Paraconsistent Set Theory
Talk Summary

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1 Introduction

In set theory, if we allow every predicate to determine a set we run into contradictions such as the Russel's paradox. In classical logic every statement can be derived from a contradiction resulting in a theory that is trivial. One approach to avoid this is to admit contradictions but work with a sufficiently weak underlying logic such that the infrence $\Phi,\ \neg\Phi \vdash \Psi$ does not hold for every sentence Φ and Ψ. Such a logic is called paraconsistent.

In his Paper "Transfinite Numbers in Paraconsistent Set Theory" [1] Zach Weber gives an axiomatic development of naive set theory in paraconsistent logic. Here I present some of the results from his paper.

2 Logic

We start by fixing the language and logic. For the language we use the standard language of first order set theory: with primitives $\wedge$, $\neg$, $\forall$, $=$ and $\in$ and variables x,y,z... and formulas Φ, Ψ, Υ... built up by standard formation rules. We use the shorthand $\Phi \wedge \Psi$ for $\neg(\neg\Phi \vee \neg\Psi)$, $\Phi \leftrightarrow \Psi$ for $(\Phi \rightarrow \Psi) \wedge (\Psi \leftrightarrow \Phi)$ and $\exists$ for $\neg\forall\neg$. We use the following paraconsistent logic:

Axioms

I $\Phi \rightarrow \Phi$

IIa $\Phi \wedge \Psi \rightarrow \Phi$

IIb $\Phi \wedge \Psi \rightarrow \Psi$

III $\Phi \wedge (\Psi \vee \Upsilon) \rightarrow (\Phi \wedge \Psi) \vee (\Phi \wedge \Upsilon) \quad (distribution)$

IV $(\Phi \rightarrow \Psi) \wedge (\Psi \rightarrow \Upsilon) \rightarrow (\Phi \rightarrow \Upsilon) \quad (conjuncivesyllogism)$

V $(\Phi \rightarrow \Psi) \wedge (\Phi \rightarrow \Upsilon) \rightarrow (\Phi \rightarrow \Psi \wedge \Upsilon)$

VI $(\Phi \rightarrow \neg\Psi) \rightarrow (\Psi \rightarrow \neg\Phi) \quad (contraposition)$

VII $\neg\neg\Psi \to \Psi$ $\quad$ (*doubenegationelimination*)
VIII $(\Phi \to \Psi) \to \neg(\Phi \wedge \neg\Psi)$ $\quad$ (*counterexaple*)
IXa $(\Phi \to \Psi) \to [(\Psi \to \Upsilon) \to (\Phi \to \Upsilon)]$
IXb $(\Phi \to \Psi) \to [(\Upsilon \to \Phi) \to (\Upsilon \to \Psi)]$ $\quad$ (*hypotheticalsyllogisms*)
X $\forall \Phi \to \Phi(y/x)$
XI $\forall x(\Phi \to \Psi) \to (\Phi \to \forall x \Psi)$
XII $\forall x(\Phi \vee \Psi) \to \Phi \vee \forall x \Psi$

In axiom X, y is free for x. In axioms XI and XII x is not free in Φ.

Rules

I $\Phi, \Psi \vdash \Phi \wedge \Psi$ $\quad$ (*adjunction*)
II $\Phi, \Phi \to \Psi \vdash \Psi$ $\quad$ (*modusponens*)
III $\Phi \to \Psi, \Upsilon \to \Delta \vdash (\Psi \to \Upsilon) \to (\Phi \to \Delta)$
IV $\Phi \vdash \forall x \Phi$
V $x = y \vdash \Phi(x) \to \Phi(y)$ $\quad$ (*substitution*)

We also add the meta rule:

$$\text{If } \Phi \vdash \Psi, \text{ then } \Phi \vee \Upsilon \vdash \Psi \vee \Upsilon.$$

Note that $\Phi \vdash \Psi \to \Phi$ is **not** valid for all Φ and Ψ.

t-rules

We introduce a new constant t with the rules $\Phi \vdash t \to \Phi$ and $t \to \Phi \vdash \Phi$. We also introduce the shorthand:

$$\Phi \mapsto \Psi := \Phi \wedge t \to \Psi.$$

The main appeal of $\mapsto$ is that

$$\Phi \vdash \Psi \mapsto \Phi;\ \Phi \to \Psi \vdash \Phi \mapsto \Psi \text{ and } \Phi, \Phi \mapsto \Psi \vdash \Psi.$$

for all Φ.

Proof. I only prove that $\Phi, \Phi \mapsto \Psi \vdash \Psi$, the other two are similar.

1. Φ $\quad$ (assumption)
2. $\Phi \mapsto \Psi$ $\quad$ (assumption)
3. $\Phi \wedge t \to \Psi$ $\quad$ (def. of $\mapsto$)
4. $t \to \Phi$ $\quad$ (1, t-rule)
5. $t \to t$ $\quad$ (AI)
6. $t \to \Phi \wedge t$ $\quad$ (4,5 AV)
7. $t \to \Psi$ $\quad$ (3,6 AIV)
8. Ψ $\quad$ (7, t-rule).

□

3 Set Theory

The axioms of our set theory are as follows:

Extensionality: $x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)$ and

Comprehension:[1] $\exists y \forall x(x \in y \leftrightarrow \Phi(x))$.

