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Sheaves in Topology (Master)
Summer Semester 2025

Sheet 4

Question 4.1

Define an abelian group structure on the morphisms in a derived category (using the description of morphisms as zig-zags $\bullet \overset{\simeq}{\leftarrow} \bullet \to \bullet$).

Question 4.2

Let $Z \subset X$ be a closed subspace and $\mathcal{A}$ an abelian category. Show that $D(\mathsf{Sh}(Z, \mathcal{A}))$ is a full subcategory of $D(\mathsf{Sh}(X, \mathcal{A}))$.

Question 4.3

We consider the sheaf of rings $\mathcal{C}^\infty$ on $\mathbb{R}$.

(a) Show that $i_*\mathbb{R}$ is a sheaf of $\mathcal{C}^\infty$-modules.

(b) An $\mathcal{R}$-module $\mathcal{F}$ on a space with a sheaf of rings $\mathcal{R}$ is called *flat* if $\mathcal{F} \otimes_{\mathcal{R}} -$ preserves acyclic complexes. Show that $i_*\mathbb{R}$ is not a flat $\mathcal{C}^\infty$-module.

(c) Give a flat resolution of $i_*\mathbb{R}$ and compute $\mathrm{Tor}_i^{\mathcal{C}^\infty}(i_*\mathbb{R}, i_*\mathbb{R})$ (assuming the content of Remark 3.50).

Question 4.4

Let $\mathcal{A}$ be an abelian category.

(a) Show an object $A \in D(\mathcal{A})$ is isomorphic to 0 if and only if $H^i(A) = 0$ for all i.

(b) Let $B = (\mathbb{Z} \xrightarrow{z \mapsto 2z} \mathbb{Z})$ and $C = (\mathbb{Z} \xrightarrow{z \mapsto [z]} \mathbb{Z}/3)$ be two complexes in $\mathsf{Ch}(\mathbb{Z})$, both concentrated in degrees 0 and 1.

Find a map $f : B \to C$ with $H^i(f) = 0$ for all i but f is not the zero map in $D(\mathcal{A})$.

(c) Also find a map $g : D \to E$ in $\mathsf{Ch}(\mathbb{Z})$ which is the zero map in $D(\mathbb{Z})$ but not in $K(\mathbb{Z})$.

These questions will be discussed in the exercise class on 9 May 2025.

Questions with an asterisk are more challenging.