Unbeatable Strategies

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Game theory

Game theory is an extremely diverse subject, with applications in

- Mathematics
- Economics
- Social sciences
- Computer science
- Logic
- Psychology
- etc.

What we will focus on

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  - Finite games
  - Finite-unbounded games
  - Infinite games

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- **Part II.** Applications of games in analysis, topology and set theory.

We will see a gradual **Paradigm shift:**

Use mathematical objects to study games	$\Longrightarrow$	Use (infinite) games to study mathematical objects

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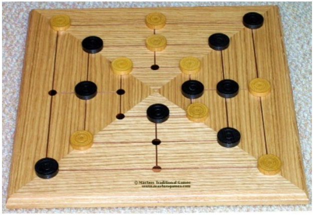
Two-player, perfect information, **zero sum** game

- There are two players, Player I and Player II. Player I starts by making a move, then II makes a move, then I again, etc.
- At each stage of the game, both players have full knowledge of the game.
- Player I wins iff Player II loses and vice versa.

Games we want to model



Games we do not want to model

We will **not** consider games with:

- An element of chance



Games we do not want to model

Specifically we will not consider games with:

- Moves taken *simultaneously*

Games we do not want to model

Specifically we will not consider games with:

- Players possessing information of which others are unaware

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2. **Finite-unbounded game:** the outcome of the game is decided at a finite stage, but when this happens is not pre-determined.
3. **Infinite game:** the game goes on forever, and the outcome is only decided "at the limit".

Part I

1. Finite games

Chess

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- Is it finite?

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- Chess is a two-player, perfect information game.
- Is it zero-sum? Let's just say: a **draw** is a **win by Black**.
- Is it finite? Yes, assuming the *threefold repetition rule.* There are 64 squares, 32 pieces, so at most 64^{33} unique positions. So chess ends after $3 \cdot 64^{33}$ moves.

  (We could easily find a much lower estimate, but we don't care).

Coding chess

Assign a unique natural number $\leq 64^{33}$ to each position of chess.

White:	
Black:	

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Each game has length n for some $n \leq 3 \cdot 64^{33}$. Let LEGAL be the set of those sequences which correspond to a sequence of legal moves according to the rules of chess. Let WIN $\subseteq$ LEGAL be those sequences that end on a win by White.

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Then "chess" is completely determined by the two sets LEGAL and WIN.

General finite game

Definition (Two-person, perfect-information, zero-sum, finite game)

Let N be a natural number (the **length** of the game), let $A \subseteq \mathbb{N}^{2N}$. The game $G_N(A)$ is played as follows:

- Players I and II take turns picking one natural number at each step of the game.

I:	x_0		x_1		$\dots$		x_{N-1}	
II:		y_0		y_1		$\dots$		y_{N-1}

  The sequence $s := \langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1} \rangle$ is called a **play of the game** $G_N(A)$.

- Player I wins the game $G_N(A)$ iff $s \in A$, otherwise Player II wins.
- $A =$ **pay-off set** for Player I; $\mathbb{N}^{2N} \setminus A =$ **pay-off set** for Player II.

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Notice two conceptual changes:

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Note: the number of possible options at each move can be infinite!

Strategies

Definition (Strategy)

A **strategy for Player I** is a function $\sigma : \bigcup_{n<N} \mathbb{N}^{2n} \longrightarrow \mathbb{N}$.

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- If $s = \langle x_0, \ldots, x_{N-1} \rangle$ then $s * \tau$ is the play of the game $G_N(A)$ in which II plays according to τ and I plays s.

Example

Example: a play of $G_N(A)$ where I uses σ and II plays $t := \langle y_0, \dots, y_{N-1} \rangle$.

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II:		y_0		y_1		$\dots$

The result of this game is denoted by $\sigma * t$.

Winning strategies

Definition (Winning strategy)

A strategy σ is **winning** for Player I iff $\forall t \in \mathbb{N}^N\, (\sigma * t \in A)$.

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Definition (Determinacy)

The game $G_N(A)$ is **determined** iff either Player I or Player II has a winning strategy.

Determinacy of finite games

Theorem (Folklore)

Finite games are determined.

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But then, Player I **does not** have a winning strategy iff

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But this holds iff II has a winning strategy in $G_N(A)$. □

Back to real chess

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Define two games:

- "White-chess" = draw is a win by White.
- "Black-chess" = draw is a win by Black.

