Advanced Set Theory: Infinite Games and Determinacy
Assignment 1

Exercise 1 (Basic game principles and notation).

(a) Consider the standard infinite game in which Player I and Player II play integers and z is the result of an infinite run of the game. The winning condition is given informally:

- Player I wins if and only if for every $n \in \omega$, there are infinitely many $i \in \omega$ such that $z(i) = n$.

Formalize this game as a standard infinite game of the form $G(A)$. Which player has a winning strategy? Describe (informally) that winning strategy.

(b) Let T be a tree, intuitively representing the *legal moves* in an infinite game. Let $A \subseteq T$ be the pay-off set and let $G_T(A)$ be the game in which Players I and II play integers in T, that is, every finite position $t := \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle$ of the game must be an element of T. Show that this game can be formalized as a standard game of form $G(A')$, i.e., that there is an $A' \subseteq \omega^\omega$ such that $G_T(A)$ is equivalent to $G(A')$ in the sense that Player I has a winning strategy in $G(A')$ if and only if he has a winning strategy in $G_T(A)$, and the same for Player II. Show that if A is a Borel set then A' is also a Borel set.

Exercise 2 (Winning strategies and complements).

Consider the following notation: for a set $A \subseteq \omega^\omega$ and $n \in \omega$, let

$$A/_{\langle n \rangle} := \{x \in \omega^\omega : \langle n \rangle ^\frown x \in A\}.$$

(a) Prove that for every A, Player I has a winning strategy in $G(A)$ iff for some n, Player II has a winning strategy in $G(\omega^\omega \setminus A/_{\langle n \rangle})$.

(b) Prove that for every A, Player II has a winning strategy in $G(A)$ iff for every n, Player I has a winning strategy in $G(\omega^\omega \setminus A/_{\langle n \rangle})$.

(c) Prove that for every $n \in \omega$, the function f_n given by

$$f_n(x) := \langle n \rangle ^\frown x$$

is continuous.

(d) Suppose $\mathbf{\Gamma}$ is a class of sets of reals, closed under continuous pre-images. Define the *dual pointclass* $\breve{\mathbf{\Gamma}}$ to be the collection of complements of sets in $\mathbf{\Gamma}$, i.e.,

$$\breve{\mathbf{\Gamma}} := \{A \ : \ (\omega^\omega \setminus A) \in \mathbf{\Gamma}\}.$$

Prove that $\mathbf{\Gamma}$-determinacy holds iff $\breve{\mathbf{\Gamma}}$-determinacy holds (in particular $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ determinacy are equivalent).

Exercise 3. (AD_{all})

Recall that AD is the statement that for all $A \subseteq \omega^\omega$, $G(A)$ is determined and that this contradicts AC (the axiom of choice).

Extend the definition of a game in such a way that the players need not play natural numbers, but can play *any* sets whatsoever (i.e., x_0, y_0, x_1, y_1, etc. are any sets). The games still have length ω, thus a *payoff* set can be any proper class $\mathcal{A} \subseteq \mathbb{V}^\omega$. Let AD_{all} be the principle that all such games are determined.

Show that AD_{all} implies AC, and conclude that AD_{all} is inconsistent even with ZF.