ROOTED TREES APPEARING IN PRODUCTS AND CO-PRODUCTS

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Abstract. We review basic concepts related to rooted trees and their combinatorics as they are needed for the introduction of several bialgebra or Hopf algebra structures on vector space bases of rooted trees. In view of the applications, we have to distinguish between several notions of such trees, such as abstract/planar, partially labeled/non-labeled etc. We present a unifying approach, inspired by the theory of operads. Then we focus on important operations on trees, such as grafting and cutting trees, which appear in various examples.

1. Introduction

Rooted trees are examples of combinatorial objects which appear in a lot of different contexts in mathematics. Their appearance as algebra bases or vector space bases for various interesting Hopf algebra structures is one aspect, which still leads to a lot of new research. We mention here the Hopf algebras of A. Connes and D. Kreimer (cf. [2]), the Hopf algebras of R. Grossman and R. G. Larson (cf. [5]), the dendriform Hopf algebras of J.-L. Loday and M. Ronco (cf.[14]), and Hopf algebra structures introduced by C. Brouder and A. Frabetti (cf. [1]).

The aim of this note is not to delve into Hopf algebras, but to describe the foundations, i.e. to present an approach to rooted trees useful for the study of the mentioned Hopf algebras.

Our goal in Section 2 is to put several different types of rooted trees – planar (ordered), or abstract (unordered), usually partially labeled – under a common roof, which is inspired by operad theory. In Section 3 we look at products or operations on trees, which are related to the concatenation of words. In particular, we look at parenthesized words, which correspond directly to binary trees. The relevant operations like the grafting product on trees extend to operations $V \otimes V \to V$ or $(V \otimes \ldots \otimes V) \to V$, where V is some vector space spanned by trees or forests of trees. Dually defined are co-operations on V (into tensor products of V). In Section 4, we look at the substitution operation for trees. There are various related operations. We use a right comb presentation of planar binary trees, and we give an explicit bijection between planar forests with $n-1$ vertices and corresponding planar binary trees with n leaves. Then we concentrate on the concept of cuts, as it is used for the coproduct of the Connes-Kreimer Hopf algebra. We finish with some remarks on nonsymmetric operads and the Stasheff polytopes.

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2. Types of rooted trees - an overview

We distinguish between several types of trees. First we recall the notions of rooted trees and planar rooted trees (compare [21, 22]). We skip the definition of graphs (see [6]). A naive notion of a graph will suffice. For a more sophisticated notion, involving half-edges, see [16], §5.3.

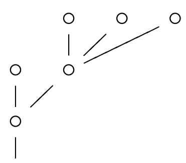
2.1. **Rooted Trees.** A finite connected graph $\emptyset \neq T = (\mathrm{Ve}(T), \mathrm{Ed}(T))$ with a distinguished vertex ρ_T is called a rooted tree, if the following condition holds: For every vertex $\lambda \in \mathrm{Ve}(T)$ there is exactly one path connecting λ and ρ_T (equivalently, T has no cycles, compare the definition given in the appendix of [21]).

The vertex ρ_T is called the root of T. Thinking of the edges as oriented towards the root, at each vertex there are incoming edges and one outgoing edge. The standard convention is that the root has no outgoing edges, but here we add to the root an outgoing edge that is not connected to any further vertex. If we want to exclude this edge, we speak of the other edges as inner edges.

If λ and λ' in $\mathrm{Ve}(T)$ are connected by an edge oriented from λ' to λ, then λ is called the father of λ', and λ' is called a child of λ.

The vertices with no child are called leaves, and the set of leaves of T is denoted by $\mathrm{Le}(T)$.

2.2. **Example.** The following graph is a rooted tree:



Here we draw the root at the bottom. If the number $\#\mathrm{Ve}(T)$ of vertices of T is at least 2, T is uniquely described by the unordered list of non-empty rooted trees $T_1, \dots T_r$, $r \geq 1$, of the full subtrees whose roots are the childs of the root ρ_λ of T (cf. Definition 2.16).

2.3. **Planar rooted trees.** Let T be a rooted tree. For any given vertex λ of T, we say that λ is an r-ary vertex, if the number of incoming edges is r.

Binary trees are rooted trees where all vertices are 2-ary, except for the leaves, which are 0-ary. Analogously, m-ary trees are defined.

For T a rooted tree of any type, we write the set $\mathrm{Ve}(T)$ of vertices as a disjoint union $\bigcup_{r\in\mathbb{N}} \mathrm{Ve}^r(T)$, where $\mathrm{Ve}^r(T)$ consists of all r-ary vertices of T. The elements of $\mathrm{Ve}^*(T) = \bigcup_{r\geq 1} \mathrm{Ve}^r(T) = \mathrm{Ve}(T) - \mathrm{Le}(T)$ are called internal vertices of T.

The height of a vertex $\lambda \in \mathrm{Ve}(T)$ is the number of edges separating it from ρ_T. The height of a rooted tree T is the maximum height of its vertices.

A rooted tree T together with a chosen order of incoming edges at each vertex is called a planar rooted tree (or ordered rooted tree). In our drawings, this is an ordering from left to right, see Example (2.4).

Planar binary trees and planar m-ary trees are defined as well. One also considers incomplete planar m-ary trees, defined recursively as follows: Every such tree is either the empty tree or uniquely determined by the ordered list of m possibly empty incomplete planar m-ary trees, see Example (2.5).

