Set Theory Project: Introduction to Forcing

Assignment 3

Part A: The forcing relation

1. Prove the following facts about the "semantic" forcing relation.[1]

   (a) Suppose that $\forall p \in \mathbb{P}\ \exists q \leq p\ (q \Vdash \phi)$. Then $\mathbf{1} \Vdash \phi$.

   (b) Similarly, suppose $p_0 \in \mathbb{P}$ is such that $\forall p \leq p_0\ \exists q \leq p\ (q \Vdash \phi)$. Then $p_0 \Vdash \phi$.

   (c) The following are equivalent:

      i. $p_0 \Vdash \phi$,

      ii. $\forall p \leq p_0\ (p \Vdash \phi)$, and

      iii. $\{p \leq p_0 \ :\ p \Vdash \phi\}$ is dense below p_0.

2. For $p \in \mathbb{P}$ and ϕ in the forcing language, we say p *decides* ϕ if $p \Vdash \phi$ or $p \Vdash \neg\phi$. Show that for every $p \in \mathbb{P}$ and every ϕ, there is $q \leq p$ which decides ϕ.

3. Prove the inductive rule for forcing a disjunction: $p \Vdash \phi \vee \psi$ iff $\{q \leq p \ :\ q \Vdash \phi \vee q \Vdash \psi\}$ is dense below p. You may do this either by reducing $\phi \vee \psi$ to a statement with conjunctions and negations, or use the definition of the semantic relation and prove the statement directly.

[1] I.e., using the definition: $(p \Vdash \varphi)^M \ :\Leftrightarrow\ \forall G$ $\mathbb{P}$-generic over M if $p \in G$ then $M[G] \models \varphi$, not using the "syntactic" forcing relation $\Vdash^*$.

Part B: Properties of forcing

1. Let τ be a name such that $p \Vdash \tau \in \check{\omega}$. Show that there exists $q \leq p$ and $n \in \omega$ such that $q \Vdash \tau = \check{n}$. We say that q *decides* τ.

2. A forcing partial order $\mathbb{P}$ is called *σ-closed*, if for any decreasing sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ there exists a condition $q \in \mathbb{P}$ such that $p_n \geq q$ for all n (not all forcings are σ-closed, in fact, most of the forcings appearing in this section of Kunen are *not* σ-closed).

   Let $\dot{f}$ be a $\mathbb{P}$-name such that $p_0 \Vdash (\dot{f} : \omega \to \omega)$. Prove that there exists a function $g : \omega \to \omega$ in M and a $q \leq p$ such that $q \Vdash \dot{f} = \check{g}$.

   Conclude form this that if $\mathbb{P}$ is a σ-closed forcing and G is $\mathbb{P}$-generic over M, then $\omega^\omega \cap M = \omega^\omega \cap M[G]$ (i.e., $\mathbb{P}$ does not add new functions from ω to ω).

Part C: Cohen forcing

1. Consider the partial order $Fn(\omega, \omega)$, i.e., finite functions p with $\text{dom}(p), \text{ran}(p) \subseteq \omega$ ordered by $q \leq p$ iff $q \supseteq p$ (the standard partial order for adding a new real). This forcing is typically called *Cohen forcing*.

   Let G be Cohen-forcing-generic over M. Show that $f_G := \bigcup G$ has the following property: for every $x \in \omega^\omega \cap M$, there are infinitely many $n \in \omega$, such that $x(n) < f_G(n)$ (we say that f_G *is an unbounded real over* M).

   Hint: for every $x \in \omega^\omega \cap M$ and every $k \in \omega$, define appropriate dense sets $D_{x,k} = \dots$

2. Let $a, S \subseteq \omega$ be infinite sets. We say that S *splits* a if both $a \cap S$ and $a \setminus S$ are infinite (so S "splits" a into two infinite parts). If $M \subseteq M[G]$ is a generic extension and $S \in M[G]$, then we say that S *is a splitting real over* M, if for every $a \in [\omega]^\omega \cap M$, S splits a. Clearly, a *splitting real* S cannot be in M. Show that if f_G is as above, then $\{n : f_G(n) = 0\}$ is a splitting real over M.

   Hint: for every infinite $a \subseteq \omega$ and every $k \in \omega$, define appropriate dense sets $D_{a,k}$.