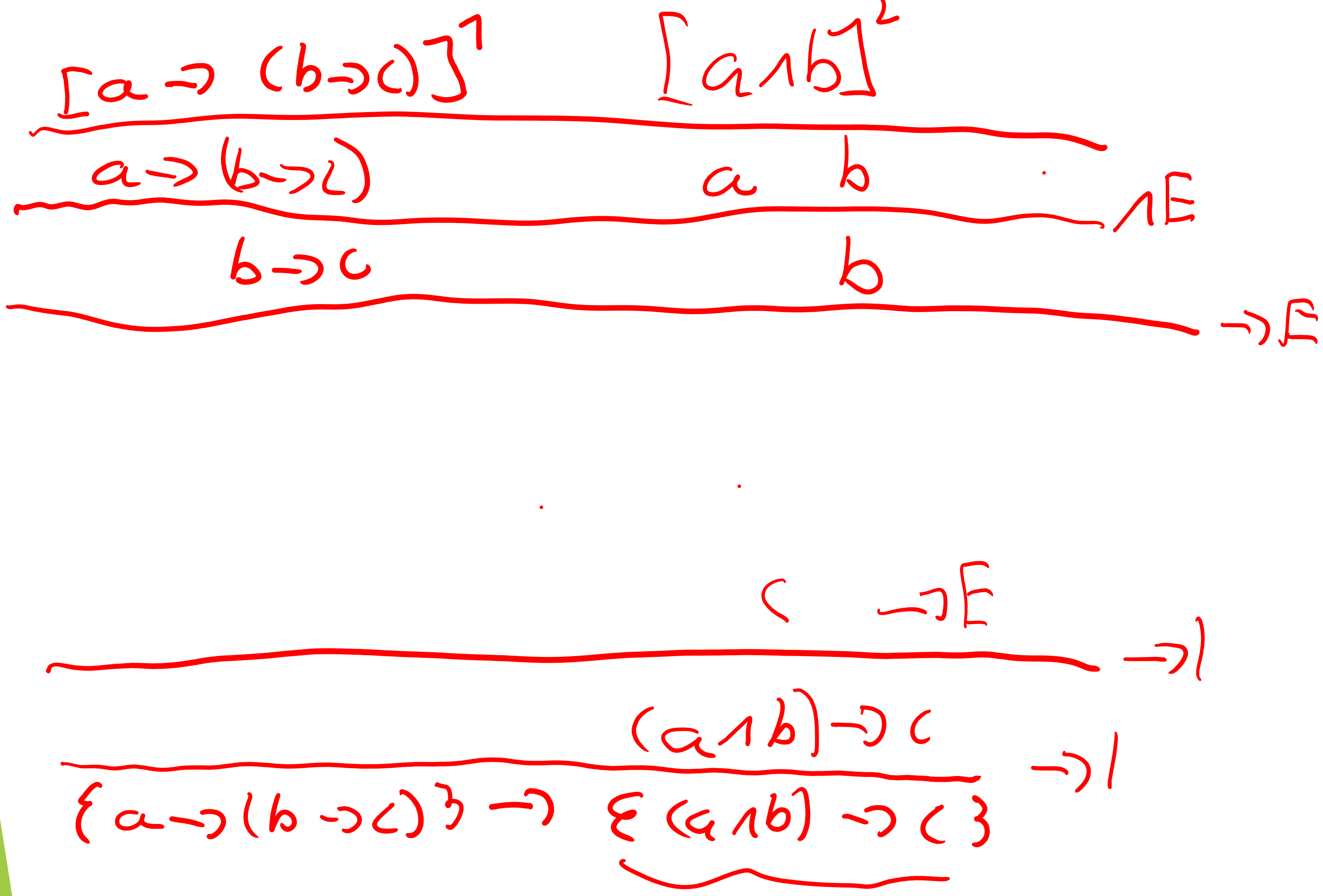
{a ∧ b →c} → {a →(b →c)}





Proof strategy

exercise :{a →(b →c)} → {a ∧ b →c}



INTRODUCTION RULES	ELIMINATION RULES
$(\wedge I)\quad \dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\,\wedge I$	$(\wedge E)\quad \dfrac{\varphi \wedge \psi}{\varphi}\,\wedge E \qquad \dfrac{\varphi \wedge \psi}{\psi}\,\wedge E$
$(\to I)\quad \dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\,\to I$	$(\to E)\quad \dfrac{\varphi \quad \varphi \to \psi}{\psi}\,\to E$

Hypothesis

$$(\to I)\qquad \dfrac{\begin{array}{c}[\varphi]\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\to I$$

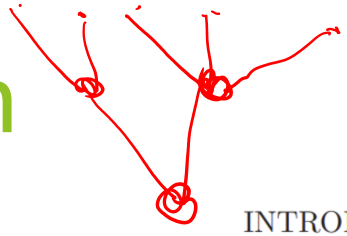
- hypothesis is cancelled
- no need of hypothesis
- hypothesis may maintain

