# Rational Homotopy Theory

**Master's Course**
**Winter Semester 2020/21**


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# Contents

These are the lecutre notes as of November 18, 2021.

An up to date version of these notes can be found at `http://www.math.uni-hamburg.de/home/holstein/lehre/RHTnotes.pdf`.

# 1. Introduction

## 1.1. Overview

This is an introduction to Rational Homotopy theory.

Homotopy theory is the study of topological spaces up to homotopy equivalence. It is very hard.

Algebra is very useful, invariants like cohomology and homotopy groups help distinguish homotopy type and contain topological information about a given space. But these invariants are still imperfect. The most fundamental invariants are homotopy groups, but they are incredibly hard to compute! (I don't think anybody knows $\pi_{42}(S^2)$, for example. We do know $\pi_{42}(S^{29})$ though. It is $\mathbb{Z}/504\mathbb{Z}$.)

Dream: Associate to every topological space $X$ an algebraic invariant $A(X)$ such that we can recover the homotopy type of $X$ from $A(X)$, read off homotopy groups, compute mapping spaces, etc.

The dream comes true if we are willing to restrict our attention to *rational homotopy theory*.

What does that mean? The key idea is to work one prime at a time.

This idea is of course borrowed from number theory. It is not obvious what it means to consider a topological space at a prime, but we will make sense of it.

The actual primes are very hard. So let us understand prime broadly: As prime ideals of the integers. The easiest prime ideal is (0). Localising $\mathbb{Z}$ at (0) is just another way of saying we invert all nonzero integers, in other words we form fractions and obtain $\mathbb{Q}$.

Localising a space at (0) we obtain its *rationalization*, which will be defined later.

Another way of looking at rationalization is that we want to discard *torsion*. An element $g$ in a group is *torsion* if $g^n = e$ for some $n$. While $\mathbb{Z}$ itself is torsion-free it has torsion quotients, so if we want to avoid all torsion phenomena we need to replace it by $\mathbb{Q}$.

Here is an outline of the course as it is planned at the moment. There may well be changes.

1. Introduction.

2. Differential graded algebras and their minimal models.

3. Model categories: Axiomatic homotopy theory.

4. The model category of dg algebras.

5. Simplicial sets: A combinatorial model for topological spaces.

6. The Polynomial de Rham functor and the adjunction between spaces and algebras.

7. Interlude on spectral sequences.
8. Some background on homotopy theory.
9. Main result: The Quillen-Sullivan equivalence.
10. Rationalisation and homotopy groups.
11. Applications.

This is an advanced graduate course, it should be roughly your third course on topology.

The course is not complete in the sense that I will leave out some details and use some highly non-trivial results from the literature. But I will give a fairly detailed proof of the main theorems of the course.

I will introduce background material as needed and will give examples of the concepts and some sketches of proofs when there is no time to give full proofs. I will of course give references, but it is unfortunately not realistic to cover in detail all that one would need to prove the main theorem from scratch.

You can influence the pace and focus of the course somewhat by asking questions or telling me to slow down or speed up (via email).

## 1.2. A few words on the history and literature

The idea of considering homotopy theory "modulo torsion" goes back to ideas of Serre in the 1950s.

The modern approach to the subject was introduced by Dan Quillen's seminal 1969 paper [Qui69] first establishing the deeply algebraic nature of rational homotopy theory.

Later Dennis Sullivan simplified the algebraic structures involved by recognising the role the de Rham complex could play [Sul77]. This construction was already suggested in 1959 René Thom (unpublished).

The abstract theory of localization of spaces at a class of equivalences was suggested by J. Frank Adams and developed by Bousfield and Kan [BK72] in a book affectionately know as "The yellow monster".

We will follow the exposition in [BG76] which makes Sullivan's construction functorial and combines the geometric intuition and algebraic clarity of the two approaches.

For the necessary background on homotopy theory we will refer to [May99] and [MP11].

If you want to complement this course with some different perspectives, a good book introducing rational homotopy theory is [GM81]. Many interesting results and applications can be found in [FHT12] which favours a less abstract approach. At the other extreme a very modern perspective on rational homotopy theory is given by [Lur11a].

Finally, a number of mathematicians have taught courses on the subject and there are some interesting lecture notes around.

## 1.3. Introducing Homotopy Theory

Recall that for any topological space $X$ with a base point $x_0$ we have the following homotopy invariants:

1. $H_*(X, \mathbb{Z})$
2. $H^*(X, \mathbb{Z})$ with its graded ring structure
3. $\pi_1(X, x_0)$
4. $\pi_i(X, x_0)$ for $i \geq 2$.

I will assume that you know the first three (i.e. have computed them in some examples). Let us recall higher homotopy groups.

I will denote by $I$ the interval $[0, 1]$.

We denote by $[(X, A), (Y, B)]$ the set of homotopy classes of continuous maps from $X$ to $Y$ that send $A$ to $B$. In particular we define $f$ and $g$ to be homotopic if there is a homotopy $H$ from $f$ to $g$ that sends $A \times I$ to $B$.

**Definition 1.1.** Given $(X, x_0) \in \mathsf{Top}_*$ and $n \geq 1$ we define $\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$. This becomes a group by concatenating in the last variable, i.e. $(f * g)(t_1, \dots t_n) = g(t_1, \dots, t_{n-1}, 2t_n)$ if $t \leq 1/2$ and $(f * g)(t_1, \dots t_n) = f(t_1, \dots, t_{n-1}, 2t_n - 1)$ if $t \geq 1/2$.

By the Eckmann-Hilton argument this group structure is abelian and it does not matter, which coordinate we use to define it.

**Definition 1.2.** A map $f : (X, x_0) \to (Y, y_0)$ is called a *weak homotopy equivalence* if $\pi_n(f)$ is an isomorphism for all $n \geq 0$.

We will be considering weak homotopy equivalences throughout this course as a natural class of equivalence between topological spaces.

That seems optimistic, as it seems we have replaced our topological problem, of determining which spaces are homotopy equivalent, with an (almost) algebraic one.

The justification is given by Whitehead's theorem. We will not prove it, and we will not directly use it to prove our main results, but it provides important motivation.

Recall first that a *CW complex* is a space $X$ that is the union of subspaces $X_n$, where $n \in \mathbb{N}$, such that $X_0$ is discrete and $X_n$ is obtained from $X_{n-1}$ by attaching a set of $n$-cells.

More categorically (and more precisely) we have $X = \operatorname{colim}_{n \in \mathbb{N}} X_n$ and every $X_n$ is a pushout $X_{n-1} \amalg_{f, \amalg S^{n-1}} \amalg D^n$ where $S^{n-1} \to D^n$ is the usual inclusion and $f : \amalg S^{n-1} \to X_{n-1}$ is some continuous attaching map.

**Theorem 1.3** (Whitehead's theorem)**.** *Any weak homotopy equivalence between CW complexes is a homotopy equivalence.*

**Warning 1.4.** This does not mean that a CW complex is determind by its homotopy groups! A counterexample is given by $S^2 \times \mathbb{R}P^3 \not\simeq \mathbb{R}P^2 \times S^3$. The existence of a map inducing the isomorphism is a serious condition.

We also have the following.

**Theorem 1.5** (CW approximation theorem)**.** *Let $X$ be a topological space. Then there exists a CW complex $Z$ and a weak equivalence $f : Z \to X$.*

So studying CW complexes up to homotopy is equivalent to studying all topological spaces up to weak equivalences.

As the class of CW complexes is large enough to contain all spaces one is usually interested in, this seems like good category to study. CW complexes and homotopy classes of maps between them form what is usually called the *homotopy category*.

So we may build our study of homotopy theory on the higher homotopy groups.

I have tried to convince you that homotopy groups are very difficult to compute, so you might worry we have not gained much. But while homotopy groups are hard to compute, they can be computed with. And one thing one can show is the following result.

**Proposition 1.6.** *Let $f : X \to Y$ be a map between connected CW complexes which induces an isomorphism of fundamental groups and isomorphisms $H_n(X, \mathbb{Z}) \cong H_n(Y, \mathbb{Z})$ for all $n$. Then $X$ and $Y$ are weakly homotopy equivalent and thus homotopy equivalent.*

So if we understand fundamental groups and homology groups (which are all very computable) then we can detect weak equivalences.

They key ingredient is that while homotopy groups and homology groups are very different, the first nonzero homotopy and homology group of a space agree (this is *Hurewicz' theorem*). Thus a space with no non-trivial homology groups also has no homotopy groups and is contractible. The idea is to apply this to the homotopy fibre of our map $f$.

If you have not met these results before, you should not worry. They just serve as motivation for now, that algebraic invariants are a promising way to study the homotopy type of topological spaces.

## 1.4. Rationalization

With this brief overview in mind we may define a rational space as one whose algebraic invariants are rational.

There are some subtleties here as it's not clear what it means for the fundamental group to be rational. So we will make a seemingly naive definition just considering the homology groups:

**Definition 1.7.** Let $X$ be a connected topological space. We say that $X$ is *rational* if $H_n(X, \mathbb{Z})$ is a $\mathbb{Q}$-vector space for every $n \geq 1$.

**Remark 1.8.** This definition turns out to be a very good definition, in particular if $X$ satisfies one additional condition, that it is *nilpotent*. We will return to this important point.

You have probably never met a rational space! Let us build an example.

We first define the pointed mapping cylinder of a pointed map $f : (X, x_0) \to (Y, y_0)$ of topological spaces as $X \times I \amalg_f Y$, which is defined as the quotient of $X \times I \amalg Y$ by the relation identifying every $(x, 1)$ with $f(x)$ and the relation identifying the subset $\{x_0\} \times I$ with a single point (also identified with $f(x_0) = y_0$).

**Example 1.9.** We will inductively construct spaces $X_k$, each equipped with a map $i_j : S^1 \to X_k$ from the circle.

$X_1$ is just the circle $S^1$ with base point given by the unit 1, and $i_1$ is the identity map.

Now for each positive integer we consider the pointed map $w_k : S^1 \to S^1$ that sends $t$ to $t^k$, so it wraps the first circle $k$ times around the other. On fundamental groups it induces multiplication by $k$.

Let $W_k$ be the pointed mapping cylinder of $w_k$. $W_k$ contains two canonical copies of $S^1$, given by inclusion of the source $s_k : S^1 \to S^1$ and target $t_k : S^1 \to W_k$ of the map $w_k$.