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- "White-chess" = draw is a win by White.
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Both games are determined, so:

	White wins White-chess	Black wins White-chess
White wins Black-chess		
Black wins Black-chess		

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- "Black-chess" = draw is a win by Black.

Both games are determined, so:

	White wins White-chess	Black wins White-chess
White wins Black-chess	White wins chess	
Black wins Black-chess		

Back to real chess

What about the **draw** in actual chess?

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Back to real chess

Corollary

In Chess, either White has a winning strategy or Black has a winning strategy or both White and Black have "drawing strategies"

Back to real chess

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In Chess, either White has a winning strategy or Black has a winning strategy or both White and Black have "drawing strategies"

Of course, this is a purely theoretical result, and only tells us that one of the above must exist. It does not tell us **which one it is**.

2. Finite-unbounded games

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- We cannot extend all games to some fixed length N.
- We must specify when a game has been completed.

General finite-unbounded games

Notation: $\mathbb{N}^* := \bigcup_n \mathbb{N}^n$ (finite sequences of natural numbers).

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Definition (Two-person, perfect-information, zero sum, finite-unbounded game)

Let A_{I} and A_{II} be disjoint subsets of $\mathbb{N}^*$. The game $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ is played as follows:

- Players I and II take turns picking numbers at each step.

I:	x_0		x_1		x_2		$\dots$	
II:		y_0		y_1		y_2		$\dots$

- Player I **wins** $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ iff for some n, $\langle x_0, y_0, \dots, x_n, y_n\rangle \in A_{\mathrm{I}}$ and Player II **wins** $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ iff for some n, $\langle x_0, y_0, \dots, x_n, y_n\rangle \in A_{\mathrm{II}}$.
- The game is **undecided** iff $\langle x_0, y_0, \dots, x_n, y_n\rangle \notin A_{\mathrm{I}} \cup A_{\mathrm{II}}$ for any $n \in \mathbb{N}$.
- A_{I} = **pay-off set** for Player I, A_{II} = **pay-off set** for Player II.

Strategies

Definition (Strategy)

A **strategy for Player I** is a function $\sigma : \{s \in \mathbb{N}^* \mid |s| \text{ is even } \} \longrightarrow \mathbb{N}$.

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"Perpetual check" in chess = non-losing but not winning strategy.

Winning/non-losing strategies

Notation:

- $\mathbb{N}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{N}\}$ (infinite cartesian product of copies of $\mathbb{N}$).
- For $x \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, $x \restriction n :=$ initial segment of x of length n.

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Definition (Non-losing strategy)

Let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ be a finite-unbounded game.

1. A strategy ∂ is **non-losing** for Player I iff $\forall t \in \mathbb{N}^* \, (\sigma * t \notin A_{\mathrm{II}})$.
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Definition (Winning strategy)

1. A strategy σ is **winning** for Player I iff $\forall y \in \mathbb{N}^{\mathbb{N}} \, \exists n \, ((\sigma * (y \restriction n)) \in A_{\mathrm{I}})$.
2. A strategy τ is **winning** for Player II iff $\forall x \in \mathbb{N}^{\mathbb{N}} \, \exists n \, (((x \restriction n) * \tau) \in A_{\mathrm{II}})$.

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What does **determinacy** mean in the finite-unbounded context?

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A game $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ is **determined** if either I has a winning strategy, or II has a winning strategy, or both I and II have non-losing strategies (in which case the game will remain undecided *ad infinitum*).

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Theorem (Zermelo-König-Kalmár? Gale-Stewart?)

Finite-unbounded games are determined.

Towards the proof...

Actually, we prove a stronger result:

Lemma

Let $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ be a finite-unbounded game.

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Before proving the lemma, a question: suppose I does not have a winning strategy in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$. **Will this always remain the case?** I.e., will I never have a winning strategy at any stage of the game?

Towards the proof... (continued)

After all, Player II might make a mistake, so that Player I will **obtain** a winning strategy due to the mistake II made.

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Definition

If $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ is a finite-unbounded game and $s \in \mathbb{N}^{2n}$, then $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s)$ denotes the game **starting with position** s, i.e., assuming that the first n moves are given by s.

