1 Axioms and Stratified Comprehension

Axiom 1 (Extensionality)**.** *For all sets A and B, if for every object x, $x \in A$ if and only if $x \in B$, then $A = B$.*

Axiom 2 (Atoms)**.** *If x is an atom, then $y \notin x$ for any object y.*

Axiom 3 (The Universal Set)**.** *The universal set $V = \{x \mid x = x\}$ exists.*

Axiom 4 (Complements)**.** *For every set A, the set $A^{\mathsf{c}} = \{x \mid x \notin A\}$ exists.*

Definition 1. $\emptyset = V^{\mathsf{c}}$

Axiom 5 (Boolean Unions)**.** *For all sets A and B, the set $A \cup B = \{x \mid x \in A \vee x \in B\}$ exists.*

Definition 2. $A \cap B = (A^{\mathsf{c}} \cup B^{\mathsf{c}})^{\mathsf{c}}$

Definition 3. $A \setminus B = A \cap B^{\mathsf{c}}$

Axiom 6 (Set Union)**.** *For every set A of sets, the set $\bigcup A = \{x \mid \exists y : x \in y \in A\}$ exists.*

Axiom 7 (Singletons)**.** *For every object x, the set $\{x\} = \{y \mid y = x\}$ exists.*

Axiom 8 (Ordered Pairs)**.** *For all objects x and y, the ordered pair (x, y) exists. Furthermore, $(x, y) = (u, v)$ if and only if $x = u$ and $y = v$.*

Remark 1. *We include ordered pairs as primitives for convenience. We could also use the standard Kuratowski pairs for many things, but it causes complications because its relative type is greater than its projections.*

Definition 4. *A set R is a relation if every element of R is an ordered pair (as defined above).*

Axiom 9 (Cartesian Products)**.** *For all sets A and B, the set $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$ exists.*

Axiom 10 (The Diagonal)**.** *The set $[=] = \{(x, x) \mid x \in V\}$ exists.*

Axiom 11 (Projections)**.** *The sets $\pi_1 = \big\{\big((x, y), x\big) \mid x, y \in V\big\}$ and $\pi_2 = \big\{\big((x, y), y\big) \mid x, y \in V\big\}$ exist.*

Axiom 12 (Inverses)**.** *For every relation R, the relation $R^{-1} = \{(y, x) \mid xRy\}$ exists.*

Axiom 13 (Compositions)**.** *For all relations R and S, the relation $R \circ S = \{(x, z) \mid \exists y : xRySz\}$ exists.*

Axiom 14 (Domains)**.** *For every relation R, the set $\mathsf{dom}(R) = \{x \mid \exists y : xRy\}$ exists.*

Axiom 15 (Singleton Images)**.** *For every relation R, the relation $\mathsf{SI}(R) = \big\{\big(\{x\}, \{y\}\big) \mid xRy\big\}$ exists.*

Definition 5. $\mathcal{P}_1(A) = \mathsf{dom}\big(\mathsf{SI}(A \times A)\big) = \big\{\{x\} \mid x \in A\big\}$

Axiom 16 (Inclusion)**.** *The relation $[\subseteq] = \{(x, y) \mid x \subseteq y\}$ exists.*

Definition 6. $\mathcal{P}(A) = \mathsf{dom}\big((V \times \{A\}) \cap [\subseteq]\big)$

Remark 2. *$\mathcal{P}(V)$ is the set of all sets, and $V \setminus \mathcal{P}(V)$ is the set of all atoms. Is $V \setminus \mathcal{P}(V)$ nonempty? (Yes, if you assume AC).*

Theorem 1. *The set $[\in] = \{(x, y) \mid x \in y\}$ does not exist.*

Proof. Assume to the contrary that $[\in]$ exists. Then we can construct the set $R = \mathsf{dom}([=] \setminus [\in])$. Note that $R = \{x \mid x \notin x\}$. This leads to a contradiction. Hence, $[\in]$ does not exist. ☐

Definition 7. *A first-order formula φ with the relations $=$, $\in$, π_1, and π_2, is stratified if one can assign a natural number to every variable in φ (called its type) such that the following hold:*

1. *Each variable has the same type wherever it appears.*
2. *The types of x and y are the same if $(x = y)$, $(x\ \pi_1\ y)$, or $(x\ \pi_2\ y)$ appears in φ.*
3. *The type of y is one higher than the type of x if $(x \in y)$ appears in φ.*

Theorem 2 (Stratified Comprehension)**.** *For every stratified formula φ, the set $\{x \mid \varphi(x)\}$ exists.*

Proof. Let $\varphi(v_1)$ be a stratified formula with m variables and largest type N.