Define $X_k$ by gluing $X_{k-1}$ and $W_k$ along $S^1$ using the maps $i_{k-1}$ and $s_k$, so $X_k$ is the following colimit:

$$\begin{tikzcd} S^1 \arrow[r, "s_k"] \arrow[d, "i_{k-1}"] & W_k \arrow[d] \\ X_{k-1} \arrow[r] & X_k \end{tikzcd}$$

Then define $i_k$ to be the composition of $t_k$ with the inclusion $W_k \to X_k$.

Let $X = \operatorname{colim} X_i$.

We note that every $X_i$ is homotopy equivalent to a circle, but we will see that $X$ is not. While the fundamental groups of all the $X_k$ are iomorphic to $\mathbb{Z}$, the point of this construction is that $X_{k-1} \to X_k$ induces the map $1 \mapsto k \cdot 1$ on fundamental groups.

We compute higher homology groups of $X$ first. Let $n \geq 2$. Any map from an $n$-simplex factors through some $X_i$ (since $\Delta^n$ is compact), so any closed $n$-chain is a boundary since $H_n(X_i) \cong H_n(S^1) = 0$ for $n \geq 2$. Then $H_n(X, \mathbb{Z}) = 0$ if $n \geq 2$.

Next we compute $\pi_1(X, 1)$. The loop $\gamma$ around $X_1$ is non-trivial since any homotopy $S^1 \times I \to X$ factors through some $X_i$, and $[\gamma] = i! \in \pi_1(X_i, 1)$. As we can write $[\gamma] = i![\delta]$ for a generator $[\delta]$ of $\pi_1(X_i, 1)$ we see that the fundamental group contains all rational multiples of $[\gamma]$.

Moreover, any map from $S^1$ factors through some $X_i$, so all elements in $\pi_1(X, x_0)$ are rational multiples of $[\gamma]$.

So the fundamental group of $X$ is just a 1-dimensional $\mathbb{Q}$-vector space. Its abelianization is still $\mathbb{Q}$, so $H_1(X, \mathbb{Z}) = \mathbb{Q}$ and $X$ is rational in the sense of Definition 1.7.

We denote $X$ by $S^1_{\mathbb{Q}}$.

We note that our example is in fact an *Eilenberg-MacLane space* $K(\mathbb{Q}, 1)$, i.e. it has no higher homotopy groups and the fundamental group is $\mathbb{Q}$. Again we use that any map from $S^n$ to $X$ factors through some $X_i$ (because $S^n$ is compact), so it follows that $\pi_{\geq 2}(X, 1) = 0$.

So we have found the rational analogue of the circle. We could define rational spheres similarly and glue together rational CW complexes. However, there is a better way of thinking about rational spaces.

**Definition 1.10.** Let $f : X \to Y$ be a map between connected topological spaces. We say that $f$ is a *rational equivalence* if $f_* : H_n(X, \mathbb{Q}) \to H_n(Y, \mathbb{Q})$ is an isomorphism for all $n \geq 1$.

**Definition 1.11.** A *rationalization* of a topological space $X$ is a rational space $X_{\mathbb{Q}}$ together with a rational equivalence $X \to X_{\mathbb{Q}}$.

**Example 1.12.** The natural inclusion $\iota : S^1 \to S^1_{\mathbb{Q}}$ is a rational equivalence and $S^1_{\mathbb{Q}}$ is a rationalization.

Let us verify this: The higher homology groups are zero on both sides and we have $H_1(S^1_{\mathbb{Q}}, \mathbb{Q}) \cong H_1(S^1_{\mathbb{Q}}, \mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$ by the universal coefficient theorem. Of course $H_1(S^1, \mathbb{Q}) \cong \mathbb{Q}$.

To check that the isomorphism is induced by $\iota$ we note that the generator of $H_1(S^1, \mathbb{Q})$ is sent to the vector space generator $[\gamma]$ of $H_1(S^1_{\mathbb{Q}}, 1)$ from Example 1.9.

Any simply connected topological space is rationally equivalent to a rational space, i.e. has a rationalization. To show this we could take a CW complex and replace all of the cells and attachment maps by rationalized versions. I won't provide details as we will concentrate on a different approach in this course.

Instead of considering the subcategory of rational topological spaces (which looks odd and unwieldy) one may examine the quotient category of topological spaces, where we identify all topological spaces which are rationally equivalent. This is sometimes called the *localization* at the category of spaces at rational equivalences.

**Definition 1.13.** The equivalence class of a topological space $X$ up to rational equivalence is called the *rational homotopy type* of $X$.

**Example 1.14.** Let $\mathbb{R}P^2$ be the projective plane. Then the rational homology groups of $\mathbb{R}P^2$ all vanish and the projection from $\mathbb{R}P^2$ to a point is a rational equivalence.

Note however, that $\pi_2(\mathbb{R}P^2, *) \cong \mathbb{Z}$, which has rationalization $\mathbb{Q}$ rather than 0. If we had defined rational equivalence in terms of homotopy groups instead of homology groups, this would not be a rationalization. This shows that we made a meaningful choice in our definition of rationalization. It turns out that the presence of the fundamental group is what is making this more subtle.

**Remark 1.15.** Because fundamental groups present difficulties people often restrict to simply connected spaces, but there is a lot of interesting mathematics to do with the role of fundamental groups in rational homotopy theory.

We will in due course restrict to *nilpotent spaces*, which have nilpotent fundamental groups that moreover act nilpotently on the higher homotopy groups.

More complicated fundamental groups create a genuine problem for rational homotopy theory.

## 1.5. The de Rham algebra

We recall from Proposition 1.6 that for a simply connected space we may consider homology rather than homotopy groups to understand its homotopy type. And homology groups are much easier to compute.

So our first, very optimistic, plan is to study the rational homotopy type of $X$ by studying its rational homology groups. We can capture a lot more information by studying cohomology instead, as there is a graded commutative ring structure.

But this is still not very promising, as the homotopy type contains more information than just the cohomology ring.

One key insight of modern homological algebra is to not reduce a complex to its cohomology groups. So our third guess is to use the cochain complex which computes rational cohomology.

The usual singular cochains have a cup product inducing the product on cohomology, but it is not commutative but only "homotopy commutative". This makes life much harder. (It is also a very large algebra.)

However, as long as we are working over a field in characteristic zero, we can use the de Rham complex to compute cohomology, and the de Rham complex is naturally (graded) commutative. This is our fourth guess, and amazingly, it works.

Let us recall the de Rham complex $\mathscr{A}(X)$ of a smooth manifold $X$.

**Definition 1.16.** Define on $\mathbb{R}^n$ the algebra

$$\mathscr{A}(\mathbb{R}^n) = \mathscr{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}\langle dx_1, \dots, dx_n \rangle,$$

where the first factor is given by smooth functions on $\mathbb{R}^n$ and the second tensor factor is a vector space on symbols $dx_{i_1} dx_{i_2} \cdots dx_{i_k}$ with $i_j < i_{j+1}$ equipped with an anticommutative multiplication $dx_i dx_j = -dx_j dx_i$. This is graded by putting $dx_i$ in degree 1. Then differentiation map $f \mapsto \sum \frac{\partial f}{\partial x_i} dx_i$ induces a differential.

It has a graded commutative product structure coming from the multiplication on the $dx_i$ and the pointwise multiplicatino of functions.

For a general manifold $X$ with an open cover $\{U_i\}_{i \in I}$ with $U_i \cong \mathbb{R}^n$ we can glue the $\mathscr{A}(\mathbb{R}^n)$ to obtain the *de Rham algebra* $\mathscr{A}(X)$.

The cohomology of this complex is the de Rham cohomology, which is isomorphic to singular cohomology or Čech cohomology.

**Remark 1.17.** I am of course skipping some details on how to glue. Another definition is that the de Rham algebra is the algebra of smooth sections of exterior powers of the cotangent bundle on $X$. This would also need some more definitions. We will get to rigorous definitions in due course.

The de Rham complex is real, rather than a rational, algebra, and it is only defined for manifolds. But these issues can be overcome.

*Commutative differential graded algebras over $\mathbb{Q}$ admit their own homotopy theory, and it is equivalent to the homotopy theory of (nice) rational spaces via (a version of) the de Rham functor.*

*Moreover, we can write down explicit models for the de Rham algebra of a topological space X, that allow us to compute the homotopy groups of X and much more.*

# 2. Commutative differential graded algebras

## 2.1. Definitions

In this section we will introduce commutative differential graded algebra and begin examing their homotopy theory.

Let $k$ be a field. It will soon be $\mathbb{Q}$.

**Definition 2.1.** A *commutative differential graded algebra* (aka *cdga*) over $k$ is a $k$-algebra $A$ such that:

1. $A$ is *graded*, i.e. $A = \oplus_{n\in\mathbb{Z}} A^n$. If $a \in A^n$ we say $a$ is homogenous of degree $n$ and write $|a|$ for $n$.

2. $A$ has a product such that $|ab| = |a| + |b|$.

3. $A$ is *graded commutative*, i.e. $a.b = (-1)^{|b||a|} b.a$ if $a, b$ are homogeneous elements.

4. $A$ has a *differential* $d : A^i \to A^{i+1}$ for each $i$ satisfying $d^2 = 0$.

5. The differential $d$ is a graded derivation, i.e. $d(ab) = da.b + (-1)^{|a|} a.db$. This is the *graded Leibniz rule*.

We denote the unit map $k \to A$ by $e_A$ or $e$. It is clear that $1_A = e_A(1) \in A^0$. We call $a$ an *odd* or *even* element according to whether $|a|$ is odd or even.

If we forget the multiplicative structure then we are left with a differential graded $k$-module, which is just a cochain complex.

The cohomology of a cdga $A$ with respect to $d$ is a commutative graded algebra which we denote by $HA$, or $H^i(A)$ for the piece in degree $i$.

**Example 2.2.** Any commutative algebra may be viewed as a cdga concentrated in degree 0.

**Example 2.3.** Let $X$ be a manifold. Then the de Rham complex equipped with the wedge product is naturally a cdga over $\mathbb{R}$.