Formally, $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s) = G_{<\infty}(A_{\mathrm{I}}/s, A_{\mathrm{II}}/s)$ where

$$A_{\mathrm{I}}/s := \{t \in \mathbb{N}^* \mid s^\frown t \in A_{\mathrm{I}}\}$$

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Proof. We only prove 1. Suppose I has no w.s. We will define ρ such that for any $s \in \mathbb{N}^*$, I does not have a w.s. in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s * \rho)$, by induction on the length of s.

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Suppose ρ is defined on all s of length $\leq n$ and I does not have a w.s. in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; s * \rho)$. Fix s with $|s| = n$.

Claim.

$\forall x_0 \, \exists y_0$ such that I does not have a w.s. in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; (s * \rho)^\frown \langle x_0, y_0 \rangle)$.

Proof (continued)

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Otherwise, $\exists x_0$ such that $\forall y_0$ I has a w.s., say σ_{x_0, y_0}, in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}}; (s * \rho)^\frown \langle x_0, y_0 \rangle)$.

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"play x_0, and for any y_0 which II plays,
continue playing according to strategy σ_{x_0,y_0}".

This contradicts the I.H. □

Proof (continued)

Now extend ρ by defining, for every x_0, $\rho((s * \rho)^\frown \langle x_0 \rangle) := y_0$, for the y_0 given by the Claim. So ρ is defined on sequences of length $n+1$ and satisfies I.H.

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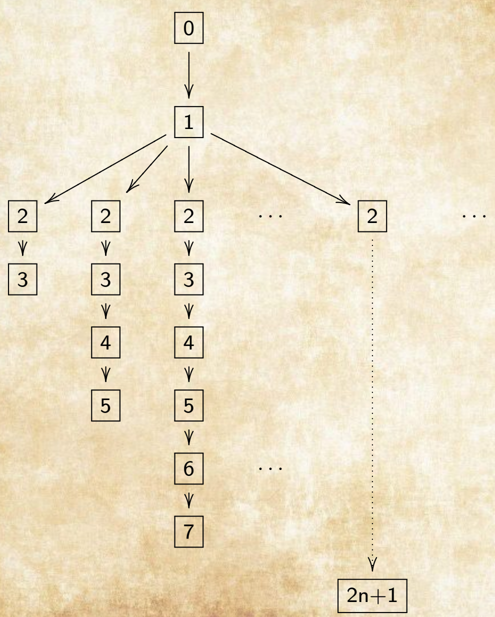
Corollary (Zermelo-König-Kalmár? Gale-Stewart?)

Finite-unbounded games are determined.

Upper bound on number of moves

Question (Zermelo, 1912). Assuming a player **has** a w.s., is there one (uniform) $N \in \mathbb{N}$ such that this player can win in at most N moves, regardless of the moves of the opponent?



There is a chip at field 0. The two players take turns in moving it one field ahead each time. Player I starts. The first who cannot make a valid move loses

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Theorem (Zermelo/König)

Assume I has a w.s. σ in $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$. Assume that, at each stage, there are **at most finitely many** *legal moves II can make. Then there is $N \in \mathbb{N}$ such that I wins in at most N moves. Similarly for Player II.*

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History: This was claimed by Zermelo, but the proof contained a gap which König filled by introducing the now well-known **König's Lemma**: "every finitely branching tree with infinitely many nodes contains an infinite path".

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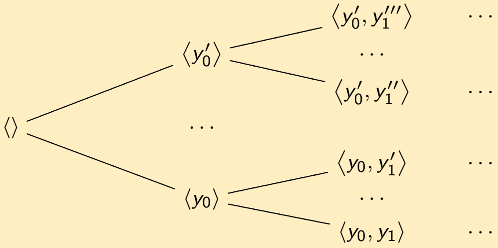
Let σ be a fixed w.s., and assume, towards contradiction, that the claim is false. Let T be the tree of all finite sequences $t \in \mathbb{N}^*$ such that $\sigma * t \notin A_{\mathrm{I}}$, ordered by end-extension.



