1. Want to show $ZFC \nvdash CH$.

what does that mean?

$ZFC + \neg CH$ is <u>consistent</u>.

There is a model of $ZFC + \neg CH$

Prove $M \models ZFC + \neg CH$

$ZFC \vdash \exists M \models ZFC + \neg CH$

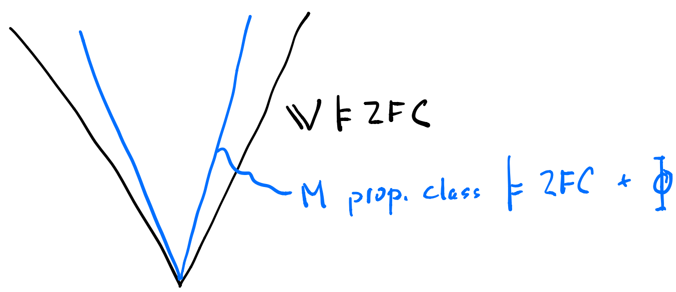
$ZFC \vdash$ it's own consistency <u>Problem</u>

2. Instead:

IF ZFC is consistent THEN $ZFC + \Phi$ is consistent

$Con(ZFC) \to Con(ZFC + \Phi)$

<u>Intuition</u>:



Q: What does $M \models \varphi$ mean for proper classes M?

A: <u>Relativization</u>

φ^M = replace all occ. of "$\forall x$" by "$\forall x \in M$" and "$\exists x$" by "$\exists x \in M$"

$\varphi \rightsquigarrow \varphi^M$ effective syntactic procedure

Now: we **say**: $M \models ZFC + \Phi$ (M transitive, proper class)

we **mean**: for any formula φ in $ZFC + \Phi$,

$$ZFC \vdash \varphi^M$$

⚠ **NOTE**: you cannot say $ZFC \vdash \forall \varphi \in ZFC + \Phi : \varphi^M$

If so: $Con(ZFC) \to Con(ZFC + \Phi)$

Why? **Intuitively**: $V \models ZFC \Rightarrow$ in V, we have $M \models ZFC + \Phi$

Formally: If $ZFC + \Phi \vdash \perp$

Then $\varphi_1 \cdots \varphi_n$ in $ZFC + \Phi$, we have

$$\vdash (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \perp$$

Then (clear) $\vdash \big((\varphi_1 \wedge .. \varphi_n) \to \perp \big)^M$

Then since $ZFC \vdash \varphi_i^M$ for $i = 1 \sim n$

$ZFC \vdash \perp^M$

$ZFC \vdash \perp$ $\square$

Talking about φ^M: does φ^M **mean** the same as φ?

Ex: $(x \in y)^M \longleftrightarrow x \in y$? yes!

Ex: $(x = P(y))^M \leftrightarrow x = P(y)$? No!

Def: M is a (prop. class) model, φ formula, φ is **absolute** for M, if $\varphi^M \leftrightarrow \varphi$

$ZFC \vdash \varphi^M \leftrightarrow \varphi$ (or some part of ZFC)

Lemma: If φ is Δ_0-formula (all quantifier are of form $\forall x \in y$ or $\exists x \in y$)

then φ is absolute for all transitive models M.

Example of absolute: $x \subseteq y$, $x \cap y = z$, $x = \bigcup y$, $x = \omega$, $x < \omega$,

f is a function, α is an ordinal, α is a succ/limit ord.

R is a rel, R is transitive, ...

$\varphi(\dots a, \dots b \dots)$

φ^M is only defined if the param a, b are in M.

Example of non-absolute: $x = P(y)$

κ is a cardinal

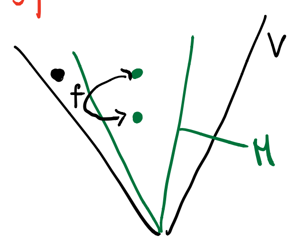
$|x| = |y|$

$2^\kappa = \lambda$

cardinal arithmetic

- R is a set
- $R \in M$
- "R is a rel" $\leftrightarrow$ ("R is a rel")^M

$\exists f : X \cong Y$



⚠ $M \subseteq V$ prop. class

you will have lots of subsets $X \subseteq M$ which are $X \notin M$

⚠ M ctbl, $X \subseteq M$ $\omega \in M$

$X = \{x_0\ x_1, x_2 \dots\}$

$f = \{(0, x_0), (1, x_1), \dots\} \subseteq M$

But: $f \notin M$

Reflection!