**Example 2.4.** Let $\Omega(n)^0 = \mathbb{Q}[t_0, \ldots, t_n]/(\sum t_i = 1)$. This is the algebra of polynomial functions on the $n$-simplex. Now let $\Omega(n)^*$ be the cdga generated over $\Omega(n)^0$ by generators $dt_0, \ldots, dt_n$ in degree 1 satisfying $\sum dt_i = 0$. The differential sends $t_i$ to $dt_i$ and $dt_i$ to 0. By the Leibniz rule it follows that a polyonimal $f$ satisfies $df = \frac{\partial f}{\partial t_i} dt_i$.

This is the polynomial de Rham algebra of the $n$-simplex and we will meet it again. For simplicity we will from now on replace it by the isomorphic subalgebra without generators $t_0$ and $dt_0$.

**Remark 2.5.** The signs can easily get confusing. It is best to remember the *Koszul rule of signs*, saying that whenever we commute an object of degree $m$ past an object of degree $n$ we pick up a sign of $(-1)^{mn}$. This boils down to: Whenever we commute two odd object past each other, we pick up a minus sign. I am using the vague term "object" because this rule applies to both elements and operations.

The Koszul rule explains why $b.a = (-1)^{|b||a|}a.b$ and also why the second summand of $d(ab)$ is $(-1)^{|a|}a.db$ (since the operator $d$ has degree 1).

**Remark 2.6.** We are using a grading and differential where the differential has degree +1, i.e. raises degree, we call this a *cohomological grading*. Some authors use the opposite convention, i.e. a homological grading.

A map $f : A \to B$ that preserves grading, product and differential is called a *homomorphism*.

We denote the category whose objects cdga's over $k$ and whose morphisms are homomorphisms by $\mathsf{cdgA}_k$, where we may drop the subscript if the field is clear from context.

A cdga is *nonnegatively graded* if $A^i = 0$ if $i < 0$. The full subcategory of nonnegatively graded cdga's over $\mathbb{Q}$ will be denoted $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$.

Motivated by Whitehead's theorem (and the theory of chain complexes) we define:

**Definition 2.7.** A homomorphism of cdga's that induces isomorphisms on all cohomology groups is called a *quasi-isomorphism*

## 2.2. Free and semi-free cdga's

We will now consider cdga's whose multiplicative structure is easy.

We first recall the *symmetric algebra* on a vector space $V$ over $k$, denoted $S(V)$. It has a vector space decomposition $S(V) = \oplus_p S^p(V)$ where $S^p(V) = V^{\otimes p}/S_p$ for each $p$, i.e. it is the quotient of the $p$-fold tensor power of $V$ by the natural action of the symmetric group $S_p$ permuting the factors. The product is obtained in the obvious way by considering an element $v_1 \otimes \cdots \otimes v_p$ as a monomial $v_1 \cdots v_p$ and multiplying them together by concatenation ("writing next to each other").

Explicitly if $V$ has a basis $\{x_1, \dots, x_n\}$ then $S(V)$ is the polynomial algebra $k[x_1, \dots, x_p]$ and $S^p(V)$ consists of homogeneous polynomials of degree $p$.

We can extend this construciton to graded vector spaces $V$, but the naive symmetric algebra on a graded vector space is not commutative, as for two elements in odd degree we should have $ab = -ba$.

We define the *free graded commutative algebra* $S(V) = \oplus_p S^p(V)$, where $S^p(V) = V^{\otimes p}/S_p$ but $S_p$ acts such that every transposition of odd elements changes the sign, so $(12).(v \otimes w) = -w \otimes v$ if $v, w$ have odd degree. $S(V)$ has a grading where each monomial $v_1 \dots v_p$ has degree $|v_1| + \cdots + |v_n|$. It has a product just as in the ungraded case.

Explicitly if $V$ consists of $V^{ev} = \oplus_{2k} V^{2k}$ and $V^{odd} = \oplus_{2k+1} V^{2k+1}$ then $S(V) = S(V^{even}) \otimes \Lambda(V^{odd})$ where $\Lambda(V^{odd})$ denotes the *exterior algebra* on $V^{odd}$, with one generator for each basis element of $V^{odd}$ and the relation $ab = -ba$.

We write $A = B\langle x_1, \ldots, x_k\rangle$ for the free graded commutative $k$-algebra with generators $\{x_1, \ldots, x_k\}$.

If now $V$ is a cochain complex with a differential $d$ then we can define a differential $d_{SV}$ on any monomial $v_1 \ldots v_n$ by the Leibniz rule: $d(vw) = dv.w + (-1)^{|v|} dw$ etc.

**Definition 2.8.** The cdga $(S(V), d_{SV})$ as above is called the *free commutative differential graded algebra* on the cochain complex $V$.

**Example 2.9.** The cdga $\Omega(n)$ is the free cdga on the complex $\mathbb{Q}^n \to \mathbb{Q}^n$ with differential given by the identity. (Just label two bases as $\{t_i\}$ and $\{dt_i\}$) to see this.)

**Remark 2.10.** One can define the condition of freeness in a more systematic way. Note that there is an obvious forgetful functor $U$ from the category $\mathsf{cgA}$ of commuative graded algebras to the category $\mathsf{gVect}$ of graded vector spaces. By general considerations (the adjoint functor theorem) it has a left adjoint and the free graded commutative algebras are given by the essential image of this functor.

Similarly the forgetful functor $U : \mathsf{cdgA} \to \mathsf{dgMod}$ has a left adjoint $S$ taking a cochain complex $V$ to the free cdga on $V$.

The following weakening of the definition of a free cdga will be very useful to us:

**Definition 2.11.** A cdga $(A, d)$ is called *semi-free* if the graded algebra $(A, 0)$ obtained by forgetting the differential is a free graded commutative algebra.

**Remark 2.12.** This is analogous to considering projective resolutions, like chain complexes that are free (as modules) after forgetting the differential.

## 2.3. Augmented cdga's

**Definition 2.13.** An *augmentation* of a differential graded $k$-algebra is a map $\epsilon : A \to k$ such that $\epsilon \circ e = \mathbf{1}_k$. If $A$ has an augmentation it is called *augmented*. A choice of augmentation is similar to a choice of basepoint for a topological space, as we will see later.

A morphism between augmented dga's is a homomorphism that also preserves the augmentation. The category of augmented cdga's over $k$ is denoted by $\mathsf{cdgA}_{/k}$, and the category of non-negative graded augmented cdga's over $\mathbb{Q}$ is $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$.

**Definition 2.14.** For an augmented differential graded algebra we can introduce the *augmentation ideal* $\bar{A} := \ker(\epsilon : A \to k)$. Furthermore let the *indecomposables* of $A$ be $IA := \bar{A}/(\bar{A} \cdot \bar{A})$.

We can augment $\Omega(n)$ by the evaluation map at the origin, sending $f(t)$ to $f(0)$ and $dt_i$ to 0. Then the indecomposables of $\Omega(n)$ are all the $t_i$ and $dt_i$.

**Definition 2.15.** Given $A, B \in \mathsf{cdgA}$ we define the *tensor product* $A \otimes B$ as the linear span of symbols $a \otimes b$ with the usual tensor product relations. We equip this with the grading $|a \otimes b| = |a| + |b|$ for homogenous elements $a \in A$ and $b \in B$ and let the differential by $d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$. The product is $(a \otimes b).(a' \otimes b') = (-1)^{|b||a'|} a.a' \otimes b.b'$.

We then have $(A \otimes B)^n = \oplus_{p+q=n} A^p \otimes B^q$.

We note that this agrees precisely with the tensor product of cochain complexes. (Or if you haven't met the tensor product of cochain complexes before, use this as a definition.)

## 2.4. Minimal models

Recall we are trying to describe topological spaces in terms of their de Rham algebra consisting of smooth differential forms.

The de Rham complex is in general very large and unwieldy as an algebra.

On the other hand the cohomology ring does not carry enough information. So we look for a class of cdga's that one can work with and that is richer than the cohomology ring.

They will be semi-free and satisfy a further condition, which ensures they can be built inductively from the ground field.

We say a cdga $B$ is *connected* if it is nonnegatively graded and the unit map $k \to B^0$ is an isomorphism.

We make some auxiliary definitions: Let $B(n)$ be the subalgebra generated by $B^i$ for $0 \leq i \leq n$ and $dB^n$. Also, let $B(-1)$ be $k.1$. Now define $B(n, 0)$ to be $B(n-1)$ and $B(n, p)$ to be the subalgebra generated by $B(n, p-1)$ and all the elements $x \in B^n$ with $dx \in B(n, p-1)$.

**Definition 2.16.** A semi-free connected cdga $M$ is called *minimal* if it satisfies $M(n) = \bigcup_p M(n, p)$ for all $n$. A *minimal model for a cdga $B$* is a minimal cdga $M$ together with a quasi-isomorphism $e : M \to B$.

This definition implies that the differential on a minimal model $M$ is decomposable, i.e. $dm \subset \bar{M}.\bar{M}$ for all $m \in M$. We say that a cdga $A$ is *simply connected* if $A$ is connected and $A^1 = 0$). A simply connected semi-free cdga $A$ is minimal if and only if the differential is decomposable.

It follows from the definition that a minimal algebra is connected and hence it has a unique augmentation $\epsilon : M \to k$ that is the identity on $M^0$ and kills everything else.

Let us now connect this with topology:

**Definition 2.17.** A *minimal model for a manifold $X$* is a minimal model for the de Rham algebra of $X$.

**Example 2.18.** Let us calculate the minimal model $M(S^n)$ for the $n$-sphere. The de Rham cohomology of $S^n$ is generated by the volume form $dvol$ of degree $n$. (Even if you have never seen the de Rham complex before this should not be surprising! You know that the cohomology of the de Rham complex is the cohomology of the sphere, so there must be some

$n$-form that generates it! This is exactly the $n$-fom that one may integrate to find the volume of the sphere, hence the name.)

Thus to find $M(S^n)$ we need a generator $x$ in degree $n$, with $dx = 0$.

If $n$ is odd, the obvious map $\mathbb{R}\langle x\rangle \to A(S^n)$ induces an isomorphism on cohomology. As $\mathbb{R}\langle x\rangle$ is minimal it is a minimal model and $M(S^n)$ is the exterior algebra on one generator.

If $n$ is even, $x$ generates a polynomial algebra. Since $M(S^n)$ must be free the only way to kill $x^2$ is to introduce an element $y$ of degree $2n-1$ with $dy = x^2$. Then the cdga $M(S^n) = \mathbb{R}\langle x, y \mid dy = x^2\rangle$ is minimal. We define $M(S^n) \to A(S^n)$ by $x \mapsto dvol$ and $y \mapsto 0$. This is a quasi-isomorphism, so we have found a minimal model.