$$
\begin{aligned}
(v_i^n \,\square\, v_j^n)' &= v_i \ \mathsf{SI}^{N-n}\square\ v_j \\
(v_i^n \in v_j^{n+1})' &= v_i \ \mathsf{SI}^{N-(n+1)}[\subseteq]\ v_j \\
(\neg\varphi)' &= \neg\varphi' \\
(\varphi \vee \psi)' &= \varphi' \vee \psi' \\
(\exists v_i^n \varphi)' &= \exists v_i : v_i \in \mathcal{P}_1^{N-n}(V) \wedge \varphi' \\
\widehat{\varphi}(v_1^n) &= v_1 \in \mathcal{P}_1^{N-n}(V) \wedge \varphi' \\
\{x \mid \varphi(x^n)\} &= \bigcup{}^{N-n}\{x \mid \widehat{\varphi}(x)\}
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{i+1}^m &= (\pi_2^i \circ \pi_1) \cap (V^m \times V) \\
\Pi_m^m &= \pi_2^{m-1} \\
(v_i \ \mathsf{SI}^k\square\ v_j)! &= \mathsf{dom}\big((\Pi_i^m \circ \mathsf{SI}^k\square) \cap \Pi_j^m\big) \\
\big(v_i \in \mathcal{P}_1^k(V)\big)! &= \mathsf{dom}\big(\Pi_i^m \circ \mathsf{SI}^k[=]\big) \\
(\neg\varphi)! &= V^m \setminus \varphi! \\
(\varphi \vee \psi)! &= \varphi! \cup \psi! \\
(\exists v_i \varphi)! &= \mathsf{dom}\Bigg((V^m \times \varphi!) \cap \bigcap_{j \neq i}\big[\Pi_j^m \circ (\Pi_j^m)^{-1}\big]\Bigg) \\
\{x \mid \varphi(x^n)\} &= \bigcup{}^{N-n}\mathsf{dom}(\widehat{\varphi}!)
\end{aligned}
$$

□

2 Theorem of Infinity

Remark 3. *Every natural number (as they are typically defined) exists, but we cannot define the entire set of natural numbers in the usual way because any formula containing $n \cup \{n\}$ cannot be stratified. Instead, we will define the natural numbers as sets of sets with a given cardinality.*

Definition 8. $0 = \{\emptyset\}$ *(i.e. the set of all sets with zero elements)*

Definition 9. *For every set A of sets, let $A + 1 = \big\{x \cup \{y\} \mid y \notin x \in A\big\}$.*

Remark 4. *This means that 1 is the set of all sets with one element, and likewise for 2, 3, 4, etc.*

Definition 10. $\mathsf{Succ} = \big\{(A, A+1) \mid A \in \mathcal{P}\big(\mathcal{P}(V)\big)\big\}$

Definition 11. $\mathbb{N} = \bigcap\{A \mid 0 \in A \wedge \mathsf{Succ}[A] \subseteq A\}$

Corollary 1. *Mathematical induction for stratified properties.*

Definition 12. $\mathsf{Fin} = \bigcup\mathbb{N}$ *(i.e. the set of all finite sets)*

Theorem 3 (Infinity)**.** $\mathbb{N} \notin \mathsf{Fin}$

Proof. We want to show that $\mathbb{N} \notin n$ for any $n \in \mathbb{N}$, but first, we need the following lemmas:

Lemma 1. *For all $m, n \in \mathbb{N}$, if $m + 1 = n + 1$, then $m = n$.*

Proof. By induction (on m using the property "for all $n \in \mathbb{N}$, if $m + 1 = n + 1$, then $m = n$"). □

Lemma 2. *If $n \in \mathbb{N} \setminus \{0\}$, then $n = m + 1$ for some $m \in \mathbb{N}$.*

Proof. By induction. □

Corollary 2. $\mathsf{Succ}|_{\mathbb{N}} : \mathbb{N} \sim \mathbb{N} \setminus \{0\}$*, so $\mathbb{N} \sim \mathbb{N} \setminus \{m\}$ for all $m \in \mathbb{N}$.*

Lemma 3. *If $A \sim B \in n \in \mathbb{N}$, then $A \in n$.*

Proof. By induction. □

Now we can prove the theorem by induction. Clearly $\mathbb{N} \notin 0$. If $\mathbb{N} \in n+1$, then $\mathbb{N} \setminus \{m\} \in n$ for some $m \in \mathbb{N}$. Since $\mathbb{N} \sim \mathbb{N} \setminus \{m\} \in n$, it follows that $\mathbb{N} \in n$. Therefore, if $\mathbb{N} \notin n$, then $\mathbb{N} \notin n + 1$. Hence, the theorem follows by induction. □

