ZFC+¬CH

Ruiting Hu

1 A Model of ZFC+ ¬CH

Theorem 1.1. *There is M[G] such that M[G]* $\models 2^{\aleph_0} > \aleph_1$.

Let $\mathbb{P}$ be the set of all finite functions p such that:

- dom(p) is a finite subset of $\omega_2 \times \omega$
- ran(p) $\subset \{0,1\}$

$p < q$ iff $q \subset p$ and for any generic G, $f = \bigcup G$.

Lemma 1.2. f *is a function; dom(f)=*$\omega_2 \times \omega$

Proof. Assume for RAA that f is not a function, then there are $p, q \in G$ such that for some (α, n), $p(\alpha, n) \neq q(\alpha, n)$. WLOG, say $p(\alpha, n) = 0$, $q(\alpha, n) = 1$, then since G is a filter, $p < p' = ((\alpha, n), 0)$ and thus $p' \in G$. Similarly, $q < q' = ((\alpha, n), 1)$ and thus $q' \in G$. It follows that $p' \cap q' = \emptyset \in G, \perp$.

For the second part, let $D_{\alpha,n} = \{p \in \mathbb{P} : (\alpha, n) \in dom(p)\}$. $D_{\alpha,n}$ is dense in $\mathbb{P}$ because for any $p \in \mathbb{P}$, we could extend p by adding $p(\alpha, n) = 0/1$. Thus, $(\alpha, n) \in dom(f)$ for all $(\alpha, n) \in \omega_2 \times \omega$.

□

Let $f_\alpha : \omega \to \{0,1\}$ be defined as $f_\alpha(n) = f(\alpha, n)$ for all $\alpha < \omega_2$. Let $h : \omega \to \{0,1\}^\omega$ be defined as $h(\alpha) = f_\alpha$

Lemma 1.3. h *is 1-1*

Proof. Assume $\alpha \neq \beta$, we show that $f_\alpha \neq f_\beta$.

Let $D = \{p \in \mathbb{P} : p(\alpha, n) \neq p(\beta, n)$ for some $n\}$.

D is dense in $\mathbb{P}$ because for any $p \in \mathbb{P}$, we could extend p by adding $p(\alpha, n) = 1, p(\beta, n) = 0$. Since G is a filter, $G \cap D \neq \emptyset$ □

Each f_α is a characteristic function of $a_\alpha \subset \omega$. The a_αs are called *Cohen generic reals* . We have added $\aleph_2^M$ many *Cohen generic reals* to M.

2 Preservation of Cardinals

It remains to be shown that $\aleph_2^M = \aleph_2^{M[G]}$. This is not trivial since M[G] might allow more bijections than M and lead to $\omega_n^M < \omega_n^{M[G]}$.

Definition 2.1 (Cardinality Preservation)**.** For any forcing poset $\mathbb{P} \in M$ $\mathbb{P}$ preserves cardinals iff for all generic G, $(\beta$ is a cardinal$)^M$ iff $(\beta$ is a cardinal$)^{M[G]}$ for all $\beta < o(M)$.

$\mathbb{P}$ preserves cofinalities iff for all generic G, $\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)$ for all limit $\gamma < o(M)$.

Definition 2.2 (Cofinality Preservation)**.** For any forcing poset $\mathbb{P} \in M$ $\mathbb{P}$ preserves cofinalities iff for all generic G, $\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)$ for all limit $\gamma < o(M)$.

We prove two lemmas regarding the conditions under which $\mathbb{P}$ preserves cofinality and cardinaltiy.

Lemma 2.1. *$\mathbb{P}$ preserves cofinality iff for all generic G: for all limit β such that $\omega < \beta < o(M)$, $(\beta$ is regular$)^M \to (\beta$ is regular$)^{M[G]}$.*

Proof. $\to$ is trivial from Definition 1.2.

$\leftarrow$: Assume for all generic G: for all limit β such that $\omega < \beta < o(M)$, $(\beta$ is regular$)^M \to (\beta$ is regular$)^{M[G]}$, for any limit $\gamma < o(M)$, let $\beta = cf^M(\gamma)$, we show that $\beta = cf^{M[G]}(\gamma)$.

Let $X \in \mathcal{P}(\gamma) \cap M$ be such that $type(X) = \beta$ and $sup(X) = \gamma$. Since $\beta = cf^M(\gamma)$, $(\beta$ is regular$)^M$ and by assumption $(\beta$ is regular$)^{M[G]}$.

Since $X \subseteq \gamma$, $sup(X) = \gamma$, then $cf^{M[G]}(\gamma) = cf^{M[G]}(type(X)) = cf^{M[G]}(\beta) = \beta$ □

Lemma 2.2. *If $\mathbb{P}$ preserves cofinality, then $\mathbb{P}$ preserves cardinality.*

Proof. By Lemma 2.1, M and $M[G]$ have the same regular cardinals. ZFC implies that every cardinal is either regular or $\leq \omega$ or a supremum of regular cardinals. □

3 Countable Chain Condition and Preservation of Cardinality

We have proven that preservation of cofinality implies preservation of cardinality. To show that $\mathbb{P}$ preserves cardinality it suffices to show that $\mathbb{P}$ preserves cofinality. We prove this by proving that $\mathbb{P}$ satisfies c.c.c. and that c.c.c. implies preservation of cofinality.

Definition 3.1. A forcing notion $\mathbb{P}$ satisfies the *countable chain condition* (c.c.c.) if every antichain in $\mathbb{P}$ is at most countable.

Theorem 3.1. *If $\mathbb{P}$ satisfies c.c.c., then $\mathbb{P}$ preserves cofinality.*

Proof. By lemma 2.1, it suffices to show that if $\mathbb{P}$ satisfies c.c.c., then for any regular cardinal κ^M, $\kappa^{M[G]}$ is regular. It suffices to show that for any $\lambda < \kappa$, every function $f^{M[G]} : \lambda \to \kappa$ is bounded.

Let $\dot{f}$ be a name, $p \in \mathbb{P}$. Assume:

$p \Vdash \dot{f}$ is a function from $\check{\lambda}$ to $\check{\kappa}$.

For every $\alpha < \lambda$, let $A_\alpha = \{\beta < \kappa : \exists q < p, q \Vdash \dot{f}(\alpha) = \beta\}$

If $W = \{q_\beta : \beta \in A_\alpha\}$ is a set of witness to $\beta \in A_\alpha$, then W is an antichain. Because if not, then there are $r \in \mathbb{P}$, $\beta \neq \theta$ such that $r \leq q_\beta$ and $r \leq q_\theta$. It follows that for any generic filter G containing r, $M[G] \Vdash \dot{f}(\alpha) = \beta$, $M[G] \Vdash \dot{f}(\alpha) = \theta$ and $M[G] \Vdash \beta \neq \theta$, $\perp$.

By c.c.c., W is countable.

Since κ is regular, $\bigcup_{\alpha<\kappa} A_\alpha$ is bounded by $\gamma < \kappa$. Thus, for all $\alpha < \lambda, p$ forces $\dot{f}(\alpha) < \gamma$. $\square$

Theorem 3.2. $\mathbb{P}$ *chosen in section 1 has c.c.c.*

Proof. By lemma 3.7, $\mathbb{P} = Fn(I, J)$ where $I = \omega_2 \times \omega$ and $J = \{0, 1\}$. Since J is countable, $\mathbb{P}$ has c.c.c. $\square$