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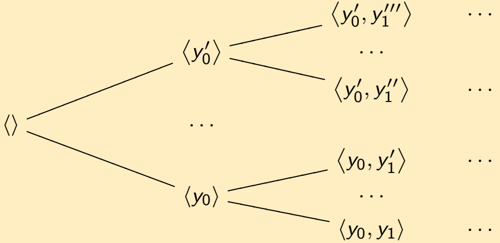

Since II has finitely many options, the tree is **finitely branching**. Since for every N, I does not win in at most N moves, the tree has **infinitely many nodes**. By **König's Lemma**, it has an infinite branch, which generates $y := \langle y_0, y_1, y_2, \dots \rangle \in \mathbb{N}^{\mathbb{N}}$.

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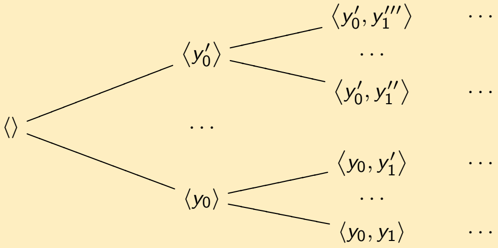

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But then, $\sigma * (y \restriction n)$ is not in A_{I} **for any** $n \in \mathbb{N}$! So σ is not a winning strategy. □

3. Infinite games

Motivation

The finite-unbounded formalism was somewhat clumsy, because we needed infinite sequences $x \in \mathbb{N}^{\mathbb{N}}$ to formulate winning strategies correctly, yet we insisted on games being decided at a **finite** stage.

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Definition (Two-person, perfect-information, zero-sum, infinite game)

Let $A \subseteq \mathbb{N}^{\mathbb{N}}$. The game $G(A)$ is played as follows:

- Players I and II take turns picking numbers at each step.

I:	x_0		x_1		x_2		$\dots$	
II:		y_0		y_1		y_2		$\dots$

- Let $z := \langle x_0, y_0, x_1, y_1, x_2, y_2, \dots \rangle \in \mathbb{N}^{\mathbb{N}}$ be the **play of the game** $G(A)$. Player I wins if and only if $z \in A$, otherwise II wins.
- A = **pay-off set** for Player I; $\mathbb{N}^{\mathbb{N}} \setminus A$ = **pay-off set** for Player I.

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A **strategy for Player I** is a function $\sigma : \{s \in \mathbb{N}^* \mid |s| \text{ is even } \} \longrightarrow \mathbb{N}$.
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A strategy σ is **winning** for Player I iff $\forall y \in \mathbb{N}^N\ (\sigma * x \in A)$.
A strategy τ is **winning** for Player II iff $\forall x \in \mathbb{N}^N\ (x * \tau \notin A)$.

Examples

We have seen examples of finite games (chess, checkers, etc.) and finite-unbounded games (chess without the threefold repetition rule, games on infinite boards etc.) What is an interesting example of an infinite game?

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- Player I wins iff infinitely many 5's have been played.

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- Player I wins iff infinitely many 5's have been played.
- Player I wins iff $\sum_{i=0}^{\infty} \left(\frac{1}{x_i+1} + \frac{1}{y_i+1} \right) < \infty$.

Examples

We have seen examples of finite games (chess, checkers, etc.) and finite-unbounded games (chess without the threefold repetition rule, games on infinite boards etc.) What is an interesting example of an infinite game?

I:	x_0		x_1		x_2		$\dots$	
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- Same as above, but with the additional condition that II must play a bigger number than I's previous move.

Some cardinality arguments

Lemma

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Proof.

Let $\{a_0, a_1, a_2, \ldots\}$ enumerate A. Let τ be the strategy "at your i-th move, play $a_i(2i+1)+1$". Let $z := x * \tau$ for some x. By construction, for each i, $z(2i+1) \neq a_i(2i+1)$. Hence, for each i, $z \neq a_i$. □

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This is only relevant if CH is false (otherwise it follows from the previous lemma).

Determinacy

Definition (Determinacy)

The game $G(A)$ is **determined** iff either Player I or Player II has a winning strategy.

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Theorem (Mycielski-Steinhaus)

Assuming AC, *there exists an* $A \subseteq \mathbb{N}^{\mathbb{N}}$ *such that* $G(A)$ *is not determined.*

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Assuming AC, *for every set* X *there exists a well-ordered set* $(I, \leq)$, *such that*

1. $|I| = |X|$, *and*
2. $\forall\alpha \in I, |\{\beta \in I \mid \beta < \alpha\}| < |I| = |X|$.

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Proof.