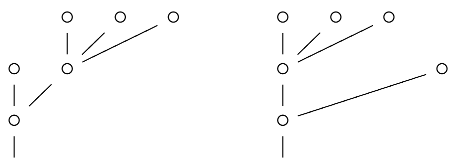
Ordinary rooted trees are also called abstract (or unordered) rooted trees to stress that they are non-planar.

Consequently, we denote by PTree the set of planar rooted trees, and by ATree the set of abstract rooted trees. Some authors also use the notations $OT :=$ PTree and $UT :=$ ATree; OT for ordered trees, UT for unordered trees.

By A_mTree and P_mTree we denote the corresponding sets of m-ary (complete) rooted trees, and in the binary case $m = 2$ we also use the notation ABTree and PBTree.

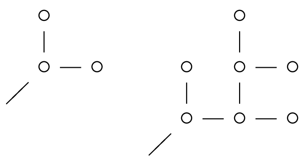
2.4. **Example.** For every vertex of a planar rooted tree, the chosen order of incoming edges corresponds to an ordering of edges from left to right. Every drawing of a rooted tree provides us with a planar structure, which we have to forget when dealing with abstract rooted trees.

The drawings

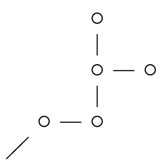


represent the same abstract rooted tree T (of height 2), but different planar rooted trees T^1, T^2.

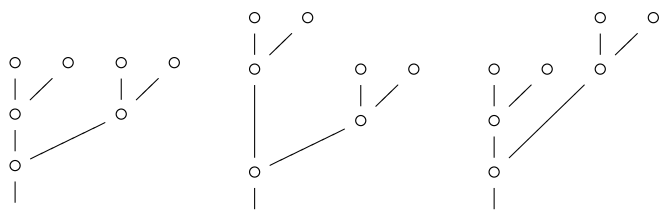
2.5. **Example.** Examples for binary trees can be drawn as follows:



An incomplete binary tree is represented by the following picture:



2.6. **Leveled trees.** We mention that one can also associate levels to all vertices of a given rooted tree, to distinguish for example between the rooted trees



Such leveled trees and also incomplete trees are not going to be considered in the rest of this article.

2.7. **Integer sequences.** For $n, p \in \mathbb{N}$, let

$$\mathrm{PTree}_n := \{T \in \mathrm{PTree} : \#\mathrm{Ve}(T) = n\}$$

be the set of planar rooted trees with n vertices, and let

$$\mathrm{PTree}^p := \{T \in \mathrm{PTree} : \#\mathrm{Le}(T) = p\}$$

be the set of planar rooted trees with p leaves.

Furthermore, let

$$\mathrm{PTree}_n^p := \mathrm{PTree}_n \cap \mathrm{PTree}^p.$$

We make the analogous definitions for abstract rooted trees, and also for the abstract binary and for the planar binary rooted trees.

The number $\#\mathrm{PTree}_n$ of planar rooted trees with n vertices is the n-th Catalan number

$$c_n = \frac{(2(n-1))!}{n!(n-1)!}.$$

The numbers c_n, for $n = 1, \dots, 11$, are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796.$$

with generating series $f(t) = \sum_{n=1}^{\infty} c_n t^n$ given by

$$\frac{1 - \sqrt{1-4t}}{2}.$$

The numbers c_n also count planar binary rooted trees with n leaves.

The numbers $a_n = \#\mathrm{ATree}_n$ of abstract rooted trees with n vertices, for $n = 1, \dots, 11$, are

$$1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842.$$

The generating series $f(t) = \sum_{n=1}^{\infty} a_n t^n$ fulfills the equation

$$f(t) = \frac{t}{\prod_{n \geq 1}(1 - t^n)^{a_n}}$$

or equivalently the equation

$$f(t) = t \exp\Big(\sum_{k\geq 1} \frac{f(t^k)}{k}\Big)$$

(cf. [6]).

The numbers $b_n = \#\mathrm{ABTree}^n$ of abstract binary rooted trees with n leaves, for $n = 1, \dots, 11$, are

$$1, 1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451$$

with generating series $f(t)$ given by (cf. [18]) the equation

$$f(t) = t + \frac{1}{2} f(t)^2 + \frac{1}{2} f(t^2).$$

The series can thus be written in the form

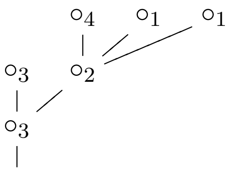
$$\begin{aligned} f(t) &= 1 - \sqrt{1 - f(t^2) - 2t} \\ &= 1 - \sqrt{\sqrt{1 - f(t^4) - 2t^2} - 2t} = \dots \end{aligned}$$

Information on these integer sequences can be found in the On-Line Encyclopedia of Integer Sequences [20] (see Sequence A000108 for c_n, Sequence A000081 for a_n, Sequence A001190 for b_n).

2.8. **Labeled trees.** Let M be a set and T a rooted tree (of a given type). The set M is considered to be a set of labels (or colors).

Then a labeling of T is a map $\nu : \mathrm{Ve}(T) \to M$. The rooted tree T together with such a labeling is called a labeled rooted tree (of the given type).

2.9. **Example.** In our drawing, we can put labels at the vertices:



Here we have used the set $M = \{1, 2, 3, 4\}$.

2.10. **Admissible labelings.** Let M be a labeling $\nu : \mathrm{Ve}(T) \to M$ as above. Suppose that $M = \bigcup_{r\in\mathbb{N}} M_r$, where $M_0, M_1, M_2, \dots$ is a given sequence of sets.

Then the labeling ν is called (M)-admissible, if $\nu(\lambda) \in M_r$ for every r-ary vertex λ (for every r).