Recall from Model Theory: If $|M| \geq \omega$ then there is $N \preceq M$ and $|N| = \omega$ (down. Löw. Skolem)

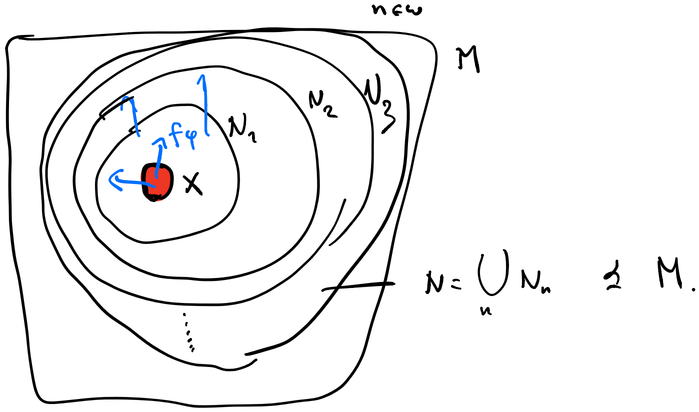
More generally: $X \subseteq M$, there is $X \subseteq N \preceq M$ s.t.

$$|N| = \max(\aleph_0, |X|)$$ (N = Skolem Hull of X in M)

You add Skolem functions: $f_\varphi(\vec{x})$ such that

if $M \models \exists y\, \varphi(\vec{x}, y)$ then $M \models \varphi(\vec{x}, f_\varphi(\vec{x}))$

Then
$$N_0 = X$$
$$N_{n+1} = N_n \cup \{ f_\varphi(\vec{x}) \mid \vec{x} \in N_n \}$$
$$N = \bigcup_{n \in \omega} N_n$$



? Analogue for set theory: V infinite model, there is $M \preceq V$ and $|M| = \omega$ (then $M \models ZFC$)

Not Quite!

Instead: Start with an arbitrary *finite* fragment $ZFC^* \subseteq ZFC$

Reflection Theorem: $ZFC^* = \{\varphi_1 \dots \varphi_n\} \subseteq ZFC$

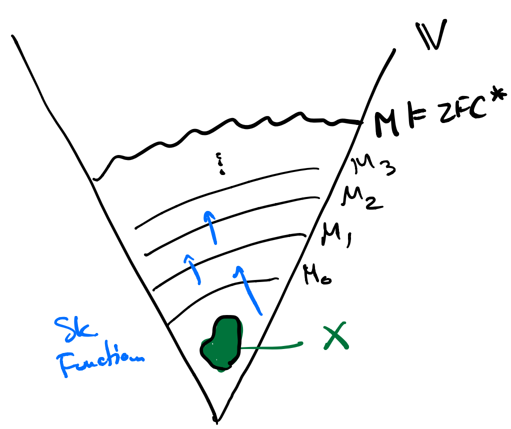
1. For any X there is M such that $X \subseteq M$ and $M \models ZFC^*$ and $|M| = \max(\aleph_0, |X|)$
   (M may not be transitive)

2. For any X there is a transitive M ($M = V_\alpha$) s.t. $M \models ZFC^*$ and $X \subseteq M$

3. If X transitive then there is transitive M s.t. $M \models ZFC^*$ and $X \subseteq M$ and $|M| = \max(\aleph_0, |X|)$
   (Apply Mostowski Collapse after 1)

$\Longrightarrow$ Take $X = \emptyset$.
Then there is a countable, transitive $M \models ZFC^*$.

Proof-idea:



LAST THING:

- $V_\lambda \models ZFC \setminus$ Replacement (λ limit ord.)
- $H_\kappa \models ZFC \setminus$ Power Set (κ regular card.)
- If κ is str. inacc. cardinal, then $V_\kappa = H_\kappa$

$\Rightarrow V_\kappa = H_\kappa \models ZFC$

Refl. 3

$X = \{\omega_1\} \subseteq M$ M ctbl, not transitive.

$\{\omega_1\} \in M$

$\omega_1 \in M$

Collapse: $\pi: M \to \bar{M}$. Then clearly $X \notin M$

bec. o.w. $\omega_1 \in M$

$\omega_1 \in M$