If you are familiar with transfinite ordinals: take $I := \kappa$, where $\kappa = |X|$, i.e., κ is the smallest ordinal in bijection with X. □

Proof

Proof of theorem. First, notice that a strategy is a function from $\mathbb{N}^*$ to $\mathbb{N}$ and $\mathbb{N}^*$ is countable. So there are $2^{\aleph_0}$ strategies. Use I with $|I| = 2^{\aleph_0}$ to enumerate the strategies of I and II:

$$\{\sigma_\alpha \mid \alpha \in I\}$$

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For each $\alpha \in I$, let

$$\text{Plays}(\sigma_\alpha) := \{\sigma_\alpha * y \mid y \in \mathbb{N}^\mathbb{N}\}$$

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We will produce two disjoint subsets of $\mathbb{N}^\mathbb{N}$: $A = \{a_\alpha \mid \alpha \in I\}$ and $B = \{b_\alpha \mid \alpha \in I\}$, by induction on $\alpha \in I$.

Proof (continued)

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Since $\{b_\beta \mid \beta < \alpha\}$ is in bijection with $\{\beta \in I \mid \beta < \alpha\}$, it has cardinality $< 2^{\aleph_0}$. But as we saw, $|\mathrm{Plays}(\tau_\alpha)| = 2^{\aleph_0}$. Hence, there is at least one element in $\mathrm{Plays}(\tau_\alpha) \setminus \{b_\beta \mid \beta < \alpha\}$, so pick some a_α from there.

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Do the same for $\{a_\beta \mid \beta < \alpha\} \cup \{a_\alpha\}$. This also has cardinality $< 2^{\aleph_0}$ so we can pick b_α in $\text{Plays}(\sigma_\alpha) \setminus (\{a_\beta \mid \beta < \alpha\} \cup \{a_\alpha\})$.

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By construction, $A \cap B = \varnothing$.

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Similarly, if τ is a strategy for II then $\tau = \tau_\alpha$ for some α. Then $a_\alpha \in \text{Plays}(\tau_\alpha)$, so again τ_α cannot be winning. □

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By a similar argument $G(B)$ is not determined either. □

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The most convenient way to measure "complexity" of subsets of $\mathbb{N}^{\mathbb{N}}$ is **topology**.

Topology on the Baire space

Notation: $s \lhd x$ means "s is an initial segment of x".

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Definition

1. For every $s \in \mathbb{N}^*$, let $O(s) := \{x \in \mathbb{N}^{\mathbb{N}} \mid s \lhd x\}$.
2. The **standard topology** on $\mathbb{N}^{\mathbb{N}}$ is generated by $\{O(s) \mid s \in \mathbb{N}^*\}$. The corresponding space is called **Baire space**.

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Equivalently: use the metric defined by

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ 1/2^n & \text{where } n \text{ is least s.t. } x(n) \neq y(n) \end{cases}$$

Some properties of this topology

Some properties:

- $\mathbb{N}^\mathbb{N}$ is a **Polish space** (second-countable, completely metrizable).
- $\mathbb{N}^\mathbb{N}$ is Hausdorff; in fact it is **totally separated**
  ($\forall x \neq y$ there are open U, V such that $x \in U$, $y \in V$ and $U \cap V = \mathbb{N}^\mathbb{N}$.)
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Set theorists typically prefer working with $\mathbb{N}^\mathbb{N}$ instead of $\mathbb{R}$ (in fact we call elements of $\mathbb{N}^\mathbb{N}$ **real numbers**).

Gale-Stewart Theorem

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Proof: Suppose A is open and I has no w.s. Then, as we did before, construct a strategy ρ for II such that I still has no w.s. in the game $G(A; (s * \rho))$ for any $s \in \mathbb{N}^*$. But now ρ must be winning, because, if not, then there is some y such that $\rho * y \in A$. But **since A is open**, there is a basic open set $O(s) \subseteq A$ such that $\rho * y \in O(s)$. But this means $s \lhd (\rho * y)$, so I **does** have a w.s. (the trivial strategy) in $G(A; s)$: contradiction.

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Similar argument for closed A. □

Finite-unbounded vs. open/closed

In fact, there is a **precise correspondence** between finite-unbounded games $G_{<\infty}(A_{\mathrm{I}}, A_{\mathrm{II}})$ and infinite games $G(A)$ with open pay-off sets A.