Hypothesis

$$
\text{(RAA)} \quad \frac{\begin{array}{c}[\neg\varphi] \\ \vdots \\ \bot\end{array}}{\varphi}\,\text{RAA}
$$

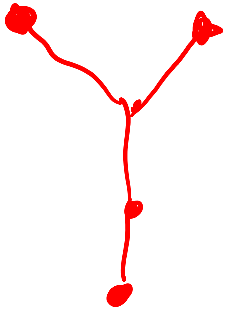
- hypothesis is cancelled
- hypothesis is wrong

Structure of derivation

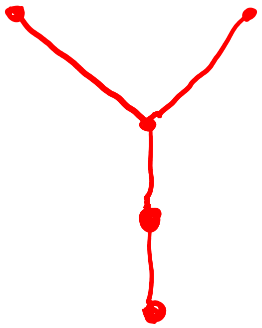


- ¬ ¬a ↔ a

$$\dfrac{\dfrac{\dfrac{[\neg a \to \bot]^1 \quad [\neg a]^2}{\bot} \to E}{a}\ RAA}{\neg\neg a \to a} \to I$$



$$\dfrac{\dfrac{\dfrac{[a]^1 \quad [a \to \bot]^2}{\bot} \to E}{\neg a \to \bot} \to I}{a \to \neg\neg a}$$



INTRODUCTION RULES ELIMINATION RULES

$$(\wedge I)\ \dfrac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge I \qquad (\wedge E)\ \dfrac{\varphi \wedge \psi}{\varphi} \wedge E \quad \dfrac{\varphi \wedge \psi}{\psi} \wedge E$$

$$(\to I)\ \dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi} \to I \qquad (\to E)\ \dfrac{\varphi \quad \varphi \to \psi}{\psi} \to E$$

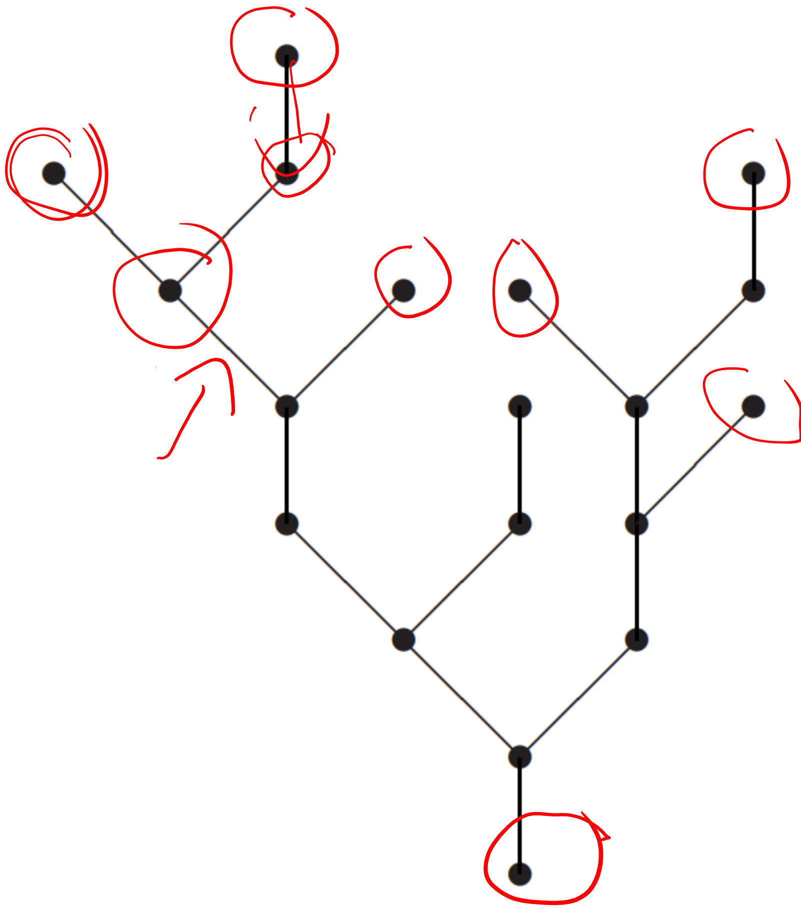
We have two rules for ⊥, both of which eliminate ⊥, but introduce a formula.

$$(\bot)\ \dfrac{\bot}{\varphi} \bot \qquad (RAA)\ \dfrac{\begin{matrix}[\neg\varphi] \\ \vdots \\ \bot\end{matrix}}{\varphi} RAA$$

