Mini-lecture on Gödel's Constructible Universe

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Forcing and Independence Proofs

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Prerequisites

Reflection Theorems

Let ZFC* be any sufficiently large finite fragment of the ZFC-axioms. Then:

- Reflection #1. For any a there is a set model M such that $a \subseteq M$, $M \models$ ZFC* and $|M| = \max(\aleph_0, |a|)$.

  (However: M is generally not transitive.)

- Reflection #2. For any a there is a set model M such that $a \subseteq M$, $M \models$ ZFC* and M is transitive.

  $M = V_\alpha$ is possible

  (However: $|M|$ may be very large.)

- Reflection #3. For any transitive set a there is a set model M such that $a \subseteq M$, $M \models$ ZFC*, M is transitive and $|M| = \max(\aleph_0, |a|)$.

E.g.: $a = \{\aleph_{20}\}$

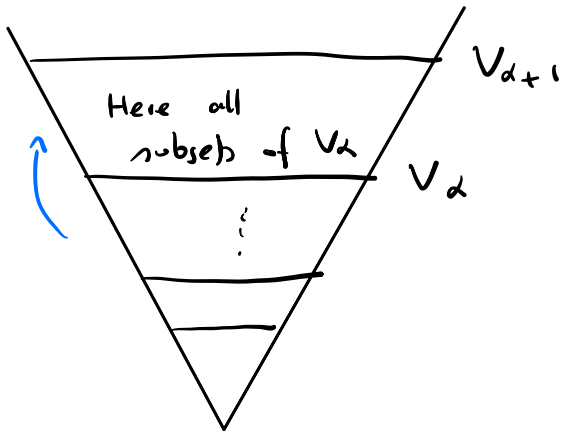
Gödel's Constructible Universe $\mathbf{L}$

Recall $\mathbb{V} = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$

$V_0 = \emptyset$

$V_{\alpha+1} = P(V_\alpha)$

$V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$



<u>Idea of L:</u> Only use <u>definable</u> subsets at stage α.

Definable $\Rightarrow$ in FOL, lang. of set theory

Definition

- A subset $Y \subseteq X$ is definable over X if there is a formula φ and parameters $a_1 \dots a_n \in X$ such that

$$Y = \{y \in X \mid X \models \varphi(y, a_1 \dots a_n)\}.$$

- $\mathrm{Def}(X) = \{Y \subseteq X \mid Y \text{ is definable over } X\}$.

Important that X is a set, and so you can formally express "$\exists \varphi$ s.t. $X \models \varphi \dots$"

Definition

- $L_0 := \varnothing$.
- $L_{\alpha+1} := \mathrm{Def}(L_\alpha)$.
- $L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$ (for λ limit).

$$\mathbf{L} := \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$$

Let's see how V differs from L.

- $L_0 = V_0 = \emptyset$
- $L_n = V_n$ for $n < \omega$ (because fin. sets are definable)
- $L_\omega = V_\omega$
- $L_{\omega+1} \stackrel{?}{=} V_{\omega+1}$

NB: $|V_{\omega+1}| = 2^{\aleph_0}$

$|L_{\omega+1}| = \aleph_0$, because there are only ctbly many formulas

$|V_{\omega+2}| = 2^{(2^{\aleph_0})}$ while $|L_{\omega+2}| = \aleph_0$

Lemma: For $\alpha > \omega$, $|L_\alpha| = |\alpha|$.

Proof: Induction + $|Def(X)| = |X|$ if X infinite.

Facts:
- L_α is transitive
- $\alpha < \beta \Rightarrow L_\alpha \subseteq L_\beta$
- $Ord \cap L_\alpha = \alpha$ (i.e. $\beta \in L_\alpha \Leftrightarrow \beta < \alpha$)

Define: $x \in \mathbb{L} \Rightarrow \rho_L(x) :=$ least α s.t. $x \in L_{\alpha+1}$.

Theorem

$\mathbf{L} \models$ ZF

⋮

- Power Set : let $x \in \mathbf{L}$, so $x \in L_\alpha$.

  Need to find $y \in L$ s.t.