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- If $G_{<\infty}(A_{\mathsf{I}}, A_{\mathsf{II}})$ is given, let

$$\tilde{A}_{\mathsf{I}} := \bigcup \{O(s) \mid s \in A_{\mathsf{I}}\}$$

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$G(\tilde{A}_{\mathsf{I}})$ means **undecided = win for II**.
$G(\mathbb{N}^{\mathbb{N}} \setminus \tilde{A}_{\mathsf{II}})$ means **undecided = win for I**.

(recall "White-chess" and "Black-chess" in the finite context).

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  (recall "White-chess" and "Black-chess" in the finite context).

- Conversely, if A is open we can define $A_{\mathrm{I}} := \{s \mid O(s) \subseteq A\}$ and $A_{\mathrm{II}} := \{s \mid O(s) \cap A = \varnothing\}$.

Beyond open and closed

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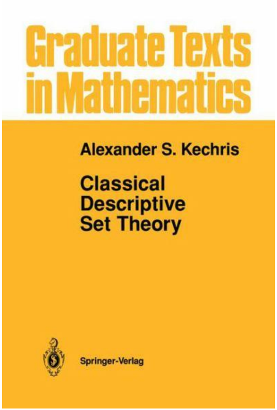
- Tony Martin, 1975: $G(A)$ is determined for Borel sets A.

Borel determinacy

Unfortunately, it is beyond the scope of this course to prove **Borel determinacy**.



If you want to read the proof, I recommend this book (pages 140–146).

Some ideas involved in the proof:

- "Unravel" complex game to one with lower complexity.
- Iterate until you reach open/closed pay-off set.
- The unraveling involves games with moves not in $\mathbb{N}$ but in $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))$ and so on (iterations of the power set all the way until ω_1).

Donald A. Martin (UCLA)

Beyond Borel

Of course, you can go further: analytic sets, coanalytic sets . . . projective sets (recursively obtained from Borel sets using **projections** (Suslin-operation) and **complements**).

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For classes of sets beyond Borel, determinacy postulates are **independent of** ZFC, i.e., they can consistently be true and false.

In set theory, it is particularly popular to look at **large cardinal axioms** (postulating the existence of “very large” objects, whose existence cannot be proved from ZFC but is thought an intuitively “natural” extension of ZFC).

Large cardinal axioms

Stronger axioms imply that larger classes are determined:

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- Martin-Steel, 1989: if there exist n **Woodin cardinals** and a **measurable cardinal** above them, then $G(A)$ is determined for every $\mathbf{\Pi}^1_{n+1}$ set A.

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- Martin-Steel, 1989: If there are infinitely many **Woodin cardinals**, then $G(A)$ is determined for every projective A.

Even further?

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AD is consistent with ZF (without choice), so we can use the theory ZF + AD instead of ZFC.

More on the Axiom of Determinacy

Why is AD so interesting? Because it implies many regularity properties for subsets of $\mathbb{R}$. For example, AD $\Rightarrow$ all sets are **Lebesgue-measurable**, have the **Baire Property** and the **Perfect Set Property**.

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However, AD can be seen in two ways:

1. ZF + AD is an alternative mathematical theory, competing with ZFC, or
2. to say that something follows from ZF + AD is just **une façon de parler** for things that hold in the definable/constructive fragment of mathematics.

What's next?

In Part II, we will look at **consequences of determinacy**. All the results will have the following structure: given a desirable property of sets (e.g. Lebesgue-measurability), construct a special game $G'(A)$, and prove that **if** $G'(A)$ is determined **then** all sets A satisfy the desired property (e.g. are Lebesgue-measurable). Typically, the moves of $G'(A)$ are not natural numbers, but some other objects that can be coded by natural numbers.

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In the context of AD, the above immediately implies that **all sets** A **satisfy the desired property**. In terms of ZFC, such a statement is meaningless.

However, these results can also be seen as postulating something about a limited class of sets. If $\boldsymbol{\Gamma}$ is a collection of subsets of $\mathbb{N}^{\mathbb{N}}$ (or the real numbers), satisfying certain closure properties (e.g., closed under continuous pre-images), then **the determinacy of all sets in** $\boldsymbol{\Gamma}$ **implies that all sets in** $\boldsymbol{\Gamma}$ **satisfy the desired property.**

End of Part I