3 Cardinals and Specker's Theorem

Definition 13. $|A| = \{B \mid A \sim B\}$

Remark 5. *The class of cardinals is a set.*

Theorem 4. *The set* $S = \big\{(x, \{x\}) \mid x \in V\big\}$ *does not exist.*

Proof. If S exists, then we can construct the set $S \circ \big((\mathcal{P}_1(V) \times V) \cap [\subseteq]\big) = [\in]$. As demonstrated before, this leads to a contradiction. □

Remark 6. *We cannot prove (in general) that* $|A| = |\mathcal{P}_1(A)|$*. It is, however, easy to show that* $|A| = |B|$ *if and only if* $|\mathcal{P}_1(A)| = |\mathcal{P}_1(B)|$*.*

Definition 14. $T|A| = |\mathcal{P}_1(A)|$

Remark 7. T *is a well-defined function on cardinals, but* T *is a proper class.* T *cannot be a set because the relative type of* $|A|$ *is less than the relative type of* $|\mathcal{P}_1(A)|$*.* T *increases relative type by one.*

Definition 15. $T^{-1}|\mathcal{P}_1(A)| = |A|$

Remark 8. T^{-1} *is only defined on cardinals* $\leq |\mathcal{P}_1(V)|$*.* T^{-1} *decreases relative type by one.*

Axiom 17 (Counting)**.** $T(n) = n$ *for all* $n \in \mathbb{N}$*.*

Corollary 3. *Mathematical induction for stratified properties, but for the purposes of stratification, we can ignore the types of variables in* $\mathbb{N}$*.*

Definition 16. $|A|^{|B|} = T^{-1}|{}^{B}A|$*. In particular,* $\mathsf{exp}|A| = 2^{|A|} = T^{-1}|\mathcal{P}(A)|$*.*

Remark 9. exp *is a function defined on cardinals* $\leq |\mathcal{P}_1(V)|$*.*

Theorem 5 (Cantor)**.** $T(\kappa) < \mathsf{exp}\big(T(\kappa)\big)$

Proof. The standard diagonalization proof (essentially). □

Corollary 4. $|\mathcal{P}_1(V)| = T|V| < \mathsf{exp}\big(T|V|\big) = |\mathcal{P}(V)| \leq |V|$

Lemma 4. $\mathsf{exp}\big(T(\kappa)\big) = T\big(\mathsf{exp}(\kappa)\big)$ *(whenever* $\mathsf{exp}(\kappa)$ *exists)*

Proof. Easy. □

Lemma 5. exp *is a monotone function.*

Proof. Easy. □

Definition 17. κ *is a Specker number if* $\mathsf{exp}^n(\kappa) = |V|$ *for some* $n \in \mathbb{N}$*.*

Theorem 6 (Specker)**.** $|\mathcal{P}(V)| < |V|$

Proof. Let μ be the least Specker number (well-ordering the cardinals requires the Axiom of Choice). Then there exists $n \in \mathbb{N}$ such that $\mathsf{exp}^n(\mu) = |V|$. Since T commutes with exp, it follows by induction that $\mathsf{exp}^n\big(T(\mu)\big) = T\big(\mathsf{exp}^n(\mu)\big) = T|V|$. (This induction argument requires the Axiom of Counting.) Assume to the contrary that $|\mathcal{P}(V)| = |V|$. Then $\mathsf{exp}\big(T|V|\big) = \mathsf{exp}|\mathcal{P}_1(V)| = |\mathcal{P}(V)| = |V|$. Therefore, $\mathsf{exp}^{n+1}\big(T(\mu)\big) = \mathsf{exp}\big(\mathsf{exp}^n\big(T(\mu)\big)\big) = \mathsf{exp}\big(T|V|\big) = |V|$, so $T(\mu)$ is a Specker number. Since μ is the least Specker number, it must be that $T(\mu) \geq \mu$, so by induction, $T|V| = \mathsf{exp}^n\big(T(\mu)\big) \geq \mathsf{exp}^n(\mu) = |V|$ (exp is a monotone function). This contradicts the corollary of Cantor's theorem. Hence, $|\mathcal{P}(V)| < |V|$. □

Corollary 5. $\mathcal{P}(V) \neq V$*, so there exist atoms.*

Corollary 6. $|\mathcal{P}(V)| < |V \times V|$*, so there are more ordered pairs than sets (i.e. it is impossible to define our ordered pairs as sets).*

Corollary 7. $NF + (\mathit{Counting}) \vdash \neg AC$

Remark 10. *Specker's original proof does not require the Axiom of Counting.*