The set of planar rooted trees $T \in \mathrm{PTree}$ with admissible labeling from $(M_r)_{r\in\mathbb{N}}$ is denoted by $\mathrm{PTree}((M_r))$ or simply by $\mathrm{PTree}(M)$. Similarly, $\mathrm{ATree}(M)$ is used for abstract rooted trees with (M)-admissible labelings.

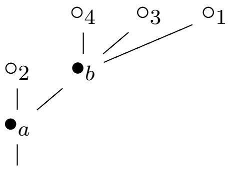
2.11. ***n*-trees.** Suppose furthermore that $M_0 = \mathbb{N}$ and that only such labelings are allowed that map the leaves of any rooted tree with n leaves bijectively on the set $\{1, 2, \ldots, n\}$. Planar or abstract rooted trees with such an (M)-admissible labeling are called (planar or abstract) n-trees.

A (planar or abstract) rooted tree T is called reduced, if $\mathrm{Ve}^1(T) = \emptyset$, i.e. if there are no 1-ary vertices in T.

The set of planar reduced rooted trees is denoted by PRTree.

2.12. **Example.** Let $M_1 = \emptyset$. Then rooted trees (of any type) with an (M)-admissible labeling are necessarily reduced.

The following drawing represents a planar n-tree with $a \in M_2, b \in M_3$:

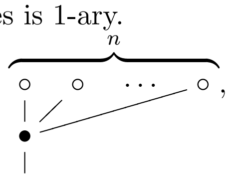


Note that, in our drawing, we used different colors/labels $\bullet_a, \bullet_b, \ldots$ for internal vertices and $\circ_1, \circ_2, \ldots$ for the leaves.

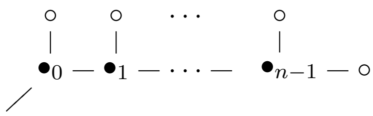
We can identify the set PTree with the set $\mathrm{PTree}((M_r = \{\circ\}, r \in \mathbb{N}))$, i.e. we consider non-labeled rooted trees as trivially labeled trees.

2.13. **Example.** The following abstract (or planar) rooted tree is not reduced: 

one of its vertices is 1-ary.

The rooted tree , called n-corolla, is reduced for every $n \geq 2$.

A planar binary tree of the form

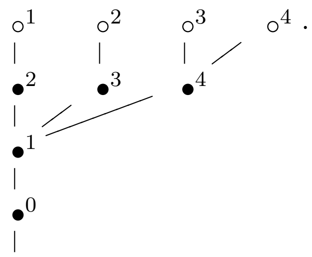


is called a right comb of height n. The right comb without labels of height n is denoted by R_n. One considers left combs as well. As abstract trees, the right comb and the left comb (of a given height) are equal.

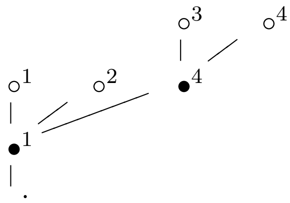
2.14. **Reduction map.** For any (planar or abstract) rooted tree T, there exists a unique rooted tree $\mathrm{red}(T)$, called the reduction of T, defined as follows: Its set of vertices is $\mathrm{Ve}(T) - \mathrm{Ve}^1(T)$, and for any pair λ, λ' of vertices of $\mathrm{red}(T)$ there is an oriented path (or, equivalently, a path not passing the root) from λ to λ' in $\mathrm{red}(T)$ if and only if there is such a path in T.

Induced is a map red : PTree → PRTree and a similar map for abstract rooted trees. Admissible labelings (of all vertices of arity $\neq 1$) are preserved.

2.15. **Example.** Consider the rooted tree

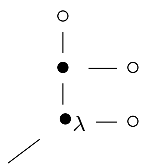


Its reduction is

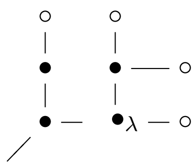


2.16. **Full subtrees.** Let T be a planar (or abstract) rooted tree, and let $\lambda \in \mathrm{Ve}(T)$ be a vertex. The vertex λ determines a subgraph T_λ of T, called the full subtree of T with root λ, such that: $\lambda \in \mathrm{Ve}(T_\lambda)$, and for every vertex $\lambda' \in \mathrm{Ve}(T_\lambda)$ all incoming edges (vertices included) of λ' in T belong also to T_λ.

2.17. **Example.**
The rooted tree



is a full subtree of



2.18. **The empty tree.** It is sometimes useful to call the empty graph $\emptyset$ a rooted tree (and to define its height to be -1). It is then formally adjoined to the sets of trees described above, e.g., we consider the vector space with basis $\mathrm{PTree} \cup \{\emptyset\}$ over the field K.

3. Operations of concatenation type

We recall the free objects of the categories of abelian semigroups, (not necessarily abelian) semigroups, and magmas. Words occur as elements of these free objects (cf. [19] as a general reference for operations and co-operations on words).

Elements of the free abelian semi-group $W_{\mathrm{Com}}(X)$ over X are commutative words $x_{i_1}^{\nu_1} x_{i_2}^{\nu_2} \cdots x_{i_r}^{\nu_r}$, $i_1 < i_2 < \ldots < i_r$, $r \geq 1$, $\nu_i \in \mathbb{N}$. Adjoining a unit 1 (empty word) we get the free abelian semi-group $W_{\mathrm{Com}}^1(X)$ with unit.

