Homework 4, due Tuesday 11 March, before 12.00

1. In the following, assume that x is not a free variable of ψ. Which of the following statements are intuitionistically valid? (If yes, give a proof, if not, give a countermodel).

   (a) $(\exists x\varphi(x) \to \psi) \to \forall x(\varphi(x) \to \psi)$

   (b) $\forall x(\varphi(x) \to \psi) \to (\exists x\varphi(x) \to \psi)$

   (c) $(\forall x\varphi(x) \to \psi) \to \exists x(\varphi(x) \to \psi)$

   (d) $\exists x(\varphi(x) \to \psi) \to (\forall x\varphi(x) \to \psi)$ [4 pts]

2. Show that $\forall x\neg\neg\varphi(x) \to \neg\neg\forall x\varphi(x)$ is valid on all frames with a finite number of worlds. [4 pts]

3. Show that the canonical model of a logic **L** between **IPC** and **CPC** has a root if and only if **L** has the disjunction property.

   (We say that a logic **L** has the disjunction property if for all ϕ, ψ we have that if $\mathbf{L} \vdash \phi \vee \psi$ then $\mathbf{L} \vdash \phi$ or $\mathbf{L} \vdash \psi$.) [5 pts]

4.* Let **CD** be the logic axiomatized by $\mathbf{IQC} + \forall x(\varphi \vee \psi(x)) \to \varphi \vee \forall x\psi(x)$. You may assume that **CD** is sound and complete with regard to models with a constant domain.

   Let P be a new unary predicate symbol. For every φ define the **relativization of φ to P**, $\varphi^{(P)}$, by induction on the complexity of φ, as follows:

   - If A is atomic, then $A^{(P)} := A$
   - $(\varphi \wedge \psi)^{(P)} := \varphi^{(P)} \wedge \psi^{(P)}$
   - $(\varphi \vee \psi)^{(P)} := \varphi^{(P)} \vee \psi^{(P)}$
   - $(\varphi \to \psi)^{(P)} := \varphi^{(P)} \to \psi^{(P)}$
   - $(\forall x\varphi)^{(P)} := \forall x(P(x) \to \varphi^{(P)})$
   - $(\exists x\varphi)^{(P)} := \exists x(P(x) \wedge \varphi^{(P)})$

   (Intuitively, $\varphi^{(P)}$ says the same as φ but with all terms ranging over P).

   Show that $\mathbf{CD} \vdash (\exists xP(x)) \to \varphi^{(P)}$ iff $\mathbf{IQC} \vdash \varphi$. [5 pts]