A word about notation: We write the minimal model for $S^{2k}$ as $\mathbb{Q}\langle x, y \mid dy = x^2\rangle$. In general we write

$$\mathbb{Q}\langle a, b, \dots \mid da = r, \dots, db = s, \dots\rangle$$

for the minimal model on generators $a, b, \dots$ with $da = r$ etc. where we leave out the terms $dx_i = 0$.

We may also write $\mathbb{Q}\langle x, dx\rangle$ for $\mathbb{Q}\langle x, y \mid dx = y\rangle$.

For example, we can write

$$\Omega(n) = \mathbb{Q}\langle t_1, \dots, t_n, dt_1, \dots, dt_n\rangle.$$

**Remark 2.19.** If you know about homotopy groups of spheres, you may know that $\pi_3(S^2) \cong \mathbb{Z}$ and more generally $\pi_{4k-1}(S^{2k}) \cong \mathbb{Z}$, and these are the only homotopy groups of spheres (other than the obvious ones) which are not torsion.

This is by no means a coincidence: We will later show that for reasonable spaces the generators in degree $n \geq 2$ of the minimal model correspond to a basis of $\pi_n \otimes \mathbb{Q}$.

**Example 2.20.** We consider another example and let $X = S^3 \vee S^3$.

First we note that this is not a manifold, so we can't take the usual de Rham complex. On the other hand, we can try a piecewise de Rham complex, where we demand that functions agree at the base point (so are continuous), but are not necessarily smooth.

Then by Mayer-Vietoris the cohomology of $X$ is concentrated in degree zero and three (2-dimensional) and it should come as no suprrise that it is generated by the volume forms of $S^3$ and $S^3$. Their product is 0 as there are no 6-forms.

We define $M(3) = \mathbb{Q}\langle x, y\rangle$ with $|x| = |y| = 3$. But then $x.y$ is a nonzero cycle in $M(3)$ so we need it to be a boundary. Introduce $z$ with $dz = x.y$ (and necessarily $|z| = 5$). But then $x.z$ and $y.z$ are cycles in degree 8, so we need two forms $a, b$ in degree 7 to make up for it, with $da = x.z$ and $db = y.z$.

This process goes on forever, but we can determine the generators of $M$ degree by degree.

Here is a question: Is there an algebraic way of expressing all the generators of $M$ in terms of $x$ and $y$? We'll hopefully come back to this later in the course, but feel free to think about it!

## 2.5. Existence of minimal models

Much like any abelian group has a free resolution we would like to have a minmial model resolving any cdga. We will only need one condition: We say a cdga B is *homologically connected* if it is non-negatively graded and $H^0 B \cong k$.

**Theorem 2.21.** *Every homologically connected cdga B has a minimal model.*

*Proof.* We will "just do it" and build the minimal model step by step. We will construct maps cdga's $M(n, p)$ and homomorphisms $f_{n,p} : M(n, p) \to B$ such that every $f_{n,p}$ induces an isomorphism on cohomology in degree up to $n-1$ and an injection in degree $n$. The $M(n, p)$ will be as in our definition of a minimal model, in particular $M(n, p) \subset M(n, p+1)$ with $dM(n, p+1) \subset M(n, p)$ and $M(n) = \cup M(n, p)$ is the sub-cdga of $M = \cup M(n)$ generated by all elements in degree at most $n$.

Let $f_{1,0} : M(1, 0) \to B$ be the unit map $\mathbb{Q} \to B$.

Now fix $n, p$ and assume by induction we have defined $f_{n,p}$. Pick cocycles $b_1, b_2, \cdots \in B^n$ representing a basis for the cokernel of $H^n(f_{n,p})$. Similarly, pick representatives $x_1, x_2, \cdots \in M(n, p)^{n+1}$ for a basis of the kernel of $H^{n+1}(f_{n,p})$. Finally we pick $c_1, c_2, \cdots \in B^n$ such that $f_{n,p} x_i = dc_i$.

Now define $M(n, p+1)$ by adding to $M(n, p)$ generators $m_1, m_2, \ldots$ in degree $n$ with $dm_i = 0$ and generators $n_1, n_2, \ldots$ in degree $n$ with $dn_i = x_i$. Define $f_{n,p+1}$ to be $f_{n,p}$ on $M(n, p)$ and $f_{n,p+1} : m_i \mapsto b_i$ and $f_{n,p+1} : n_i \mapsto c_i$.

We need to check that $f_{n,p+1}$ still induces isomorphisms on $H^{\leq n-1}$ and an injection on $H^n$. But we changed nothing in degrees smaller than $n$ and the cohomology elements we added in degree $n$ map to a basis for the cokernel, so cannot add to the kernel in degree $n$. Note that if $H^n(f_{n,p})$ is surjective then we did not add any $m_i$ and if $H^{n+1}(f_{n,p})$ happens to be injective then we did not add any $n_i$.

Now assume we have constructed $f_{n,p}$ for all $p \geq 0$ and let $M(n+1, 0) = \bigcup_p M(n, p)$ and define $f_{n+1,0}$ by $f_{n+1,0}|_{M(n,p)} = f_{n,p}$.

We claim that by construction $H^n(f_{n+1,0})$ is an isomorphism and $H^{n+1}(f_{n+1,0})$ is an injection.

Indeed, suppose $x$ represents an element in the kernel of $H^{n+1}(f_{n+1,0})$. Then $x \in M(n, p)$ for some $p$ and $H^{n+1}(f_{n,p})(x) = 0$. Then by construction of $M(n, p+1)$ we have $x = dn$ for some $n \in M(n, p+1)$, so $[x] = [dn] = 0 \in H^{n+1} M(n+1, 0)$.

For the isomorphism in degree $n$ observe first that injectivity holds for all $H^n(f_{n,p})$ by induction and thus for $H^n(f_{n+1,0})$. Surjectivity holds for $H^n(f_{n,1})$ by construction, and adding more generators to $M(n, p)$ is never going to interfere with surjectivity.

Finally let $M = \bigcup_n M(n, 0)$ and extend the $f_{n,0}$ in the obvious way to a map $f : M \to B$. It is immediate that $H^n(f)$ is an isomorphism for all $n$. □

We will see later, that minimal models are moreover essentially unique.

## 2.6. Homotopy theory in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$

We have constructed some nice models in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$, but our aim was to do homotopy theory. You are familiar with homotopies between maps of topological spaces. Homotopy depends on a notion of a contractible interval.

Our "interval cdga" will be given by the polynomial de Rham algebra on the interval, i.e. $\Omega(1) = \mathbb{Q}\langle t, dt\rangle$ is the free algebra on generators $t, dt$ with differential $d : t \mapsto dt$, see Example 2.4. The inclusion of the endpoints induces maps $\partial_0 : t \mapsto 0, \partial_1 : t \mapsto 1$ from $\Omega(1)$ to $\mathbb{Q}$.

**Definition 2.22.** A *homotopy* between two maps $f, g : A \to B$ in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ is a map $H : A \to \Omega(1) \otimes B$ with $(\partial_0 \otimes \mathbf{1}) \circ H = f$ and $(\partial_1 \otimes \mathbf{1}) \circ H = g$.

**Remark 2.23.** Note that this definition is dual to the definition of a homotopy in $\mathsf{Top}$, where a homotopy was a map $X \times I \to Y$. This was to be expected since we want to connect the two categories by a contravariant functor, like the de Rham functor.

Similarly, there is a homotopy theory for augmented cdgas. Define the homotopy as above, replacing $\otimes$ by $\tilde{\otimes}$ defined as follows:

$$\Omega(1) \,\tilde{\otimes}\, B := \mathbb{Q} \oplus (\Omega(1) \otimes \bar{B})$$

where $\bar{B}$ is the kernel of the augmentation ideal.

This phenomenon actually lives in a bigger context. The crucial ingredient in our work is to observe that $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ is a *model category*. So instead of proving specific results about homotopy theory in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ we will derive them from homotopy theory in model categories, introduced in the next chapter. Now we will just prove the following lemma:

**Lemma 2.24.** *Let $f, g : A \to B$ be homotopic in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$. Then $Hf = Hg$. The same result holds in $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0}$.*

*Proof.* Consider a homotopy $J : A \to \Omega(1) \otimes B$ between $f$ and $g$. It is enough to show that $\partial_0 \otimes \mathbf{1}$ and $\partial_1 \otimes \mathbf{1}$ are equal on cohomology, so we consider the maps

$$HA \to H(\Omega(1) \otimes B) \to HB \oplus HB$$

It's easy to see that $H\Omega(1) \cong k$, and together with an application of Lemma 2.25 this gives an isomorphism $H(\Omega(1) \otimes B) \to H\Omega(1) \otimes HB \to HB$ that is inverse to the canonical map $HB \to H(\Omega(1) \otimes B)$.

Thus the factorization $HB \to H(\Omega(1) \otimes B) \to HB \oplus HB$ of the diagonal map implies that we can factor $H(\Omega(1) \otimes B) \to HB \oplus HB$ through the diagonal map, so $H\partial_0$ and $H\partial_1$ must agree. We conclude that $Hf = H\partial_0 \circ HJ$ and $Hg = H\partial_1 \circ HJ$ must agree.

The same proof applies in $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0}$. □

Here we used the following. Recall that the tensor product of cdga's and of cochain complexes is defined in the same way, see Definition 2.15.

**Lemma 2.25** (Algebraic Künneth Theorem)**.** *Let $A, B$ be cochain complexes over a field $k$. Then we have $H^n(A \otimes B) \cong \oplus_{p+q=n} H^p(A) \otimes H^q(B)$, written concisely as $H(A \otimes B) \cong HA \otimes HB$.*

*Proof.* This is Theorem 3.6.3 in [Wei95]. As we are working over a field all Tor terms vanish. □

This implies that homotopy equivalences are quasi-isomorphisms: If $f \circ g$ and $g \circ f$ are homotopic to the identity, then $Hf$ and $Hg$ are inverse to each other.

It follows that minimal models are homotopy invariant. Given a homotopy equivalence $f : A \to B$ any minimal model $e_A : M \to A$ is a minimal model for $B$ by $f \circ e_A : M \to B$.