The free semigroup $W_{\mathrm{As}}(X)$ over X or the free monoid $W_{\mathrm{As}}^1(X)$ over X is equipped with the concatenation $W_{\mathrm{As}}^1(X) \times W_{\mathrm{As}}^1(X) \to W_{\mathrm{As}}^1(X), (v, w) \mapsto$

$v.w$ denoted by a lower dot. The elements of $W^1_{\mathrm{As}}(X)$ are words $w = w_1.w_2\dots w_r$, $w_i \in X$ for all i. Here $r \in \mathbb{N}$ is the length of w.

A magma is just a set M equipped with a binary operation (usually denoted by $\cdot : M \times M \to M$). The elements of the free magma $W_{\mathrm{Mag}}(X)$ over X are parenthesized words. We are going to identify these words with planar binary rooted trees.

3.1. **Forests.** A commutative word (not necessarily non-empty) of abstract rooted trees, written as a disjoint union $T^1 \cup T^2 \cup \dots \cup T^r$ is called a (rooted) forest.

A word $T^1.T^2\dots T^r$ of planar rooted trees is called a planar (rooted) forest.

We denote by PForest and AForest the corresponding sets of forests. The concepts and notations of labeled trees carry over to forests.

One may consider other, less canonical, combinations like abstract forests of planar rooted trees. If the context is clear, the words abstract and planar are often omitted.

3.2. **Parenthesized strings.** There is a correspondence between planar rooted trees and (irreducible) parenthesized strings. We sketch this correspondence in the general setting of planar rooted trees with (M)-admissible labeling, where M_0 is given by$\{x_1, x_2, \dots\}$, and all other $M_r, r \geq 1$ are given by $\{y_1, y_2, \dots\}$. D. Kreimer's definition of irreducible parenthesized words in [11] is completely analogous. Here we make a difference between labels for the leaves ($x_i \in X$) and labels for the internal vertices ($y_i \in Y$), though.

Given a planar rooted tree T with (M)-admissible labeling, we recursively construct the corresponding parenthesized string.

If T consists of its root ρ_T, then ρ_T is a leaf labeled by some x_i. The corresponding parenthesized string is (x_i), i.e. an opening bracket followed by the letter x_i followed by a closing bracket.

Else, let ρ_T be labeled by y_i, and let $T^1 \dots T^n$ be the forest of labeled rooted trees which remains after removing the root with its incoming edges. Assume that for each $T_j, 1 \leq j \leq n$, we have already constructed the corresponding parenthesized string w_j. Then $(y_i\ w_1 \dots w_n)$ is the parenthesized string associated to T.

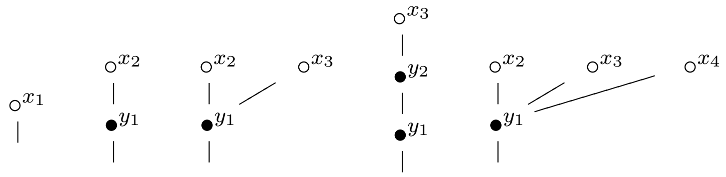
We get a string of letters and balanced brackets such that the leftmost opening bracket is matched by the rightmost closing bracket (irreducibility) and such that each letter has exactly one opening bracket on its lefthandside.

It is easy to see that the rooted tree can be reconstructed from its string.

Reducible words are defined by concatenation of irreducible ones, thus they correspond to forests.

For abstract rooted trees, there is a completely similar construction. The only difference is that some words have to be identified due to the missing order of incoming edges.

3.3. **Example.** The parenthesized strings (x_1), $(y_1(x_2))$, $(y_1(x_2)(x_3))$, $(y_1(y_2(x_3)))$, and $(y_1(x_2)(x_3)(x_4))$ represent the rooted trees



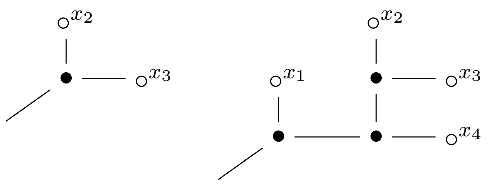
An empty pair of brackets without label is also allowed. It represents the empty tree.

3.4. **Example.** In the following we consider binary trees.

Let $M_0 = \{x_1, x_2, \ldots\}$, and let M_2 be a one element set.

For every pair of brackets, the position of the closing bracket is forced once the position of the opening bracket is given. Thus one can omit the brackets and just use a letter c to mark and replace any combination of an opening bracket followed by the common label of an internal vertex.

For example the binary trees



can be represented by the strings cx_2x_3, $cx_1c^2x_2x_3x_4$.

3.5. **Malcev representation.** Since the free magma generated by a set of variables X consists of parenthesized strings given by planar binary trees, we can call the set of planar binary trees with leaves labeled by X the free magma generated by X. The product $\cdot$ in the free magma is thus given by a map $(T, T') \mapsto \vee(T, T')$ of planar binary trees (the grafting-operation defined in the following paragraph, here in the case $n = 2$).

If a field K is given, we can pass from the free magma generated by X to the free magma algebra (similarly to passing from semi-groups or groups to semi-group algebras or group-algebras).

The representation given in Example (3.4) is the Malcev representation of the free magma algebra over $X = \{x_1, x_2, \ldots\}$ in the free associative algebra generated by $\{c, x_1, x_2, \ldots\}$. The free magma multiplication $\cdot$ corresponds to the operation $(v, w) \mapsto cvw$ in the free associative algebra.

