Universität Hamburg
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Sheaves in Topology (Master)
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Sheet 6

Question 6.1

Compute $H^i(S^1, \mathcal{C}^\infty)$ for $i > 0$ where $\mathcal{C}^\infty$ is the sheaf of smooth functions on the circle.

Question 6.2

Consider the sheaf $\mathcal{S}$ of locally constant sections of the projection $p : M \to S^1$ from the open Möbius band to the circle. Compute $H^*(S^1, \mathcal{S})$.

Question 6.3

(a) Find a double complex A^{pq} with $A^{pq} = 0$ for $q > 0$ and $p < 0$ with rows and columns exact except possibly in degree 0 such that, given $B^p = \operatorname{coker}(d : A^{p,-1} \to A^{p0})$, the natural map $\operatorname{Tot}^{\oplus}(A) \to B$ is *not* a quasi-isomorphism.

(b) Find an (unbounded) double complex A^{pq} with exact rows and columns such that $H^*(\operatorname{Tot}^{\Pi}(A)) = H^*(\operatorname{Tot}^{\oplus}(A)) = 0$.

Question 6.4 *

Let K be any subset of $X = \mathbb{R}^n$. Assume $K = \cap_i U_i$ for open sets $U_i \subset X$. Write $\mathcal{F}|_K$ for the pullback of $\mathcal{F}$ to K etc.

(a) Show carefully that $\Gamma(K, \mathcal{F}|_K) = \operatorname{colim}_i \Gamma(U_i, \mathcal{F}_{U_i})$.

(b) Show that the restriction of a flabby sheaf to an open subspace is flabby. Use (a) to show that the restriction of a flabby sheaf to K is flabby.

(c) Combine these results to show that $H^*(K, \mathcal{F}|_K) = \operatorname{colim}_i H^*(U_i, \mathcal{F}_{U_i})$

Hint: A special case of this statement was used in lectures, but the argument had gaps.

Note (a) is immediate for the presheaf $V \mapsto \operatorname{colim}_{V \subset U} \mathcal{F}(U)$, but the sheafification $\mathcal{F}|_K$ needs more care: A section of $\Gamma(K, \mathcal{F}|_K)$ is given by a collection $s_\alpha \in U_\alpha$ where $K \subset U_\alpha$ and s_α and s_β agree on $U_\alpha \cap U_\beta \cap K$. You need to find for each point an open neighbourhood in X where the sections agree. Note in $\mathbb{R}^n$ any cover has a locally finite subcover.

For point (b) note that contrary to claims in the lecture the restriction of a sheaf $\prod \mathcal{F}_x$ is not flabby a priori as products do not commute with pullbacks.

These questions will be discussed in the exercise class on 23 May 2025.