  $L \models y = P(x)$

  $L \models \forall z\,(z \in y \longleftrightarrow z \subseteq x)$

  For every $z \subseteq x$, if $z \in L$ then let

  $\alpha_z := \rho_L(z)$ $\quad (\alpha_z > \alpha)$

  Let $\beta := \sup\{\alpha_z \mid z \subseteq x \text{ and } z \in \mathbf{L}\}$

  $P(x)^{L} := \{z \mid z \subseteq x \wedge z \in \mathbf{L}\} = P(x) \cap \mathbf{L} \subseteq L_{\beta+1}$

  Then : use Comprehension in L.

- For Comprehension: you need Reflection to shift from "$L \models \varphi$" to "$L_\delta \models \varphi$".

⋮

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$\mathbf{L} \models$ AC (without assuming AC)

In fact $\mathbb{L} \models$ Global Choice, i.e. there is a (class) wellorder $<_L$ of all of $\mathbf{L}$.

Proof: Inductively. Assume L_α is wellordered by $<_\alpha$. It follows $L_\alpha^{<\omega}$ are wellordered by $<_\alpha^{lex}$.

Now: define $<_{\alpha+1}$ on $L_{\alpha+1}$:

- Given $x, y \in L_{\alpha+1}$:
  (1.) If $x, y \in L_\alpha \Rightarrow \left(x <_{\alpha+1} y \iff x <_\alpha y \right)$
  (2) If $x \in L_\alpha$, $y \in L_{\alpha+1} \setminus L_\alpha \Rightarrow x <_{\alpha+1} y$
  (3) If $x, y \in L_{\alpha+1} \setminus L_\alpha \Rightarrow$

Definable by some φ, ψ, and $a_1 .. a_n \in L_\alpha$.

$x <_{\alpha+1} y \;:\iff$ the smallest $\ulcorner \varphi \urcorner$ and $<_\alpha^{lex}$-least $\langle a_1 .. a_n \rangle$ s.t. x is def. by $\varphi(.., a_1 .. a_n)$

is $<_\alpha^{lex}$-smaller than

the smallest $\ulcorner \psi \urcorner$ and $<_\alpha^{lex}$-least $\langle b_1 .. b_k \rangle$ s.t. y is def. by $\psi(.., b_1 .. b_k)$.

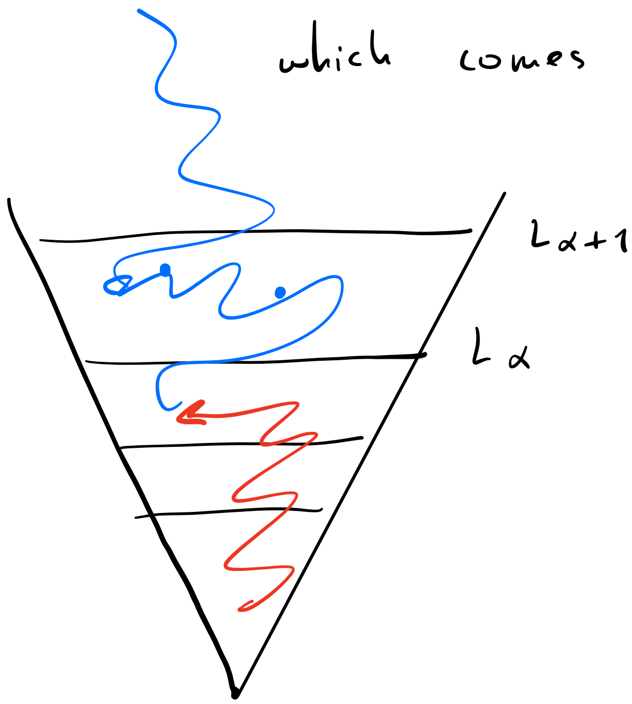
$$x = \{ z \mid L_\alpha \models \varphi(z, a_1 \ldots a_n) \}$$

$$y = \{ z \mid L_\alpha \models \psi(z, b_1 \ldots b_k) \}$$

$$\langle \ulcorner \varphi \urcorner, a_1 \ldots a_n \rangle$$

$$\langle \ulcorner \psi \urcorner, b_1 \ldots b_k \rangle$$

which comes first?



Theorem

$\mathbf{L}$ is absolute for transitive models.

More specifically, the function $\alpha \mapsto L_\alpha$ is absolute for transitive models.

Corollary: For any transitive M, $(M \models ZF^*)$

$$\alpha \in M \iff L_\alpha \in M$$

Proof: Sketchy: First-order-logic $(M \models \varphi)$ is defined by recursion using basic operations

$\Rightarrow$ absolute.