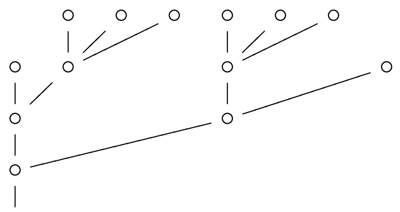
3.6. **Grafting product.** Given a forest $T^1 \ldots T^n$ of $n \geq 0$ planar rooted trees with an (M)-admissible labeling, together with a label $\rho \in M$, there is a rooted tree $T = \vee_\rho(T^1.T^2 \ldots T^n)$ defined by introducing a new n-ary root and grafting the trees $T^1, \ldots, T^n$ onto this new root. The new root gets the label ρ. The specified order in the forest determines the order of incoming edges at ρ_T. The rooted tree T is called the grafting product of $T^1 \dots T^n$ over ρ.

If there is no choice for a label ρ (i.e. there is only one label available) we simply write $\vee(T^1 \dots T^n)$.

Analogously defined is the grafting product for (forests of) abstract rooted trees. In the literature it is often denoted by $B_+(T^1 \dots T^n)$, e.g. in [12].

It is common to identify a unit for these operations with the empty tree $\emptyset$. To avoid difficulties with respect to uniqueness, we still do not allow the members T^i of a forest $T^1 \dots T^n$ to be empty. (We allow empty forests, though.)

3.7. **Example.** The grafting product $\vee(T^1, T^2)$ of T^1, T^2 from (2.4) is just



3.8. **Degrafting map.** Every planar rooted tree T is the grafting product of a forest $\neg T$ uniquely determined by T.

In the case of an n-ary root ρ, the forest $\neg T$ consists of the n full subtrees whose roots are the childs of the root ρ of T. Especially, all rooted trees in the forest $\neg T$ have heights less than the height of T.

There is a canonical de-grafting map $\neg$ from labeled non-empty planar rooted trees to forests of labeled planar rooted trees (given by deleting the root together with its label). The analogously defined operator on forests of abstract rooted trees is often denoted by B_-.

3.9. **Reversed words and mirrored trees.** For words $w = w_1.w_2 \dots w_n$ the reversed word $\bar{w}$ is defined by $w_n.w_{n-1} \dots w_1$. Similarly, given a planar rooted tree T, there is a unique rooted tree $\bar{T}$ recursively defined by

$$\overline{\vee(T^1 \dots T^n)} = \vee(\bar{T}^n \dots \bar{T}^1),$$

where $\overline{\vee(\emptyset)} = \vee(\emptyset)$ and $\bar{\emptyset} = \emptyset$. In other words, $\bar{T}$ is obtained by mirroring T along the root axis.

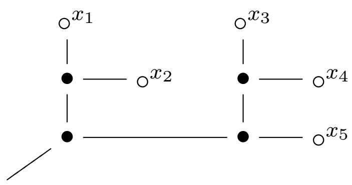
It holds that $\overline{(\bar{T})} = T$. The rooted trees T^1 and T^2 from (2.4) are in correspondence via $T \mapsto \bar{T}$.

3.10. **Splits and deconcatenation.** Dually defined to the concatenation of words is the deconcatenation. Recall that the result of the deconcatenation, applied to a word w of length r, is a sum where each summand is given by a split of w into a left part $w_{(1)}$ and a right part $w_{(2)}$.

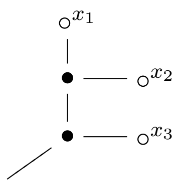
Dually defined to the grafting products on planar rooted trees are splitting co-operations. We give one example, which occurs in the case of planar binary trees as the dual co-operation of the binary grafting operation. It is studied in [10].

Let T be a planar binary tree with r leaves numbered from left to right by $1, 2, \ldots, r$. Let i, $1 \leq i < r$, be an integer. We split the tree T into two trees $T^i_{(1)}$ and $T^i_{(2)}$ by cutting in between the leaves i and $i+1$. More precisely, the tree $T^i_{(1)}$ is the reduction of the part of T which is on the left side of the path from leaf i to the root (including the path). The tree $T^i_{(2)}$ is the reduction of the analogous part on the right side of the path from leaf $i+1$ to the root.

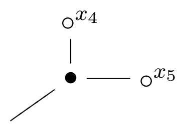
Look at $i = 3$ and $T =$



Then $T^3_{(1)} =$



, $T^3_{(2)} =$



The result of the co-operation applied to T is the sum, over all i, of the summands $T^i_{(1)} \otimes T^i_{(2)}$.

4. Operations related to the substitution

If one considers a letter which appears in a (commutative or non-commutative) word as placeholder for a word itself, and substitutes this letter by the corresponding word, one naturally obtains a new word. Similarly defined is the following substitution procedure for trees:

4.1. **Substitution.** Let T^1, T^2 be planar (or abstract) rooted trees and let b be a leaf of T^1. Then the substitution of T^2 in T^1 at b, denoted by $T^1 \circ_b T^2$, is obtained by replacing the leaf b of T^1 by the root of T^2.

4.2. **Over and under.** Given a planar binary tree T in PBTree or in PBTree(M_0, M_2), let $\alpha = \alpha(T)$ denote the first leaf of T (i.e. the leftmost leaf in a drawing which puts all leaves on one line).

Let $\omega = \omega(T)$ denote the last (i.e. rightmost) leaf of T.

Given a second planar binary tree S, we define

$$T \backslash S := T \circ_{\omega(T)} S.$$

The operation $\backslash$ is called under-operation and was introduced in [14]. Clearly $\backslash$ is associative and $S \backslash T \backslash Z$ is well-defined.

