Forcing and Independence Proofs: Assignment 4

Recall the following definition from classical topology.

Definition. *A set $X \subseteq \mathbb{R}$ is* ***dense*** *if for every rational open interval (p, q), $X \cap (p, q) \neq \varnothing$. A set $X \subseteq \mathbb{R}$ is* ***somewhere dense*** *if for some (p, q), the set $X \cap (p, q)$ is dense in (p, q). A set $X \subseteq \mathbb{R}$ is* ***nowhere dense*** *if it is not somewhere dense, i.e., if*

$$\forall p < q \in \mathbb{Q}\ \exists p', q' \in \mathbb{Q} \text{ s.t. } p \leq p' < q' \leq q \text{ and } (p', q') \cap X = \varnothing.$$

(alternatively, a set X is nowhere dense if its closure has empty interior, but this is harder to use).

By the Baire Category Theorem, the whole space $\mathbb{R}$ cannot be covered by countably many nowhere dense sets.[1] Can $\mathbb{R}$ be covered by $\aleph_1$-many nowhere dense sets? Clearly, if CH holds then $\mathbb{R} = \bigcup_{r \in \mathbb{R}} \{r\}$ is an $\aleph_1$-union of singletons, each singleton being nowhere dense. But is it true in general?

Consider the forcing $\mathbb{P}$ such that:

- The conditions are non-empty closed intervals $[p, q] \subseteq \mathbb{R}$, with $p, q \in \mathbb{Q} \cup \{-\infty, +\infty\}$ and $p < q$ (and we assume $-\infty < q < +\infty$ for all $q \in \mathbb{Q}$).
- $[p', q'] \leq [p, q]$ iff $[p', q'] \subseteq [p, q]$ (iff $p \leq p' < q' \leq q$).
- $\mathbf{1}_{\mathbb{P}} = (-\infty, +\infty) = \mathbb{R}$.

1. (a) Show that $\mathbb{P}$ has the ccc.

   (b) Write down what it means for $[p, q]$ and $[p', q']$ to be compatible.

   (c) Suppose $G \subseteq \mathbb{P}$ is a filter. Show that, in general, we could have $\bigcap G = \varnothing$.

   (d) Let $D := \{[p, q] : p, q \notin \{-\infty, +\infty\}\}$. Show that this set is dense and, if $G \cap D \neq \varnothing$, then $\bigcap G \neq \varnothing$.

   Hint: use the compactness of $[p, q]$

2. Prove the following: if $\mathsf{MA} + \neg\mathsf{CH}$ holds, then $\mathbb{R}$ cannot be covered by $\aleph_1$-many nowhere dense sets (this can be viewed as a generalization of the Baire Category Theorem).

   [Hint: start with a collection of $\aleph_1$-many nowhere dense sets. Define suitable dense sets and find a subset of $\mathbb{R}$ which is disjoint from all of them.]

[1] There are several formulations of BCT. An equivalent formulation is "non-empty open sets are not countable unions of dense sets", and another one is "countable intersections of open dense sets are dense". Countable unions of nowhere dense sets are also called **meager** or **of first category**.