$L_\alpha :=$ defined by recursion using $\mathrm{Def}(x)$ $\Rightarrow$ absolute.

Check: $\mathrm{Def}(x)^M = \mathrm{Def}(x)$

⋮

$\boxtimes$

Definition

The statement "$\forall x \exists \alpha \in \mathrm{Ord}\ (x \in L_\alpha)$" is called the Axiom of Constructiblity, and usually abbreviated by

$$\mathbf{V} = \mathbf{L}.$$

Corollary 1

$$\mathbf{L} \models (\mathbf{V} = \mathbf{L})$$

Proof: $\forall x \in L \ \exists \alpha \in Ord \quad x \in L_\alpha$ (by def).
$\iff \forall x \in L \ \exists \alpha \in Ord \quad (x \in L_\alpha)^L$
$\iff \forall x \in L \ \exists \alpha \in Ord \cap L \quad (x \in L_\alpha)^L$
$\iff [V = L]^L$

Corollary 2

If $\mathbf{M}$ is any transitive class model of ZF* with $\mathrm{Ord} \subseteq \mathbf{M}$, then $\mathbf{L} \subseteq \mathbf{M}$.

Proof: $\alpha \in M \Rightarrow L_\alpha \in M \Rightarrow L_\alpha \subseteq M \Rightarrow \mathbf{L} \subseteq M$ $\square$

This means: $\mathbf{L}$ is the <u>minimal</u> model of ZFC.

Corollary 3

If $\mathbf{M}$ is any transitive class model of ZF* $+ \mathbf{V} = \mathbf{L}$ with $\mathrm{Ord} \subseteq \mathbf{M}$, then $\mathbf{L} = \mathbf{M}$.

Proof: $M \models V = L$, then $M \models \forall x \ \exists \alpha \ (x \in L_\alpha)$
So $\forall x \in M \ \exists \alpha \in Ord \cap M \ (x \in L_\alpha)^M$
$\overset{abs}{\Longrightarrow} \forall x \in M \ \exists \alpha \in Ord \ (x \in L_\alpha)$
$\forall x \in M : x \in \mathbf{L}$
So $\mathbf{M} \subseteq \mathbf{L}$

NB: ZF* has to be enough to prove absoluteness of L_α's.

Miniature versions

Definition

If M is a transitive set model, then the height of M, denoted by $o(M)$, is defined as the least ordinal not in M (alternatively, $o(M) := \mathrm{Ord} \cap M$).

Corollary 4

$L_\delta \models (\mathbf{V} = \mathbf{L})$ for all limit δ.

Just as before.

Corollary 5

If M is any set model of ZF* with height δ, then $L_\delta \subseteq M$.

$\alpha \in M \Rightarrow \alpha < \delta \Rightarrow L_\alpha \in M$. So $L_\alpha \subseteq M$, so

$$L_\delta = \bigcup_{\alpha<\delta} L_\alpha \subseteq M$$

Corollary 6

If M is any set model of ZF* $+ \mathbf{V} = \mathbf{L}$ with height δ, then $M = L_\delta$.

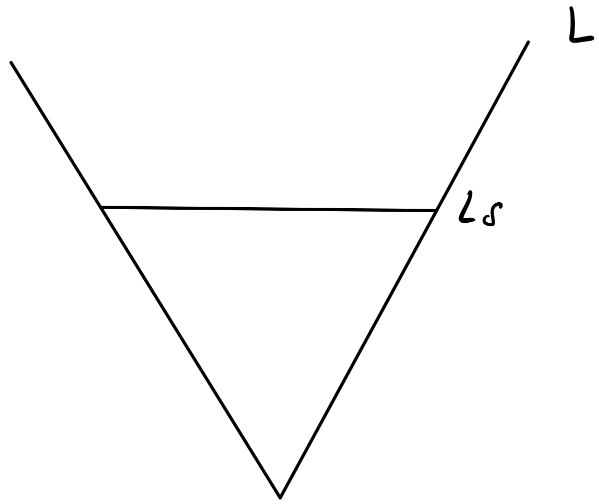
$M \models (V=L)$

$M \models \forall x \, \exists \alpha \; x \in L_\alpha$

$\forall x \in M \; \exists \alpha \in \mathrm{Ord} \cap M \; (x \in L_\alpha)^M$

$\forall x \in M \; \exists \alpha < \delta \; (x \in L_\alpha)$

So $x \in M \Rightarrow x \in \bigcup_{\alpha<\delta} L_\alpha = L_\delta$ so $M \subseteq L_\delta$.



