If $x \in M$ and φ absolute, $\exists! y\ \varphi(x,y)$

$(x \mapsto y)$

$\Rightarrow y \in M$

- Relations, functions, dom, ran, ... ordinals, new, ω.

$\boxed{\text{Absolute}}$

$\alpha \in M$ then "$\alpha \in Ord \iff M \models (\alpha \in Ord)$"

⚠ Not absolute: $P(x)$, $\{f : X \to Y\}$ $\mathbb{R}$

ω^ω, 2^ω $P(\omega)$ etc.

Cardinalities / Cardinals.

E.g.: $\alpha \in Ord$, $\alpha \in M$

$M \models (\alpha = \omega_1)$

$M \models \alpha$ is the least unctbl. ordinal.

Notation: $\omega_1^M :=$ the ordinal α such that

$(\alpha \text{ is the least unctbl ord})^M$

$P(x)^M = P(x) \cap M$

Keep in Mind: (later)

- $\mathbb{P} \in M$
- Dense sets $D \subseteq \mathbb{P}$ that are in M
  $D \in M$
- G = generic object: $G \subseteq \mathbb{P}$, $(G \subseteq M)$
  but $G \notin M$.

In some proofs: $D := \{ p \in \mathbb{P} \mid p \Vdash^* \phi \}$ (Compreh. in M)

So $D \in M$ because $\Vdash^*$ is definable in M.

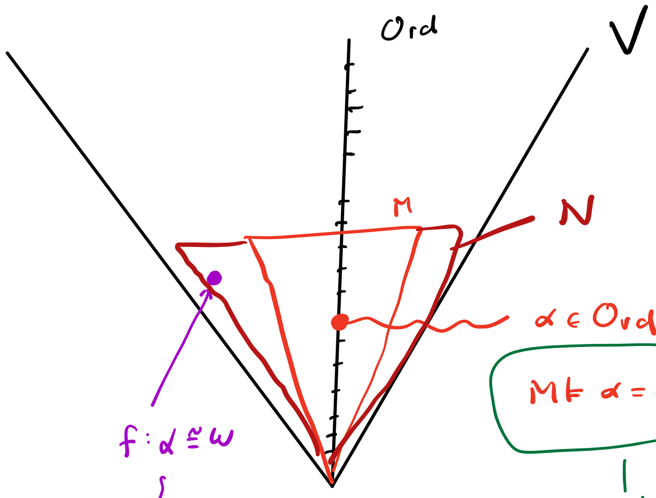
General: M ctbl. trans. model.

$\omega \in M$ and $\omega \subseteq M$

Take any $a \subseteq \omega$: $a \subseteq M$

not necessarily $a \in M$

otherwise $P(\omega) \subseteq M$ $\lightning$



$f \notin M$

$f \in N$

$N \models \alpha$ is ctbl.

$\alpha \in \mathrm{Ord}$

$f : \alpha \cong \omega$

$\alpha = \omega_1^M$

$\boxed{M \models \alpha = \omega_1}$

misleading: $M \models (\alpha = \text{least unctbl ord.})$

$\alpha = \boxed{\omega_1^M}$

$\emptyset$

$P(x)^M$

You will not see κ^M (κ cardinal)

You will see $\aleph_2^M$ $\aleph_\omega^M$

$(\aleph_\omega^+)^M$ ..

... ω_1 ...

.. ω_2 ...

. κ ...

$\forall \kappa$

Thm: ... ω_1 ...

Thm: ... κ ....

ZFC $\vdash$ ......

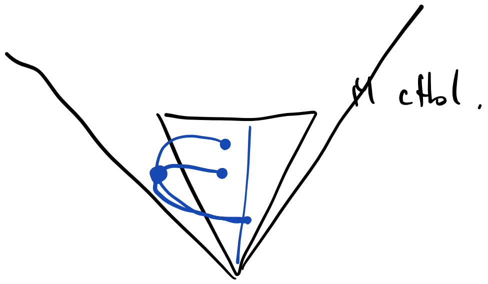
ZFC $\vdash$ $\omega_1 \leq 2^{\aleph_0}$