The analogous operation $\circ_\alpha$ given by $T \circ_\alpha S = T \circ_{\alpha(T)} S$ plays the role of an associative multiplication in a Hopf algebra defined by C. Brouder and A. Frabetti (cf. [1]).

The opposite multiplication $\circ_\alpha^{op}$, defined by $\circ_\alpha^{op}(S,T) := \circ_\alpha(T,S)$, is the over-operation $S/T := T \circ_{\alpha(T)} S$ of [14].

Using the mirror-operation $T \mapsto \bar{T}$, one can express S/T as $\overline{(\bar{T})\backslash(\bar{S})}$.

The tree $\stackrel{\circ}{|}$ consisting of the root serves as a unit for all these operations, e.g. $S\backslash \stackrel{\circ}{|} = S = \stackrel{\circ}{|}\backslash S$.

4.3. **Right comb presentation.** Given planar binary trees $T^1, \dots, T^n$, $n \geq 1$, and a sequence w of n labels from M_2, we define

$$\vee\!\!\!\rightarrow_w(T^1 \dots T^n)$$

to be the planar binary tree which can be obtained from the right comb R_n of height n as follows: We replace the first leaf $\alpha(R_n)$ by T^1, the second by T^2 and so on, leaving the $(n+1)$-th leaf (i.e. $\omega(R_n)$) unaltered. We just write $\vee\!\!\!\rightarrow(T^1 \dots T^n)$ if there is no choice of labels.

We define $\vee\!\!\!\rightarrow(\emptyset) = \stackrel{\circ}{|}$. In particular, R_n can then be written as

$$\vee\!\!\!\rightarrow(\underbrace{\stackrel{\circ}{|} \dots \stackrel{\circ}{|}}_{n}).$$

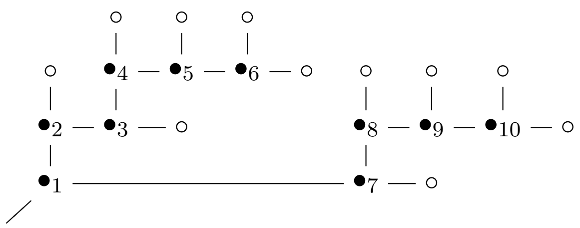
for every n.

It is easy to see that the smallest set which contains $\stackrel{\circ}{|}$ and is closed under $\vee\!\!\!\rightarrow_w$ operations contains all planar binary trees.

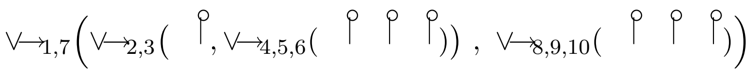
For every planar binary tree with (M)-admissible labeling, this right comb presentation is unique. The left comb presentation is similarly defined. The right (or left) comb presentation induces a map φ_r (φ_l respectively) from planar binary trees to planar forests of planar rooted trees (not necessarily binary) such that

$$\varphi_r(\stackrel{\circ}{|}) = \emptyset, \quad \varphi_r\Big(\vee\!\!\!\rightarrow_{w_1 \dots w_n}(T^1 \dots T^n)\Big) = \vee_{w_1}\big(\varphi_r(T^1)\big) \dots \vee_{w_n}\big(\varphi_r(T^n)\big)$$

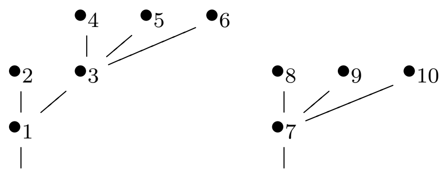
4.4. **Example.** The following planar binary tree in $\mathrm{PBTree}^{11}(M_2 = \{1, \dots, 10\})$



can be written in right comb presentation as



Its image under φ_r is this forest:



4.5. **Corollary.** Let $M = (M_0, \emptyset, M_2, \emptyset, \emptyset, \ldots)$, where M_0 is a one element set and M_2 is arbitrary, and let $\tilde{M}$ be the constant sequence $(M_2, M_2, \ldots)$.

Then the map φ_r (or φ_l) provides a bijection from the set $\mathrm{PBTree}^n(M)$ of planar binary trees with (M)-admissible labelings and n leaves onto the set of planar forests with (overall) $n-1$ vertices, with $(\tilde{M})$-admissible labelings.

The number of planar forests with $n-1$ vertices and $(\tilde{M})$-admissible labelings as well as the number of the corresponding planar binary trees is given by

$$c_n \cdot (\#M_2)^{n-1}$$

where c_n is the n-th Catalan number.

The bijection between non-labeled planar binary trees with n leaves and non-labeled planar trees with n vertices occurs as a special case (when we graft the corresponding forest onto a new root).

4.6. **Cutting trees.** Let T be a (planar or abstract) rooted tree with root ρ, and $C \subseteq \mathrm{Ve}(T)$. We call C an admissible cut of T, if for every vertex $\lambda \in C$ all vertices of the full subtree with root λ are also in C. The case $C = \emptyset$ is called the empty cut. The case $C = \mathrm{Ve}(T)$ is called the full cut.

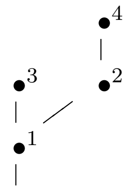
Given such an admissible cut, let $R^C(T)$ be the not necessarily non-empty tree (with root ρ, if $R^C(T) \neq \emptyset$), obtained by removing all vertices of C (together with their outgoing edges).

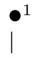
From T we can remove (the subgraph) $R^C(T)$ to get a (planar or abstract) forest $C(T)$ with set of vertices C.

