Homework 7, due Tuesday 15 April, 12:00

1. (a) Let $\mathfrak{M}$ and $\mathfrak{N}$ be two **IPC**-models, and f a **frame**-p-morphism from $\mathfrak{M}$ to $\mathfrak{N}$. Let $\psi_1, \ldots, \psi_n$ be such that for each $w \in W$ and each $i \leq n$, $w \models \psi_i$ iff $f(w) \models p_i$. Prove that for each $\varphi(p_1, \ldots, p_n)$ we have $w \models \varphi(\psi_1, \ldots, \psi_n)$ iff $f(w) \models \varphi(p_1, \ldots, p_n)$. [5 pts]

   (b) Assume that $\mathfrak{M}$ is a model with a root w_0 such that $w_0 \models \neg\neg\psi$ and $w_0 \not\models \psi$. Define a **frame**-p-morphism f from $\mathfrak{M}$ to $\mathcal{RN}_{w_2}$ in such a way that, for each $w \in W$, $w \models \psi$ iff $f(w) \models p$. [4 pts]

2. Prove that $\vdash_{\mathbf{IPC}} \varphi$ implies $\vdash_{\mathbf{S4}} \varphi^{\Box}$ in the following manner:

   Assume $\mathfrak{M}$ on $\mathfrak{F}$ is an **S4**-countermodel to $\varphi^{\Box}$. Take the frame $\mathfrak{G}$ that is obtained from $\mathfrak{F}$ by replacing each cluster (collection of nodes that are pairwise accessible from each other) by a single node. (Try to define this exactly.) There is an obvious function from $\mathfrak{F}$ onto $\mathfrak{G}$. Show that it is a p-morphism. Define a valuation on $\mathfrak{G}$ in such a way that the resulting model $\mathfrak{N}$ is an **IPC**-model. Show, by induction on the length of $\psi(p_1, \ldots, p_n)$ that, for each $w \in W$, $\mathfrak{M}, w \Vdash \psi^{\Box}$ (as an **S4**-model) iff $\mathfrak{N}, f(w) \models \psi$ (as an **IPC**-model). Finally conclude that $\mathfrak{N}$ (as an **IPC**-model) falsifies φ. [9 pts]