Only $M \models ZF + (V = L)$

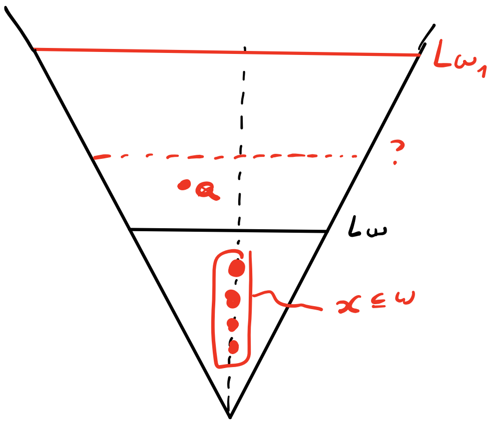
are L_δ's

where $\delta = o(M)$

Condensation Lemma

Suppose $\mathbf{V} = \mathbf{L}$. Any $x \subseteq \omega$ is contained in L_{ω_1}.

More generally, any $x \subseteq \kappa$ is contained in L_{κ^+}.



Proof: Let $x \subseteq \omega$ and let

$a = \omega \cup \{x\}$ (to make it transitive)

By Reflection #3: there is $M \models ZF^* + V=L$, s.t.

- $a \subseteq M$
- $|M| = \max(|a|, \aleph_0) = \aleph_0$
- M transitive

But then: $M = L_\delta$, for $\delta = o(M)$.

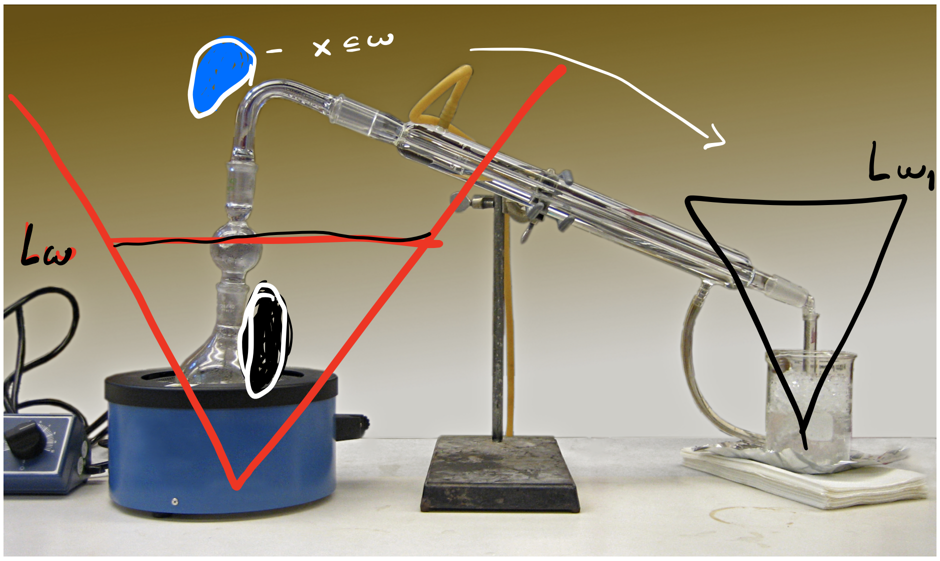
What's $o(M) < \omega_1$ (a countable ordinal)

($o(M) = M \cap \mathrm{Ord}$ and $|M| = \omega$)

So $\delta < \omega_1$.

So $a \subseteq L_\delta \subseteq L_{\omega_1}$. So $x \in a \subseteq L_{\omega_1}$.

So $x \in L_{\omega_1}$. $\square$



Corollary

$\mathbf{L} \models$ GCH

Proof: Assume $V=L$.

Since any $x \subseteq \kappa$ is in L_{κ^+}, so

$$P(\kappa) \subseteq L_{\kappa^+}$$

$$2^\kappa = |P(\kappa)| \leq |L_{\kappa^+}| = |\kappa^+| = \kappa^+ \leq 2^\kappa$$

$\square$