# 3. Model categories

## 3.1. Definition

We want to describe more formally what we mean by "a homotopy theory".

The notion of a model category was introduced by Dan Quillen in the 60's to formalize this notion, and it has been developed since then.

Model categories like $\mathsf{Top}$, $\mathsf{dgMod}_k$ or $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ are the natural habitat for homotopies, as we shall see in the following. We will only have time to establish the language here. For a very readable introduction see [DS95], for a more complete account see [Hov07].

**Remark 3.1.** Nowadays people will argue that the natural habitat for homotopy theory are not model categories but $(\infty, 1)$-categories, for example as formalized by Jacob Lurie. Indeed, $(\infty, 1)$-category have a very rich theory and satisfy many desirable properties. For example, $(\infty, 1)$-categories naturally form an $(\infty, 1)$-category, while model categories do not form a model category.

But, as we will see, model categories have a lot of computational power, and they are being used extensively to this day. An imperfect analogy is with linear algebra: A sophisticated person will surely prefer to argue about linear maps, but every now and then you do have to pick a basis and compute with matrices!

We will need the following terminology.

**Definition 3.2.** Given a commutative diagram of solid arrows

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed, "q"] & P \end{tikzcd}$$

we say that $i$ has the *left lifting property* (LLP) with respect to $p$ if the dotted arrow $q$ exists. In the same situation we say $p$ has the *right lifting property* (RLP) with respect to $i$.

For example, in a category of modules, the map from 0 to a projective module has the LLP with respect to all surjections.

**Definition 3.3.** A map $f : A \to B$ is a *retract* of a map $g : A' \to B'$ if there exist factorisations of the identity $A \to A' \to A$ and $B \to B' \to B$ making the obvious diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathbf{1}_A"] & A' \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathbf{1}_B"'] & B' \arrow[r] & B \end{tikzcd}$$

**Definition 3.4.** A *model category* is a category $\mathscr{M}$ with special classes $\mathscr{W}$ (*weak equivalences*), $\mathscr{F}$ (*fibrations*) and $\mathscr{C}$ (*cofibrations*) of morphisms such that the axioms MC1 to MC5 hold. We call $\mathscr{F} \cap W$ the *acyclic fibrations* and $\mathscr{C} \cap W$ the *acyclic cofibrations*.

MC 1 Small limits and colimits exist in $\mathscr{M}$.

MC 2 If $f$ and $g$ are maps such that $gf$ is defined and if two out of $f, g, gf$ are in $\mathscr{W}$ then so is the third. (This is called the "two-out-of-three" property.)

MC 3 If $f$ is a retract of $g$ and $g$ is in $\mathscr{F}$, $\mathscr{C}$ or $\mathscr{W}$ then so is $f$.

MC 4
- (i) Any cofibration has the LLP with respect to all acyclic fibrations.
- (ii) Any acyclic cofibration has the LLP with respect to all fibrations.

MC 5 Any map $f$ can be functorially factored in two ways:
- (i) $f = pi$, where $p$ is a acyclic fibration and $i$ is a cofibration.
- (ii) $f = qj$ where $q$ is a fibration and $j$ is a acyclic cofibration.

Note that MC 4(i) is equivalent to saying any acyclic fibration has RLP with respect to all cofibrations, similarly for MC 4(ii).

Here functorial factorisation means that you can factor without making choices, and without breaking any commutative diagrams. Formally, there is a functor from the category $\mathscr{M}^{\to}$ of morphisms in $\mathscr{M}$, with morphisms given by commutative diagrams, to the category $\mathscr{M}^{\to} \times \mathscr{M}^{\to}$ satisfying the conditions in MC 5(i) or MC 5(ii).

Since model categories have finite limits and colimits, they have an initial object 0 and a terminal object 1. We call an object $X$ *fibrant* if $X \to 1$ is a fibration and *cofibrant* if $0 \to X$ is a cofibration.

## 3.2. Examples

The fundamental example for homotopy theorists is the following:

**Example 3.5.** The category $\mathsf{Top}$ of topological spaces is a model category if we define the following:

- $\mathscr{W}$ is the class of all weak homotopy equivalences,
- $\mathscr{F}$ consists of all *Serre fibrations*, which are maps which have the RLP with respect to all inclusions $D \times \{0\} \to D \times I$ for $D$ a CW complex.
- $\mathscr{C}$ consists of maps with the LLP with respect to Serre fibrations which are weak equivalences (these are the retracts of relative CW complexes)

You have probably met the right lifting property with respect to inclusion like $\{0\} \to I$ and $I \times \{0\} \to I \times I$ when considering covering spaces in a first topology course. These are fibrations with discrete fibre.

Here is an example, which you will have met implicitly if you have done homological algebra:

**Example 3.6.** Let $R$ be a commutative ring. The category $\mathsf{dgMod}_R^{\leq 0}$ of non-positively graded cochain complexes of $R$-modules is a model category if we define the following:

- $\mathscr{W}$ is the class of all quasi-isomorphisms,
- $\mathscr{F}$ consists of all chain maps $f$ such that $f_n$ is surjective whenever $n < 0$
- $\mathscr{C}$ consists of all chain maps $f$ such that every $f_n$ is injective with projective cokernel.

Note that fibrations need not be surjective in degree 0.

**Remark 3.7.** There is a similar model structure on unbounded chain complexes. The fibrations are all levelwise surjective chain maps, but the description of cofibrations is more subtle.

There is also a dual model structure on $\mathsf{dgMod}_R^{\geq 0}$ whose cofibrations are the injections in positive degrees and whose fibrations are surjections with injective kernel. Again, this generalises to $\mathsf{dgMod}_R$.

Projective and injective $R$-modules are dual as follows from the definition using lifting properties. To disambiguate we call these model structures the *projective* and *injective model structure* on $\mathsf{dgMod}_R$.

Note that if $R$ is a field then every module is both projective and injective and the definition simplifies considerably!

A big motivation for us is that $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ is also a model category:

**Example 3.8.** The category $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ of commutative dg algebras over $\mathbb{Q}$ is a model category if we define the following:

- $\mathscr{W}$ is the class of all quasi-isomorphisms,
- $\mathscr{F}$ consists of all chain algebra maps $f$ such that $f_n$ is surjective,
- $\mathscr{C}$ consists of all chain algebra maps $f$ having the LLP with respect to acyclic fibrations.

We will examine this in more detail later on.
The same definition works if we drop the assumption of commutativity:

**Example 3.9.** The category $\mathsf{dgA}_{\mathbb{Q}}^{\geq 0}$ of dg algebras over $\mathbb{Q}$ is a model category if we define the following:

- $\mathscr{W}$ is the class of all quasi-isomorphisms,
- $\mathscr{F}$ consists of all chain algebra maps $f$ such that $f_n$ is surjective,
- $\mathscr{C}$ consists of all chain algebra maps $f$ having the LLP with respect to acyclic fibrations.

We will also meet the model category of simplicial sets.

**Warning 3.10.** In all of these examples it is hard work to prove, that these classes of maps do indeed satisfy the axioms for a model category!

## 3.3. Some useful properties

In Example 3.9 we made use of a redundancy in our definition:

**Lemma 3.11.** *Let $\mathscr{M}$ be a model category. Then the cofibrations are precisely the maps which have the LLP with respect to acyclic fibrations.*

The three analogous versions are also true.

*Proof.* Let $f : K \to L$ have LLP with respect to all acyclic fibrations. By MC 5(i) we can write $f = pi : K \to L' \to L$ where $i$ is a cofibration and $p$ an acyclic fibration. By assumption we can lift $\mathbf{1} : L \to L$ to a map $h : L \to L'$ with $ph = \mathbf{1}$. But then $f$ is a retract of $i$ and hence a cofibration by MC 3. □

Using Lemma 3.11 one can easily show:

**Lemma 3.12.** *Cofibrations and acyclic cofibrations are stable under pushout and coproducts. Fibrations and acyclic fibrations are stable under pullbacks and products.*

Here *stable under pushout* means that in the pushout diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"'] & B \arrow[d, "j"] \\ C \arrow[r] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

If $i$ is a cofibration then so is $j$.

**Remark 3.13.** This is an example how fibrations and cofibrations are dual. Whenever there is a statement about cofibrations, there will be a dual statement about fibrations.

Inspired by this observaton we may look back at the definition of a model category and observe immediately that if $\mathscr{M}$ is a model category with weak equivalences $\mathscr{W}$, cofibration $\mathscr{C}$ and fibrations $\mathscr{F}$ then $\mathscr{M}^{\mathrm{op}}$ is also a model category, with weak equivalences $\mathscr{W}^{\mathrm{op}}$, cofibrations $\mathscr{F}^{\mathrm{op}}$ and fibrations $\mathscr{C}^{\mathrm{op}}$, i.e. we swap the role of fibrations and cofibrations.

We will also need the following, which can be proved similarly to Lemma 3.12:

**Lemma 3.14.** *Cofibrations are stable under transfinite composition. This means given a sequence $X_0 \to X_1 \to X_2 \to \cdots$ of cofibrations then the induced map $X_0 \to \operatorname{colim} X_i$ is a cofibration.*

## 3.4. Homotopies in model categories

Now let us define homotopies in a model category.

Unfortunately I only have time to sketch the theory and will not prove the results. None of the proofs are particularly hard. See for example the book [Hov07] for all the details.

Recall that $f, g : A \to B$ between topological spaces are homotopic if there is a certain map $H : A \times [0, 1] \to B$. To generalise this we need a generalisation of the construction of $A \times [0, 1]$ from $A$.

**Definition 3.15.** A *cylinder object* for an object $A$ in a model category $\mathscr{M}$ is an object $A \wedge I$ such that the natural map $A \amalg A \to A$ factors as $A \amalg A \xrightarrow{i} A \wedge I \xrightarrow{q} A$ such that $q$ is a weak equivalence and $i$ is a cofibration.

Given maps $f, g : A \to B$ we write $f + g$ for the canonical map $A \amalg A \to B$.

In any model category the factorization axiom MC 5 ensures the existence of a cylinder object for every object, and we even have that $q : A \wedge I \to A$ is a trivial fibration.

**Definition 3.16.** A *left homotopy* between two maps $f, g : A \to B$ in $\mathscr{M}$ via a cylinder object $A \wedge I$ is map $H : A \wedge I \to B$ such that $H \circ i = f + g$. Two maps are *left homotopic*, written $f \overset{l}{\sim} g$ if there is a left homotopy between them for some cylinder object.

