$M[G] \models$ ZFC

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What do we want?

$$M \vDash \mathbf{ZFC} \Rightarrow M[G] \vDash \mathbf{ZFC}$$

Axioms:

- **Ext**
- **Foundation**
- **Pairing**
- **Union**
- **Comprehension**
- **Replacement**
- **Infinity**
- **Power set**
- **Choice**

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Ext, Foundation

Lemma

If $\mathbf{M}$ *is transitive, then* $\mathbf{M} \vDash \mathbf{Ext}$.

Lemma

If $\mathbf{M} \in \mathbf{WF}$ *and* $\mathbf{M}$ *is transitive, then* $\mathbf{M} \vDash \mathbf{Foundation}$

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Pairing

Lemma

If $\forall x, y \in \mathbf{M} \exists z \in \mathbf{M}(x \in z \wedge y \in z)$, *then* $\mathbf{M} \vDash$ **Pairing**.

Definition

$\mathbf{up}(\sigma, \tau) = \{(\sigma, \mathbf{1}), (\tau, \mathbf{1})\}$
$\mathbf{op}(\sigma, \tau) = \{\mathbf{up}(\sigma, \tau), \mathbf{up}(\sigma, \sigma)\}$

Lemma

M[G] $\vDash$ **Pairing**.

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If $\forall x \in \mathbf{M} \exists z \in \mathbf{M}(\cup x \subseteq z)$, *then* $\mathbf{M} \vDash$ **Union**

Lemma

$M[G] \vDash$ **Union**.

Proof.

Take $\tau \in M^{\mathbf{P}}$ *s.t.* $\tau_G \in M[G]$. $\bigcup dom(\tau) \in \mathbf{M}$ *is a name* π *say, s.t.* $\bigcup \tau_G \subseteq \pi_G$. □

Union

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Inf.

Lemma

If $\omega \in \mathrm{M}$, *then* $\mathrm{M} \vDash$ **Inf.**.

Lemma

$\mathrm{M[G]} \vDash$ **Inf.**.

Comprehension and buddies

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Definition

Assume $\mathrm{M} \vDash \mathbf{ZF} - \mathbf{P}$, $\mathbb{P} \in \mathrm{M}$ *is a forcing poset,* $\psi \in \mathcal{FL}_{\mathbb{P}} \cap \mathrm{M}$. *Then* $p \Vdash_{\mathbb{P},\mathrm{M}}$ *iff* $\mathrm{M}[\mathrm{G}] \vDash \psi$ *for all filters* G *on* $\mathbb{P}$ *s.t.* $p \in G$ *and* G *is* $\mathbb{P}$ *- generic over* M.

Tools

Truth Lemma, **Definability Lemma**.

Comprehension and buddies

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Tools

Truth Lemma, **Definability Lemma**.

Comp.

Lemma

If for all formulas $\varphi(x, z, \vec{w})$*:*

$$\forall z, \vec{w} \in \mathrm{M}(\{x \in z \mid \varphi^{\mathrm{M}}(x, z, \vec{w})\} \in \mathrm{M})$$

then $\mathrm{M} \vDash$ **Comprehension**

Comp.

Lemma

M[G] $\models$ **Comp.**

Proof.

Take $\varphi(x, z, \vec{w})$.

$$z = \pi_G \in \mathrm{M[G]}$$
$$w_1 = \sigma^0_G \in M[G]$$
$$\vdots$$
$$w_n = \sigma^n_G \in M[G]$$

$\Rightarrow$

$$S = \{x \in \pi_G \mid \varphi^{\mathrm{M[G]}}(x, \pi_G, \vec{\sigma^i})\}$$

$$\tau = \{(v, p) \mid v \in dom(\pi) \wedge p \in \mathbb{P} \wedge p \Vdash (v \in \pi \wedge \phi(v)\}$$

τ *exists by* **Definability Lemma** *and* $\tau_G \subseteq S$, $S \subseteq \tau_G$

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Rep.

Lemma

M[G] $\Vdash$ **Rep.**

Proof.

Take $\varphi(x, y) \in \mathcal{FL}_{\mathbb{P}} \cap \mathrm{M}$. *Assume* $\sigma_G = a \in M[G]$ *and* M[G] $\vDash \forall x \in a \exists y \phi(x, y))$.

To show: $b \in$ M[G], $rng(\phi) \subseteq b$.

Using **Definability Lemma** *and Reflection theorem (in* M*) we can take* Q *s.t.:* $\forall \pi \in dom(\sigma) \forall p \in \mathbb{P} \exists \mu \in M^{\mathbb{P}}(p \Vdash \phi(\pi, \mu)) \to \exists \mu \in Q(p \Vdash \phi(\pi, \mu))$.
$Q = \mathrm{M}^{\mathbb{P}} \cap (R(\alpha))^M$

Define $\alpha = Q \times \{\mathbf{1}\}$. □

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Power Set

Lemma

M[G] ⊨ **Power Set**.

Proof.

To show: $a \in \mathrm{M[G]} \Rightarrow \exists b \in \mathrm{M[G]} : \mathcal{P}(a) \cap M[G] \subseteq b$

Take $\tau \in M^{\mathbb{P}}, \tau_G = a$.

Define $\pi = Q \times \{\mathbb{1}\}$, where $Q = \mathcal{P}(dom(\tau) \times \mathbb{P}) \cap \mathrm{M}$. □

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Choice

Lemma

M[G] $\models$ **AC**.

Proof.

To show: $a \in \mathrm{M[G]} \Rightarrow a$ *can be well-ordered* .

Take $\tau_G = a \in \mathrm{M[G]}$ *and wellorder* $dom(\tau)$ *as* $\{\sigma^\eta \mid \eta < \alpha\}$.

Define $f = \{op(\hat{\eta}, \sigma^\eta) \mid \eta < \alpha\}$ *so that* f_G *is a function with domain* α *and* $a \subseteq ran(f)$.

Well-order by: $x \lhd y$ *iff* $min\{\eta < \alpha \mid f(\eta) = x\} < \min\{\eta < \alpha \mid f(\eta) = y\}$.

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M[G]
Comp.
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