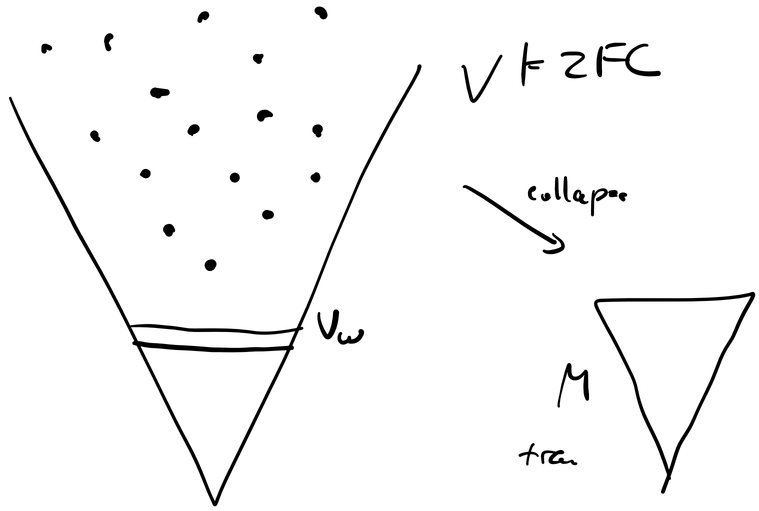
e.g. M ctbl transitive.

$M \models$ ZFC then $M \models (\omega_1 \leq 2^{\aleph_0})$

$$\omega_1^M \leq \underbrace{(2^{\aleph_0})^M}$$

$V \models$ ctbl ord α $\leq$ ctbl ordinal β





$ZFC \vdash \exists$ unctbl ordinal

$\exists x\ \phi(x)$

fix $a \in M$ $\phi(a)$

$0 \quad 1 \quad 2 \quad \dots$

$\aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \aleph_3 \quad \dots$ $\models PA$

$P(x) = y$

$\forall z\ (z \in y \Leftrightarrow z \subseteq x) \equiv \varphi(x,y)$

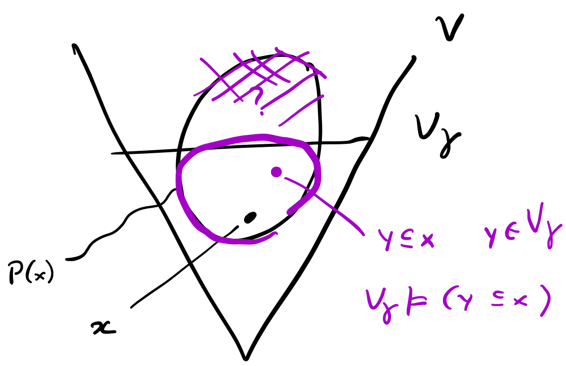
if γ = limit ordinal then $P(x)$ is absolute for V_γ

$\forall x \in V_\gamma$: if y satisfies $\varphi(x,y)$ then $y \in V_\gamma$

and $V_\gamma \models \varphi(x,y)$

To prove: suppose $x \in V_\gamma$. Let $y \subseteq x$. We must show that $y \in V_\gamma$. If this is the case for all $y \subseteq x$ then $P(x) \subseteq V_\gamma$.

Also $P(x)^{V_\gamma} = P(x) \cap V_\gamma$ (Bec "$\subseteq$" is Δ_0 $\Rightarrow$ absolute.)



"$\varphi(x)$ absolute for M"

$\forall x \in M \; (\varphi(x) \leftrightarrow \varphi^M(x))$

"$P(x)$ is absolute for M"

means: $P(x) \in M$ (and absolute)

"every $y \subseteq x$ is $y \in M$"

$$\boxed{P(x)^M = P(x)}$$

- First came forcing: $\mathbb{P}$ (1964)
- People used slightly diff. (related) forcing arguments to show $Con(ZFC + \Phi)$.

- Martin+ Solovay 1972...? 1980 ?

MA = "a lot of forcing has been done already"

$ZFC + MA \vdash \Phi$

By iterating "all possible" ccc - forcings, you get a model for ZFC + MA

Seminar Room

Wed 31: 13 – 17 (later)

Thu 1: 13 – 17 (16:00 ?)

Fri 2: 13:30 – 17:30 (until 16:00)

13:00 – 14:30
14:30 – 16:00
16:00 – 17:30 (±)

13:00 – 14:20
14:30 – 16:00 (±)

(maybe here instead)