Structure of derivation

proof: $\neg(\varphi \leftrightarrow \neg\varphi)$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          [\varphi]^1 \quad \dfrac{[\varphi \leftrightarrow \neg\varphi]^3}{\varphi \to \neg\varphi}\wedge E
        }{\neg\varphi}\to E
        \quad [\varphi]^1
      }{\bot}\to E
    }{\neg\varphi}\to I_1
    \quad
    \dfrac{[\varphi \leftrightarrow \neg\varphi]^3}{\neg\varphi \to \varphi}\wedge E
  }{\varphi}\to E
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        [\varphi]^2 \quad \dfrac{[\varphi \leftrightarrow \neg\varphi]^3}{\varphi \to \neg\varphi}\wedge E
      }{\neg\varphi}\to E
      \quad [\varphi]^2
    }{\bot}\to E
  }{\neg\varphi}\to I_2
}{
  \dfrac{\bot}{\neg(\varphi \leftrightarrow \neg\varphi)}\to I_3
}\to E
$$



Derivation: theoretical approach

set of derivation = smallest set X:

(1) The one element tree φ belongs to X for all $\varphi \in PROP$.

$\wedge$ $\rightarrow$ $\bot$

(2∧) If $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}, \begin{array}{c}\mathcal{D}'\\ \varphi'\end{array} \in X$, then $\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi'\end{array}}{\varphi \wedge \varphi'} \in X$.

If $\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array} \in X$, then $\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\varphi}, \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\psi} \in X$.

(2→) If $\begin{array}{c}\varphi\\ \mathcal{D}\\ \psi\end{array} \in X$, then $\dfrac{\begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\end{array}}{\varphi \rightarrow \psi} \in X$.

If $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}, \begin{array}{c}\mathcal{D}'\\ \varphi \rightarrow \psi\end{array} \in X$, then $\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi \rightarrow \psi\end{array}}{\psi} \in X$.

(2⊥) If $\begin{array}{c}\mathcal{D}\\ \bot\end{array} \in X$, then $\dfrac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{\varphi} \in X$.

If $\begin{array}{c}\neg\varphi\\ \mathcal{D}\\ \bot\end{array} \in X$, then $\dfrac{\begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \bot\end{array}}{\varphi} \in X$.

Sets of propositions

- $\Gamma \vdash$ a: derivation with (uncancelled) hypotheses in Γ with conclusion a
  - a derivable from Γ
- $\vdash$:turnstile
- $\Gamma = \emptyset$: $\vdash$ a, a: theorem

$\vdash a$

Sets of propositions

- $\Gamma \vdash$ a: derivation with (uncancelled) hypotheses in Γ with conclusion a
  - a derivable from Γ
- $\vdash$:turnstile
- $\Gamma = \emptyset$: $\vdash$ a, a: theorem

$$
\begin{aligned}
&(a)\ \Gamma \vdash \varphi \ \mathit{if}\ \varphi \in \Gamma,\\
&(b)\ \Gamma \vdash \varphi, \Gamma' \vdash \psi \Rightarrow \Gamma \cup \Gamma' \vdash \varphi \wedge \psi,\\
&(c)\ \Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \varphi \ \mathit{and}\ \Gamma \vdash \psi,\\
&(d)\ \Gamma \cup \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \to \psi,\\
&(e)\ \Gamma \vdash \varphi, \Gamma' \vdash \varphi \to \psi \Rightarrow \Gamma \cup \Gamma' \vdash \psi,\\
&(f)\ \Gamma \vdash \bot \Rightarrow \Gamma \vdash \varphi,\\
&(g)\ \Gamma \cup \{\neg\varphi\} \vdash \bot \Rightarrow \Gamma \vdash \varphi.
\end{aligned}
$$

Proof by using derivation (last slide)

Sets of propositions: theorem

- $\Gamma \vdash$ a: derivation with (uncancelled) hypotheses in Γ with conclusion a
- If $\Gamma = \emptyset$

  $\vdash$ a

  a: theorem

$\vdash a \to (\neg a \to b)$

$\vdash (a \to b) \to \{(b \to c) \to (a \to c)\}$

[a] [¬

$$\frac{\neg a \to b}{a \to (\neg a \to b)}$$

INTRODUCTION RULES ELIMINATION RULES

$$(\wedge I)\ \frac{\varphi \quad \psi}{\varphi \wedge \psi}\wedge I \qquad (\wedge E)\ \frac{\varphi \wedge \psi}{\varphi}\wedge E \quad \frac{\varphi \wedge \psi}{\psi}\wedge E$$

$$(\to I)\ \begin{array}{c}[\varphi]\\ \vdots \\ \dfrac{\psi}{\varphi \to \psi}\to I\end{array} \qquad (\to E)\ \frac{\varphi \quad \varphi \to \psi}{\psi}\to E$$

We have two rules for $\perp$, both of which eliminate $\perp$, but introduce a formula.

$$(\perp)\ \frac{\perp}{\varphi}\perp \qquad (\mathrm{RAA})\ \begin{array}{c}[\neg\varphi]\\ \vdots \\ \dfrac{\perp}{\varphi}\mathrm{RAA}\end{array}$$