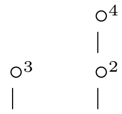
The pair $(C(T), R^C(T))$ is called result of the cut C.

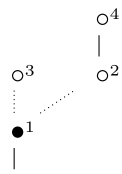
An admissible cut of T can also be defined as a non-empty subset of the set of (inner) edges of T such that for every vertex $v \in \mathrm{Ve}(T)$ on the path to the root there is at most one edge selected, cf. [2]. This definition leads to the same pair $(C(T), R^C(T))$ and is in fact equivalent, once we add the full and empty cut.

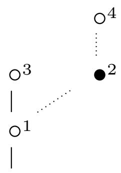
4.7. **Example.** Let T be the following planar rooted tree:



We are going to indicate (by $\circ$) which vertices are selected.
We get $R^C(T) =$ 

and $C(T) =$ 

as the result of the admissible cut 

Not an admissible cut is: 

4.8. **Hopf algebra of Connes and Kreimer.** In [11], D. Kreimer discovered a commutative Hopf algebra for the use of renormalization of quantum field theories. It was further studied by A. Connes and D. Kreimer, cf. [2], and [3].

The vector space, over a field K, of all abstract forests can be considered as a commutative polynomial algebra $K[\text{ATree}]$, graded with respect to the canonical degree function induced by

$$\deg T = n = \#\text{Ve}(T), \text{ for } T \in \text{ATree}_n,$$

and the unit 1 is identified with the empty tree $\emptyset$. The Connes-Kreimer Hopf algebra is obtained when $K[\text{ATree}]$ is provided with the coproduct Δ_{CK}, defined on the basis by

$$B_+(T^1 \dots T^n) \mapsto B_+(T^1 \dots T^n) \otimes \emptyset + (\cdot, B_+)\big(\Delta_{\text{CK}}(T^1) \otimes \dots \otimes \Delta_{\text{CK}}(T^n)\big).$$

The graded dual of this commutative Hopf algebra is isomorphic (via a graded isomorphism) to a noncommutative cocommutative Hopf algebra on trees introduced by R. Grossman and R. G. Larson (cf. [5]), see [17],[7].

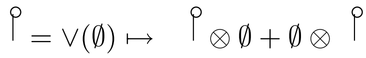
There is an alternative description of the coproduct Δ_{CK} using the concept of admissible cuts (see 4.6). Since any non-full admissible cut of $T = \vee(T^1 \dots T^n)$ corresponds to n admissible cuts (of $T^1, \dots, T^n$), it is not hard to prove by induction that the image of T under Δ_{CK} is given by

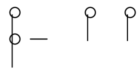
$$\sum_{C \text{admissible cut}} C(T) \otimes R^C(T),$$

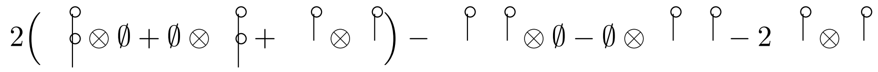
see [2].

The analogous construction for planar forests of planar trees (instead of abstract forests of abstract trees) yields a graded Hopf algebra structure which is isomorphic to the graded Hopf algebra of planar binary trees introduced by J.-L. Loday and M. Ronco in [14], see [8], [4].

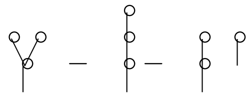
4.9. **Example.** The coproduct Δ_{CK} maps

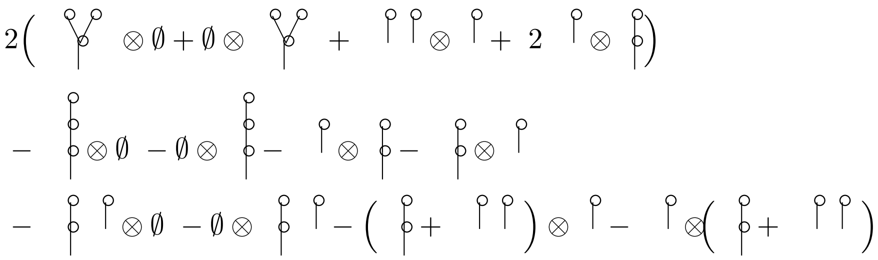


For $f = 2$  we compute that $\Delta_{\mathrm{CK}}(f)$ is given by



$= f \otimes \emptyset + \emptyset \otimes f$

For $h = 2$  we compute that $\Delta_{\mathrm{CK}}(h)$ is given by



$= h \otimes \emptyset + \emptyset \otimes h +$  $\otimes f - f \otimes$ 

4.10. **Free nonsymmetric operads.** Let a collection $(M_r)_{r\geq 2}$ of sets be given, and set $M_0 := \{\circ\}, M_1 := \emptyset$.

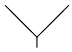
Define, for $n \geq 2$, $\underline{\Gamma(M)}(n)$ to be the vector space of all linear combinations of reduced planar trees with n leaves equipped with an (M)-admissible labeling. Let $\underline{\Gamma(M)}(0) = 0$, $\underline{\Gamma(M)}(1) = K \cdot |$, where $| :=$  is the tree consisting of the root.

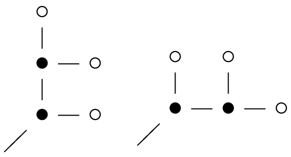
The sequence $\underline{\Gamma(M)}$ of vector spaces, together with the unit $|$ and composition maps determined by the $\circ_i$-operations $T^1 \circ_i T^2$ (given by the substitution of T^2 in T^1 at the i-th leaf, see 4.1) is the free non-Σ operad generated by the collection $M = (M_r)_{r\geq 2}$.

