# Forcing and Independence Proofs: Assignment 3

1. In the following, $\sigma, \tau, \theta$ are $\mathbb{P}$-names in $M$ and $G$ is a $\mathbb{P}$-generic filter over $M$. Are the following true or false?

   (a) If $(\sigma, \mathbf{1}) \in \tau$ then $\sigma_G \in \tau_G$.

   (b) If $(\sigma, p) \in \tau$ and $p \in G$, then $\sigma_G \in \tau_G$.

   (c) If $\sigma_G \in \tau_G$ then $(\sigma, \mathbf{1}) \in \tau$.

   (d) If $x \in \tau_G$ then there exists $(\sigma, p) \in \tau$ such that $p \in G$ and $x = \sigma_G$.

   (e) If $\sigma_G \in \tau_G$ then there exists $p \in G$ such that $(\sigma, p) \in \tau$.

   (f) If $\sigma_G \in \tau_G$ then there exists $(\theta, r) \in \tau$ such that $r \in G$ and $\theta_G = \sigma_G$.

2. A condition $p \in \mathbb{P}$ is called an *atom* if all $q, r$ extending $p$ are compatible. A forcing partial order $\mathbb{P}$ is called *atomless* if it does not contain any atoms. In Kunen, there is a proof showing that if $\mathbb{P} \in M$ is atomless than $\mathbb{P}$-generic filters over $M$ cannot exist in $M$. Prove the converse, i.e., if $\mathbb{P} \in M$ is not atomless (contains at least one atom), then there exists a $G \in M$ which is a $\mathbb{P}$-generic filter over $M$.

   (*Hint: let $G$ be the set of all $p$ which are compatible to $r$, for suitable $r$.*)

3. Let $M$ be a countable transitive model and let $\mathbb{P} \in M$ be an atomless forcing. Prove that

$$|\{G : G \text{ is a } \mathbb{P}\text{-generic filter over } M\}| = 2^{\aleph_0}.$$

4. (a) For $p \in \mathbb{P}$ and $\phi$ in the forcing language, we say that

   "*$p$ decides $\phi$*"

   if $p \Vdash \phi$ or $p \Vdash \neg\phi$. Show that for every $p \in \mathbb{P}$ and every $\phi$, there is $q \leq p$ which decides $\phi$.

   (b) Let $\tau$ be a name such that $p \Vdash \tau \in \check{\omega}$. Show that there exists $q \leq p$ and $n \in \omega$ such that $q \Vdash \tau = \check{n}$ (we say that "*$q$ decides $\tau$*").

   (c) Consider the partial order $Fn(\omega, \omega)$, i.e., finite functions $p$ with $\mathrm{dom}(p), \mathrm{ran}(p) \subseteq \omega$ ordered by $q \leq p$ iff $q \supseteq p$ (the standard partial order for adding a new real). This forcing is typically called *Cohen forcing*.

   Let $G$ be Cohen-forcing-generic over $M$. Show that $f_G := \bigcup G$ has the following property: for every $x \in \omega^\omega \cap M$, there are infinitely many $n \in \omega$, such that $x(n) < f_G(n)$ (we say that $f_G$ *is an unbounded real over $M$*).

   (*Hint: for every $x \in \omega^\omega \cap M$ and every $k \in \omega$, define appropriate dense sets $D_{x,k} = \dots$*)