Dually we can define path objects, typically written $X \to X^I \xrightarrow{p} X \times X$, which factor the diagonal map into a weak equivalence followed by a fibration $p$. Then a right homotopy between maps $f, g : A \to B$ via $B^I$ is a map $H : A \to B^I$ such that $(f, g) = p \circ H$.

We say $f$ and $g$ are homotopic and write $f \sim g$ if they are both left and right homotopic.

**Example 3.17.** In $\mathsf{Top}$ a cylinder object for $X$ is given by $X \times I$. The usual homotopies in $\mathsf{Top}$ are then left homotopies for this cylinder object. We have to be a bit careful: The existence of a left homotopy in the model category sense does not guarantee a homotopy through any specific cylinder object, so in particular not through the cylinder object we picked earlier. We will see later that in good cases the notion of homotopy will not depend on a choice of cylinder object.

There are also path objects in $\mathsf{Top}$, given by the mapping space $\mathrm{Map}(I, X)$ with the compact-open topology. Then the adjunction $\mathrm{Hom}(X \times I, Y) \cong \mathrm{Hom}(X, \mathrm{Map}(I, Y))$ shows that left homotopies via $X \times I$ and right homotopies via $Y^I$ are equivalent for any pair of topological spaces. (Since $I$ is locally compact.) Of course we want this to hold in general model categories.

We will see later that the homotopy we defined on $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ is a right homotopy.

## 3.5. Homotopy classes of maps

We define $\pi^l(A, X)$ as the equivalence classes of maps from $A$ to $X$ under the equivalence relation generated by left homotopy. As left homotopy is not necessarily transitive, we have to be a bit careful. However, between "nice" objects there is a good homotopy theory. and frequently both notions of homotopy coincide.

**Lemma 3.18.** *Let $\mathscr{M}$ be a model category, $A, B, X, Y \in \mathrm{Ob}\,\mathscr{M}$. Let $f, g : B \to X$.*

1. *If $f, g$ are left homotopic then $hf \overset{l}{\sim} hg$ for any $h : X \to Y$. If $X$ is fibrant and $f, g$ are left homotopic then $fk$ and $gk$ are left homotopic for any $k : A \to B$.*

2. *If $C$ is fibrant then there is a composition map $\pi^l(A, B) \times \pi^l(B, C) \to \pi^l(A, C)$.*

3. *If $B$ is cofibrant then left homotopy is an equivalence relation on $\mathrm{Hom}(B, X)$.*

4. *If $B$ is cofibrant and $h : X \to Y$ is an acyclic fibration or a weak equivalence between fibrant objects then $h_* : f \mapsto h \circ f$ induces a bijection $\pi^l(B, X) \to \pi^l(B, Y)$.*

*Proof.* This is Proposition 1.2.5 in [Hov07]. (Note that part 2 follows from part 1.) I will give an example of the style of argument used below. □

The dual results to Lemma 3.18 hold for right homotopies.

**Lemma 3.19.** *Let $A$ be cofibrant and $f \overset{l}{\sim} g$ for $f, g : A \to X$. Fix any path object $X^I$ for $X$. Then $f \overset{r}{\sim} g$ via $X^I$.*

*Proof.* We fix a homotopy $H : A \wedge I \to X$ from $f$ to $g$. Then we consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "r \circ f"] \arrow[d, hook, "i_0"] & X^I \arrow[d, two heads, "p"] \\ A \wedge I \arrow[r, "(f \circ q, H)"] & X \times X \end{tikzcd}$$

As $A$ is cofibrant the map $i_0 : A \to A \amalg A \to A \wedge I$ is a cofibration. (We use that $A = A \amalg 0$, and $0 \to A$ is a cofibration, so the first map is a coproduct of cofibrations!)

Moreover, $A \to A \amalg A \to A \wedge I$ is a weak equivalence as the composition with $q : A \wedge I \to A$ is the identity, so we can apply 2 out of 3.

This gives us a left $K : A \wedge I \to X^I$ by the model category axioms. With $p \circ K = (f \circ q, H)$ we find that $K \circ i_1$ is a homotopy from $f$ to $g$ via $X^I$. □

**Corollary 3.20.** *Let A be cofibrant and X fibrant. Assume that f and g are left homotopy via some cylinder object. Then f and g are right homotopic via any path object and left homotopic via any cylinder object.*

*Proof.* We apply Lemma 3.19 and its dual. □

Thus if $A$ is cofibrant and $X$ is fibrant the notions of left and right homotopy coincide and the choice of path and cylinder object is inconsequential.

We write $[A, X]$ instead of $\pi^l(A, X) = \pi^r(A, X)$.

**Theorem 3.21.** *Let $A, B$ be fibrant and cofibrant in a model category $\mathscr{M}$. Then $f : A \to B$ is a weak equivalence if and only if it is a homotopy equivalence.*

*Proof.* This is Proposition 1.2.8 in [Hov07]. □

**Example 3.22.** In the model structure on topological spaces from Example 3.5 the CW complexes are cofibrant (and fibrant, since all topological spaces are). Thus a map between CW complexes is a weak equivalence if and only if it is a homotopy equivalence. This is Whitehead's Theorem (see Theorem 1.3)!

## 3.6. Homotopy categories

We have seen that homotopy becomes much simpler if we restrict to fibrant cofibrant objects.

By MC 5 we can functorially replace any object by a fibrant object if we factor the canonical map $X \to 1$. Call this functor $R$. Similarly we call the cofibrant replacement $Q$. Then by functoriality we have natural transformations $p_X : QX \to X$ and $i_X : X \to RX$. We also notice that $RQX$ and $QRX$ are both fibrant and cofibrant, e.g. $0 \to QX \to RQX$ is a composition of cofibrations.

**Definition 3.23.** The *homotopy category* $\mathrm{Ho}(\mathscr{M})$ of a model category $\mathscr{M}$ is defined to have the same objects as $\mathscr{M}$, and with $\mathrm{Hom}_{Ho(\mathscr{M})}(X, Y) = [QRX, QRY]$ the set of homotopy classe of maps from $QRX$ to $QRY$.

The following is immediate from the definition:

**Proposition 3.24.** $\mathrm{Ho}(\mathscr{M})$ *is equivalent to the category whose objects are fibrant cofibrant objects in $\mathscr{M}$ and whose morphisms are homotopy classes of morphisms in $\mathscr{M}$.*

**Proposition 3.25.** *There is a functor $\gamma : \mathscr{M} \to$* $\mathrm{Ho}(\mathscr{M})$ *which is the identity on objects.*

*Proof.* We need to define $\gamma$ on morphisms.

It is not hard to show that there are maps $f'$ and $f''$ making the following diagram commute up to homotopy.

$$\begin{tikzcd} RQX \arrow[r, "f''"] & RQY \\ QX \arrow[u] \arrow[r, "f'"] \arrow[d] & QY \arrow[u] \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We first define $f'$ by applying the LLP of $* \to QX$ with respect to the acyclic fibration $QY \to Y$, with lower horizontal map $QX \to X \to Y$.

This is well defined up to homotopy by applying Lemma 3.18 (4) to the map $[QX, QY] \to [QX, Y]$.

Dual arguments give $f''$ and we define $\gamma(f) := [f'']$.

Functoriality is straighforward. □

**Example 3.26.** From Example 3.5 we recall the classical model structure on topological space. Its homotopy category Ho($\mathsf{Top}$) is the motivating example for the whole theory of model category, and it is often just called *the homotopy category*.

By Proposition 3.24 it is equivalent to the category of retracts of CW complexes with homotopy classes of maps between them. As every topological space has a CW approximation by Theorem 1.5 we can drop the word "‘retract" here.

**Example 3.27.** Let $R$ be a ring. Then the homotopy category of $\mathsf{dgMod}_R^{\leq 0}$ with the model structure from Example 3.6 is commonly known as the *derived category* of $R$, denoted $D^{\leq 0}(R)$, or $D(R)$ for the unbounded version.

It is the category of nonpositively graded complexes of projective modules and homotopy classes of maps between them.

## 3.7. Localization

**Definition 3.28.** Given a category $\mathscr{M}$ and a class of morphisms $\mathscr{W}$ we define the *localization* of $\mathscr{M}$ at $\mathscr{W}$ to be a category $\mathscr{M}[\mathscr{W}^{-1}]$ with a functor $Q : \mathscr{M} \to \mathscr{M}[\mathscr{W}^{-1}]$ such that $Q(w)$ is an isomorphism for any $w \in \mathscr{W}$ and which is universal with this property: Any $\mathscr{M} \to \mathscr{C}$ that sends all $w \in \mathscr{W}$ to isomorphisms factors through $Q$.

**Theorem 3.29.** *The homotopy category $Ho(\mathscr{M})$ is a localization of $\mathscr{M}$ at the class of weak equivalences. We write $\gamma : \mathscr{M} \to Ho(\mathscr{M})$ for the natural functor.*

*Sketch of proof.* First we have to show $Ho$ takes weak equivalences to isomorphisms, this follows from Theorem 3.21.

Then we need to establish the universal property. Given some $G : \mathscr{M} \to \mathscr{C}$ sending $\mathscr{W}$ to isomorphisms we need to construct $\tilde{G} : Ho(\mathscr{M}) \to \mathscr{C}$. On objects we may just use $G$, on morphisms we use the fact that $G$ sends the cofibrant and fibrant replacement functors to isomorphisms and given $f : G \to A$ represented by $f'' : RQ(A) \to RQ(B)$ we define

$$\tilde{G}(f) = G(p_B) \circ G(i_{QB})^{-1} \circ G(f'') \circ G(i_{QA}) \circ G(p_A)^{-1}.$$

We have to work to make sure this is well-defined. For details see [DS95] Section 6.2. $\square$

It is important to note that a morphism $A \to B$ in the homotopy category $\mathrm{Ho}(\mathscr{M})$ is not necessarily induced from a morphism in $\mathscr{M}$, but in general from a *zig-zag* of morphisms which can be chose of the form

$$A \xleftarrow{\sim} QA \to RB \xleftarrow{\sim} B$$

Note that all the backwards-facing maps are weak equivalences.