Families of generalized bialgebra structures on algebras over families of free nonsymmetric operads are considered in [9].

4.11. **Example.** Let M_2 consist of one generator α, and let $M_r = \emptyset$ $(r \geq 3)$.

Then all elements of $\underline{\Gamma(M)}$ are linear combinations of planar binary trees. For $n \geq 1$, we can identify a basis of $\underline{\Gamma(M)}(n)$ with the set of (non-labeled) planar binary trees with n leaves. Especially, $\dim \underline{\Gamma(M)}(n) = c_n$.

The tree  corresponds to the binary operation α, and we get ternary operations $\alpha \circ_1 \alpha$ and $\alpha \circ_2 \alpha$ as compositions:



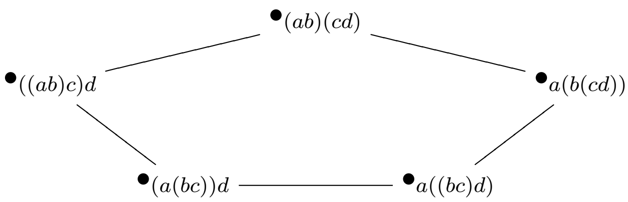
This is the non-Σ operad $\underline{\mathcal{M}ag}$ of (non-unitary) magma algebras.

4.12. **The associahedron.** Let the collection $M = (M_r)_{r \geq 0}$ be given by $M_0 := \{\circ\}, M_1 := \emptyset$, and M_r a one element set for each $r \geq 2$. Then the generated free nonsymmetric operad $\underline{\Gamma(M)}$ is known as the operad of Stasheff polytopes.

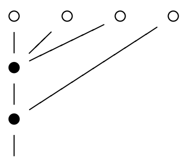
The Stasheff polytope or associahedron K_{n+1} is a convex polytope of dimension $n-1$, $n \geq 1$, with one vertex for each planar binary tree with $n+1$ leaves. More exactly, K_{n+1} is a cell complex in dimension $n-1$ with the elements of $\mathrm{PBTree}^{n+1} = \mathrm{PRTree}^{n+1}_{2n+1}$ as 0-cells.

The associahedra were created by J. Stasheff [23] to study higher homotopies for associativity. If we consider the parenthesized strings (of 3 letters) given by the two planar binary trees with 3 leaves, we can get from one to the other by shifting a bracket, in other words (cf. [16], I.1), applying an associating homotopy $h(x, y, z)$ from $x(yz)$ to $(xy)z$.

In K_4 for example, the five planar binary trees with 4 leaves have to be arranged in a pentagon

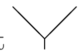


such that each side corresponds to an application of $h(x, y, z)$. These 5 sides can be labeled by the 5 reduced planar trees (with 4 leaves and 2 internal vertices) indicating the associating homotopy. For example, the edge between $(a(bc))d$ and $((ab)c)d$ corresponds to the tree



The $(n+1)$-corolla represents the top dimensional cell of the polytope K_{n+1}.

By definition of the cell complex K_{n+1} the cells of dimension k are in bijection with the elements of $\mathrm{PRTree}^{n+1}_{2n-k+1}$, for $k = 0, \ldots, n-1$.

The polytope K_2 is a point which corresponds to the unique element  of $\mathrm{PBTree}^2 = \mathrm{PRTree}^2_3$, and K_3 is an interval.

The facets (i.e. codimension one cells) of K_{n+1} are of the form $K_{r+1} \times K_{s+1}$, $r, s \geq 1$, $r + s = n$, with label obtained by grafting the s-corolla to the i-th leaf of the r-corolla, $1 \leq i \leq r$. One gets inclusion maps $\circ_i : K_{r+1} \times K_{s+1} \to K_{r+s+1}$ (cf. [16], I.1.6).

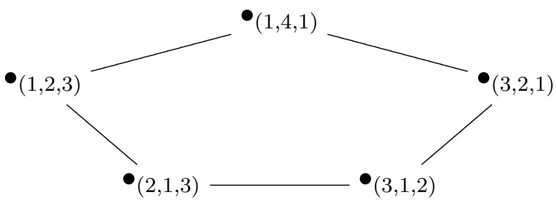
Pentagons and squares are the facets of K_5.

In fact, the realization of K_{n+1} as a convex polytope was an open problem at first. Several solutions were given (cf. [24]). A simple realization, given in [13], associates to each planar binary tree $T \in \mathrm{PBTree}^{n+1}$ a coordinate tuple $x(T)$ in (a hyperplane of) $\mathbb{R}^n$ as follows:

For $1 \leq i < n+1 = \#\mathrm{Le}(T)$, the i-th internal vertex of T is the highest internal vertex which belongs to both the paths from the i-th and the $(i+1)$-th leaf to the root.

Consider the subtree with root given by the i-th internal vertex of T, and let a_i be the number of leaves on the left side, b_i the number of leaves on the right side (of the subtree's root). Then the i-th entry of $x(T)$ is given by $a_i b_i$.

For example, the coordinate tuples we get for K_4 are:



It is shown in [13], Theorem 1.1, that the convex hull of the points $x(T), T \in \mathrm{PBTree}^{n+1}$, is a realization of the Stasheff polytope of dimension $n-1$.

It is possible to give an orientation to all the edges of the Stasheff polytope, see [15]. When 0-cells are represented as parenthesized words, arrows are directed such that they correspond to shifting a bracket from left ((xx)x) to right (x(xx)). The induced partial ordering on the set PBTree is called the Tamari order, cf. [22].

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