Forcing and Independence Proofs: Assignment 2

1. Let $\mathbb{P}$ be a (forcing) partial order and $A \subseteq \mathbb{P}$. A is called a *maximal antichain* if it is an antichain (i.e., all $p, q \in A$ are incompatible) which cannot be extended to a larger antichain (i.e., for every $p \in \mathbb{P}$ there is a $q \in A$ which is compatible to p. Show:

   (a) If $A \subseteq \mathbb{P}$ is a maximal antichain, then $\{q \in \mathbb{P} \mid \exists p \in A\,(q \leq p)\}$ is a dense subset of $\mathbb{P}$.

   (b) (AC) If $D \subseteq \mathbb{P}$ is a dense set, then there exists a maximal antichain $A \subseteq D$.

   (c) The following are equivalent for all κ:

   - if $\{D_\alpha \mid \alpha < \kappa\}$ is a collection of dense sets, then there exists a filter G, such that $G \cap D_\alpha \neq \varnothing$ for all $\alpha < \kappa$.
   - if $\{A_\alpha \mid \alpha < \kappa\}$ is a collection of maximal antichains, then there exists a filter G, such that $G \cap A_\alpha \neq \varnothing$ for all $\alpha < \kappa$.

   (Thus, in the definition of "$\mathscr{D}$-generic filter" in the statement of Martin's Axiom (and later for forcing) it does not matter whether we consider dense subsets of $\mathbb{P}$ or maximal antichains in $\mathbb{P}$.)

2. Let I be an infinite set and J an arbitrary non-empty set, and let $\mathrm{Fn}(I, J) := \{p : p$ is a finite function with $\mathrm{dom}(p) \subseteq I$ and $\mathrm{ran}(p) \subseteq J\}$. Consider the forcing

$$\mathbb{P} = (\mathrm{Fn}(I, J), \supseteq, \varnothing),$$

   i.e., $\mathbb{P}$ is the forcing with conditions from $\mathrm{Fn}(I, J)$, with the order given by $q \leq p$ iff $q \supseteq p$ (i.e., q extends p as a function), and $\mathbf{1} = \varnothing$.

   (a) Let $D_x := \{p : x \in \mathrm{dom}(p)\}$ and $R_y := \{p : y \in \mathrm{ran}(p)\}$. Show that these sets are dense, and if G is a filter which is generic for $\mathscr{D} := \{D_x : x \in I\} \cup \{R_y : y \in J\}$, then $f_G := \bigcup G$ is a surjection from I to J (i.e., it is a function, its domain is I, and its range is J).

   (b) Show that, if $|I| < |J| = \kappa$, then $\mathsf{MA}_{\mathbb{P}}(\kappa)$ is inconsistent.