**Example 3.30.** By Theorem 3.29 we can view the derived category $D^{\leq 0}(R)$ from Example 3.27 as a localization of the category of chain complexes at all weak equivalences.

We similarly view Example 3.26 as a localization of the category of topological spaces at weak equivalences.

## 3.8. Quillen adjunctions

After defining the homotopy categories it is natural to ask how we can lift a functor $F : \mathscr{C} \to \mathscr{D}$ to a functor $\mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$. In general this is not possible since $F$ need not preserve weak equivalences. In fact, preserving all weak equivalences is too strong a condition on a functor.

**Definition 3.31.** An adjunction $F \dashv G : \mathscr{M} \rightleftarrows G$ such that $F$ preserves cofibrations and $G$ preserves fibrations is called a *Quillen adjunction*, and $F$ a *left Quillen functor*.

**Remark 3.32.** It is easy to check using the lifting properties and the defining properties of an adjunction that the following are equivalent for an adjunction $F \dashv G$ between model categories:

1. $F$ preserves cofibrations and $G$ preserves fibrations,

2. $F$ preserves cofibration and acyclic cofibrations,

3. $G$ preserves fibration and acyclic fibrations.

**Example 3.33.** We recall the model categories $\mathsf{cdgA}^{\geq 0}_{\mathbb{Q}}$ and $\mathsf{dgMod}^{\geq 0}_{\mathbb{Q}}$ with the injective model structure from Remark 3.7. The forgetful functor $U : \mathsf{cdgA}^{\geq 0}_{\mathbb{Q}} \to \mathsf{dgMod}^{\geq 0}_{\mathbb{Q}}$ is right Quillen.

Firstly, it has a left adjoint, sending each cochain complex $V$ to the free cdga $S(V)$ from Definition 2.8, see Remark 2.10.

Then we need to check that $U$ preserves fibrations and trivial fibrations, but this follows straight from the definitions.

## 3.9. Derived Functors

We will again use the functorial cofibrant replacement functor $Q$ and the functorial fibrant replacement functor $R$.

**Proposition 3.34.** *Any left Quillen functor $F : \mathscr{M} \to \mathscr{N}$ induces a functor $LF : Ho(\mathscr{M}) \to Ho(\mathscr{N})$ given by $FQ$ on objects and morphisms. Dually any right Quillen functor $G : \mathscr{N} \to \mathscr{M}$ induces a functor $RG : Ho(\mathscr{N}) \to Ho(\mathscr{G})$ given by $GR$ on objects and morphisms.*

Note that the $R$ in $RG$ is a modifier, while $GR$ stands for the compositon of the functors $G$ and $R$.

*Proof.* If we can show that the composition $\gamma FQ : \mathscr{M} \to Ho(\mathscr{N})$ sends weak equivalences to isomorphisms, hence by Theorem 3.29 it induces the desired functor $RF : Ho(\mathscr{M}) \to Ho(\mathscr{N})$. But we know that $F$ sends acyclic cofibrations to acyclic cofibrations, and by Lemma 3.35 this suffices. □

We have used the following very useful fact:

**Lemma 3.35** (Ken Brown's Lemma.)**.** *Suppose a functor $F : \mathscr{M} \to \mathscr{C}$ sends acyclic cofibrations between cofibrant objects to isomorphisms. Then it sends all weak equivalences between cofibrant objects to isomorphisms.*

*Proof.* This is just playing with factorisations. Consider $f : A \to B$ a weak equivalence between cofibrant objects. Then consider the map $f + \mathbf{1}_B : A \amalg B \to B$ and factor it as a cofibration $j$ followed by an acyclic fibration $p$. As $A$ and $B$ are cofibrant the inclusions $i_A, i_B$ of $A$ and $B$ into $A \amalg B$ are cofibrations.

Now observe $f = p \circ j \circ i_A$, hence $j \circ i_A$ is a weak equivalence by MC2, and it's a cofibration. The same holds for $j \circ i_B$. Hence $F(j \circ i_B)$ is invertible and then $F(p) = F(\mathbf{1}_B) \circ F(j \circ i_B)^{-1}$ is invertible. Finally $F(f) = F(p) \circ F(j \circ i_A)$ is invertible, too. □

**Definition 3.36.** Let $F \dashv G$ be a Quillen adjunction. Then $LF : Ho(\mathscr{M}) \to Ho(\mathscr{D})$ induced by $FQ$ is the *left derived functor* of $F$ and $RG : Ho(\mathscr{N}) \to Ho(\mathscr{M})$ induced by $GR$ is the *right derived functor* of $G$.

**Remark 3.37.** The derived functor has a universal property. We recall first the natural transformation $p : Q \Rightarrow \mathbf{1}_{\mathscr{M}}$. It gives a natural transformation $Fp : FQ \to F$ where $F : \mathscr{M} \to \mathscr{C}$ is some functor. Then the pair $(LF, Fp)$ is terminal among pairs $G : Ho(\mathscr{M}) \to \mathscr{C}$ and $s : G\gamma \Rightarrow F$, also called "universal from the left".

Explicitly, whenever $G : Ho(\mathscr{M}) \to \mathscr{C}$ is a functor with a natural transformation $s : G\gamma \Rightarrow F$ then there is a natural transformation $\theta : G \Rightarrow LF$ such that $s = Fp \circ \theta_\gamma$.

## 3.10. Derived Functors II

**Example 3.38.** This was very abstract, so let us apply this to a classical example. Consider the model category $\mathsf{dgMod}_{\mathbb{Z}}^{\leq 0}$ from Example 3.6. We consider $\mathbb{Z}/p$ as a $(\mathbb{Z}, \mathbb{Z}/p)$-bimodule. (We have an action of $\mathbb{Z}$ "on the left" and an action with $\mathbb{Z}/p$ "on the right".)

We now consider the functor $M \mapsto M \otimes_{\mathbb{Z}} \mathbb{Z}/p$ that sends $M$ to the tensor product of $M$ with $\mathbb{Z}/p$.

One can check that $- \otimes_{\mathbb{Z}} \mathbb{Z}/p : \mathsf{dgMod}_{\mathbb{Z}}^{\leq 0} \to \mathsf{dgMod}_{\mathbb{Z}/p}^{\leq 0}$ is left Quillen.

The right adjoint of the tensor product is given by the Hom functor, specifically it is given by $\mathrm{Hom}_{\mathbb{Z}/p}(\mathbb{Z}/p, -)$. Note that this functor just takes a complex of $\mathbb{Z}/p$-modules and considers it as a complex of abelian groups! This clearly preserves fibrations (i.e. levelwise surjections in negative degrees) and quasi-isomorphisms, so we have a right Quillen functor and the tensor functor is left Quillen.

Thus by Proposition 3.34 there is a total derived functor $- \otimes^L \mathbb{Z}/p : D^{\leq 0}(\mathbb{Z}) \to D(Z/p)$ obtained by cofibrant replacement, so $M \mapsto \gamma(QM \otimes_{\mathbb{Z}} \mathbb{Z}/p)$.

To compute this we replace $M$ by a complex of projective (i.e. free) $\mathbb{Z}$-modules. Then the cohomology groups of $M \otimes^L \mathbb{Z}/p$ are exactly the familiar Tor-groups, $H^i(QM \otimes \mathbb{Z}/p) \cong \mathrm{Tor}^i_{\mathbb{Z}}(M, \mathbb{Z}/p)$, in particular $H^{-1}(QM \otimes^L Z/p) \cong \mathrm{Tor}_1(M, \mathbb{Z}/p)$. (Note that Tor is usually written homologically with the convention $\mathrm{Tor}^R_i = \mathrm{Tor}^{-i}_R$.)

So our notion of derived functor matches up with the familiar notion of Tor, and similar results apply for Ext.

**Remark 3.39.** Looking at this example you may argue that we worked a lot harder than in a first course on homological algebra. You would be correct. But quite often the trick in mathematics is to make an example more complicated in order to apply the underlying principles more generally.

**Remark 3.40.** It is tempting to generalise our argument to the functor $- \otimes_R N$ for a general rings $R$ and $S$ and a $(R, S)$-bimodule $N$. If you consider our argument carefully we made crucial use of the fact that $S$ was a field and thus $N$ as an $S$-module was projective.

However, this condition is very restrictive! If we had considered the functor $- \otimes \mathbb{Z}/p$ into $\mathbb{Z}$-modules rather than $\mathbb{Z}/p$-modules the argument would have broken down. The argument can be fixed with some more work.

One can define derived functors a bit more generally, using the universal property from Remark 3.37. Then a a functor $F$ can be left derived whenever it preserves weak equivalences between cofibrant objects, and both the derived functor of a left Quillen functor and the "classical" derived functor of $- \otimes^L_R N$ (for $N$ not necessarily cofibrant) are special cases.

Here is the main theorem about Quillen adjunctions:

**Theorem 3.41.** *Let $F : \mathscr{M} \rightleftarrows \mathscr{N} : G$ be a Quillen adjunction. Then there is an adjunction $LF : Ho(\mathscr{M}) \rightleftarrows Ho(\mathscr{N}) : RG$.*

*Proof.* To show there is an adjunction on the level of homotopy categories we know that $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})}(A, GRX) = [QA, GRX]$ and $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{N})}(FQA, X) = [FQA, RX]$.

Thus it suffices to show that homotopies in the two hom spaces are identified.

Let us assume $A$ is cofibrant and $X$ is fibrant and let $f, g : A \to GX$ be homotopic. Let $f^\#, g^\# : FA \to X$ be the adjoint morphisms. $GX$ is fibrant so by Lemma 3.19 we choose a left homotopy $A \wedge I \to GX$ via a cylinder object $A \wedge I$. Since $F$ preserves coproducts, cofibrations and weak equivalences between cofibrant objects, $F(A \wedge I)$ is a cylinder object for $FA$, and we have a homotopy $F(A \wedge I) \to X$ from $f^\#$ to $g^\#$ via the adjunction. □

**Definition 3.42.** If a Quillen adjunction induces an equivalence $Ho(\mathscr{M}) \cong Ho(\mathscr{N})$ we call it a *Quillen equivalence*.

## 3.11. Homotopy pushouts

There is a delicate theory about how to deal with limits and colimits in the theory of model categories. We will mainly want to understand some examples, and to keep things intuitive we consider examples in $\mathsf{Top}$ in this section.