We allow y to occur free in Φ to obtain sets like $Z = \{x : x \notin Z\}$. We let $V := \{x : \exists y(x \in y)\}$ and $\emptyset := \{x : \forall y(x \in y)\}$. We have

Theorem 3.1. *$\forall x(x \in V)$ and $\forall x(x \notin \emptyset)$.*

Assuming that our theory in not trivial[2] (meaning $\vdash \Phi$ for all Φ) we have:

Theorem 3.2. *$\nvdash \exists x(x \notin V)$ and $\nvdash \exists x(x \in \emptyset)$.*

Proof. we only proof that $\nvdash \exists x(x \in \emptyset)$ the other is similar. We show that $\exists x(x \in \emptyset) \vdash \Phi$ for any sentence Φ. We have

$$\begin{aligned} x \in \emptyset &\rightarrow \forall y(x \in y) \\ &\rightarrow x \in \{z : \Phi\} \\ &\rightarrow \Phi. \end{aligned}$$

□

I state the next theorem without proof.

Theorem 3.3. *$\exists x(x \in a \wedge x \notin b) \rightarrow a \neq b$. That is sets that differ with respect to membership are not identical.*

Russel's paradox is here a theorem.

Theorem 3.4. *$\exists x(x \neq x)$.*

Proof. By comprehension the Russel's set exist. Let $R = \{x : x \notin x\}$.

1. $\forall x(x \in R \leftrightarrow x \notin x)$ (Comprehension)
2. $R \in R \leftrightarrow R \notin R$ (1, AX)
3. $R \in R \rightarrow R \notin R$
4. $R \notin R \vee R \notin R$ (3, AVIII)
5. $R \notin R$ (4, AV)

[1] Weber uses the axiom of abstraction, $x \in \{z : \Phi(z)\} \leftrightarrow \Phi(x)$, instead comprehension. He then derives comprehension as a theorem. Doing this, however, one runs into the trouble of having to add a new term $\{x : \Phi(x)\}$ for every predicate Φ.

[2] In [2] Weber gives a similar logic where the resulting theory includes everything stated here and has been shown to be nontrivial. The reason I chose to stick with this logic is that I find it easier to work with.

6. $R \in R$ (2,5 Rule II)
7. $R \in R \wedge R \notin R$ (5,6 Rule I)

So R differs from itself with respect to membership. By Theorem 3.2 $R \neq R$ □

Definition 3.1. A set a is, with respect to $\in$:
strictly ordered iff

$$x, y, z \in a \to [x \notin x \wedge (x \in y \wedge x \notin x \to y \notin x) \wedge (y \in z \to (x \in y \to x \in z))],$$

totally ordered by $\subseteq$ iff a is strictly ordered and

$$x \in a \mapsto (y \in a \mapsto x \subseteq y \vee y \subseteq x),$$

well founded iff

$$y \subseteq a \wedge \exists z(z \in y) \mapsto \exists z(z \in y \wedge \neq \exists x(x \in z \wedge x \in y)$$

and transitive iff

$$x \in a \to x \subseteq a.$$

Definition 3.2. *On* is the set such that

$$\begin{aligned} x \in On \leftrightarrow \quad & x \text{ is well ordered} \\ \wedge \ & y \in x \to y \subseteq x \\ \wedge \ & x \subseteq On \\ \wedge \ & y \in On \mapsto (x \subseteq y \vee y \subseteq x). \end{aligned}$$

Some important properties of *On* are the following (Stated here without proof):

Theorem 3.5. *$On \in On, On \notin On$ and $On \neq On$.*

Definition 3.3. A function $f : a \to b$ is a relation with domain a and range b such that $(x, u) \in f \wedge (x, v) \in f \mapsto u = v$. A function f is invective iff $\forall x \forall y(x \neq y \mapsto f(x) \neq f(y))$.

Theorem 3.6. *The universe can be well ordered.*

Proof. Let $f : V \to On$ be the function defined by $f(x) = On$ for all $x \in V$. Now by Theorem 3.5, $On \neq On$ so $\forall x \forall y(x \neq y \mapsto On \neq On)$ and $\forall x \forall y(x \neq y \mapsto f(x) \neq f(y))$. f is therefore an injection from V to a segment of On so $\{x_{f(x)} : f(x)\}$ is a well order on V. □

4 Conclusions

We have here seen some interesting results in a paraconsistent set theory. There are however important results we did not cover such as that this set theory proves the axioms of ZFC an the Peano postulates. It also remains to be seen to what extent the theory is consistent. We do for example not know if there is a finite set a such that $a \neq a$. We do however know, by theorem 3.2, that $\emptyset = a$ cannot hold for any nonempty a, assuming this theory is not trivial.

References

[1] WEBER, Z. (2010). TRANSFINITE NUMBERS IN PARACONSISTENT SET THEORY. *The Review of Symbolic Logic*, 3(1), 71-92. doi:10.1017/S1755020309990281

[2] WEBER, Z. (2012). TRANSFINITE CARDINALS IN PARACONSISTENT SET THEORY. *The Review of Symbolic Logic*, 5(2), 269-293. doi:10.1017/S1755020312000019.