One immediate drawback of taking the homotopy category of a model category is that it does not behave well with respect to limits and colimits. For example we have the following two pushouts in $\mathsf{Top}$.

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & * \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & S^2 \end{tikzcd}$$

The diagrams are weakly equivalent but the colimits are different!

This seems disappointing, but there is a solution: The usual definiton of limits and colimits does not take into account the homotopical strucutre of our model category. Much like a Quillen functor only descends to the homotopy category after deriving it, i.e. replacing objects (co)fibrantly, so we should replace our diagrams cofibrantly.

In fact, as limits and colimits are adjoint to the constant functor, cf. Lemma A.40 one may define *homotopy limits* and *homotopy colimits* as their derived functors.

To carry this out one has to put model structures on diagram categories. We will talk more about this in the next section. Here we will begin by considering the most important examples: Homotopy pull-backs and homotopy pushouts.

**Definition 3.43.** Given a pushout diagram in a model category, we define the *homotopy pushout* to be the colimit of a weakly equivalent diagram of cofibrations:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, hook] \arrow[d, hook] & \tilde{B} \arrow[d] \\ \tilde{C} \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & H \end{tikzcd}$$

where $q : A' \to A$ is a cofibrant replacement of $A$ and we can use the model category axioms to replace $f \circ q : A' \to B$ by $A' \hookrightarrow \tilde{B} \xrightarrow{\sim} B$.

Concretely, in topological spaces, we may replace $B$ by the mapping cylinder of $f$, given by

$$Mf = (A \times I) \cup_f B = A \times I \amalg B/_\sim$$

where the equivalence relation is $(a, 1) \sim f(a)$.

If $B$ is a cofibrant one can check that the natural inclusion $A \to Mf$ is a cofibration (if it isn't then we replace $B$ cofibrantly first). The projection $Mf \to B$ is a weak equivalence (in fact $B$ is a deformation retract since we may move $(a, s)$ continuously to $(a, 1) \in B$).

In the special case $B = *$ the mapping cylinder of $f : A \to B$ is nothing but the cone on $A$ given by $CA := A \times I/(a, 1) \sim (a', 1)$.

For example $S^1 \to *$ becomes $S^1 \to CS^1 \cong D^2$. This is also an example that in practice cofibrant replaceents can sometimes be spotted by hand.

**Example 3.44.** The homotopy pushout of $* \leftarrow S^1 \to *$ is the colimit of $D^2 \leftarrow S^1 \to D^2$, i.e. $S^2$. This is the same as the colimit of $* \leftarrow S^1 \to D^2$, replacing only one map by a cofibration.

In fact, in Top it always suffices to replace one of the two arrows by a cofibration. (This is because the model category Top is left proper.)

**Definition 3.45.** The *homotopy cofiber* $C(f)$, also known as the *mapping cone* of $f$, is the homotopy pushout of $* \leftarrow A \xrightarrow{f} B$.

The homotopy cofiber can be computed to be $Mf/_\sim$ where $(a, 0) \sim (a', 0)$ for all $a, a' \in A$.

## 3.12. Homotopy pullbacks

Dually we can consider what happens to limits.

**Definition 3.46.** The *homotopy pullback* of a diagram $B \to A \leftarrow C$ is defined as the pullback of the diagram $\tilde{B} \to A' \leftarrow \tilde{C}$ obtained by replacing $A$ by a fibrant object and the two maps by fibrations.

Again we use the model category axioms to replace $B \to A'$ by $B \xrightarrow{\sim} \tilde{B} \twoheadrightarrow A'$ and similarly $C \to A'$.

The mapping path space $Ef$ of $f : A \to B$ is given as the subspace of $A \times B^I$ of pairs $(a, p)$ with $p(0) = f(a)$ and fits into $A \xrightarrow{\sim} Ef \twoheadrightarrow B$. Here $B^I$ is equipped with the compact open topology.

A special case of this is $A = *$. Then $Ef$ is given by all the paths in $B$ with starting point $f(*)$.

In fact, in Top it suffices to replace one of the two arrows by fibration. (This is because the model category Top is also right proper.)

**Definition 3.47.** The *homotopy fiber* of a map $f : X \to Y$ over $y \in Y$ is the homotopy pullback of $\tilde{X} \to Y \leftarrow \{y\}$ where $\tilde{X} \to Y$ is a fibration and $X \simeq \tilde{X}$.

The homotopy pullback can be computed by replacing $\{y\} \to Y$ by a fibration or replacinng $f$ by the fibration $Ef \to Y$. Unravelling definitions we always obtain the homotopy fiber $H(f) \subset X \times Y^I$ as given by pairs $(x, p)$ where $p$ is path from $y$ to $f(x)$.

**Example 3.48.** Consider the diagram $* \to X \leftarrow *$ where the two maps have target $x \in X$. Then the homotopy pullback is $\Omega_x X$, the space of all loops in $X$ based at $x$.

## 3.13. * General homotopy (co)limits

In this section we give the definition of homotopy limits in general and state the most important results about them.

Let $I$ be a small category and consider the category of all $I$-shaped diagrams in $\mathscr{M}$, written $\mathscr{M}^I$.

Then if the model structure on $\mathscr{M}$ is nice enough there is a model structure on $\mathscr{M}^I$ such that the diagonal functor $\Delta : \mathscr{M} \to \mathscr{M}^I$ is right Quillen.

The model category defnes weak equivalences and fibrations *objectwise*, i.e. $w : F \to G$ in $\mathscr{M}^I$ is a weak equivalence (resp. fibration) if each $w_i : F_i \to G_i$ is.

**Definition 3.49.** In the above setting, let $C$ denote the left adjoint of $\Delta$. Then the *homotopy colimit* of a diagram $F$ in $\mathscr{M}^I$ is $LC(F)$ in $Ho(\mathscr{M})$, the value of the derived functor at $F$.

Dually the derived functor of the right adjoint of $\Delta$ is the homotopy limit. However, we need a different model structure to make sure $\Delta$ is left Quillen! This time we define weak equivalences and *cofibrations* objectwise.

**Remark 3.50.** These model structures on diagram categories are called *projective model structure* (objectwise fibrations) and *injective model structure* (objectwise cofibrations).

Let us restrict attention for the remainder of this section to model categories which are *nice*, by which we mean that the projective and injective model structure exist.

If the projective model structure on $\mathscr{M}^{*\leftarrow*\rightarrow*}$ exists then the homotopy pushout from Definition 3.43 is an example of a homotopy colimits! Similarly the homotopy pullback is a homtopy limit if the relevant injective model structure exists.

Granting that our model categories are nice (showing any model category is nice will be a hard theorem!) we easily obtain the following two very useful results:

Firstly, homotopy (co)limits are well-defined up to weak equivalence:

**Corollary 3.51.** *If $F, G : I \to \mathscr{M}$ are diagrams in a model category and there is a natural transformation $\alpha : F \Rightarrow G$ such that each $\alpha_i$ is a weak equivalence then $\alpha$ induces a weak equivalence* $\operatorname{holim} F \xrightarrow{\sim} \operatorname{holim} G$.

*Proof.* By construction a derived functor sends weak equivalences to isomorphisms in the homotopy category, so $\alpha$ induces a homotopy equivalence. (I.e. the map $\lim \circ R(\alpha)$ is a homotopy equivalence.) But any map in a model category inducing an isomorphism in the homotopy category must be a weak equivalence by Theorem 3.21. □

And homotopy limits play well with Quillen functors:

**Corollary 3.52.** *Let $F : \mathscr{M} \to \mathscr{N}$ be left Quillen. Then $F$ preserves homotopy colimits, i.e. for any diagram $D : I \to \mathscr{M}$ there is a zig-zag of weak equivalences* $\operatorname{hocolim}_I F \circ D \simeq LF(\operatorname{hocolim}_I D)$.

*Proof.* Let $QD$ be a functorial cofibrant replacement of the diagram $D$. The maps $QD_i \to \operatorname{colim}_I QD$ induce an isomorphism $\operatorname{colim}_I FQD_i \to F(\operatorname{colim}_I QD)$ as $F$ commutes with colimits. We can check that $F^I : \mathscr{M}^I \to \mathscr{N}^I$ preserves cofibrant diagrams (since its right adjoint preserves objectwise fibrations), so the left hand side is weakly equivalent to the homotopy colimit via the natural maps

$$\operatorname*{colim}_I QFD_i \leftarrow \operatorname*{colim}_I QFQD_i \to \operatorname*{colim}_I FQD_i.$$

Moreover, as $\Delta$ is right Quillen the colimit of a cofibrant diagram is cofibrant and the right hand side is equivalent to $LF(\operatorname{hocolim}_I D)$ via $F(\operatorname{colim}_I QD_i) \leftarrow FQ(\operatorname{colim}_I QD_i)$. □

**Remark 3.53.** Note that there is some subtlety with the direction of arrows here. There is a lazier statement that $\operatorname{hocolim}_I F \circ D$ and $LF(\operatorname{hocolim}_I D)$ are isomorphic in the homotopy category, and that is usually good enough.

**Remark 3.54.** For all of this to make sense we need a supply of model categories which are nice!

Firstly, the model categories that we will work with, namely $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$, $\mathsf{dgMod}_R$ and $\mathsf{sSet}$ (which we are about to meet) are nice.

You can stop reading here. But for the sake of completeness I will explain a bit.

The key conditions are that $\mathscr{M}$ is a *cofibrantly generated* (which we will define later) and *accessible* (which means they are not too "large" in some sense). A model category with both properties is called *combinatorial*.

For the projective model structure (objectwise fibrations) this is fairly standard and cofibrant generation suffices, see Theorem 11.6.1 in [Hir03]. For the injective model structure (objectwise cofibrations) this is much harder, some of the relevant results are quite recent. The result is found in [Lur11b, Proposition A.2.8.2].

The model categories $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$, $\mathsf{dgMod}_R$ and $\mathsf{sSet}$ are cofibrantly generated and accessible and thus nice. In fact all the model categories we will work with to prove our main theorems, are nice.

The standard model structure on $\mathsf{Top}$ is not combinatorial and I don't think it is nice. But it is Quillen equivalent to the combinatorial model category $\mathsf{sSet}$, so that is good enough to prove theorems.

This remark should still make you a little bit wary. I seem to have swept some very hard results under the rug and the definitions I've given you don't even apply to the one example that we have thought about!

The first piece of good news is that for some categories $I$ we have injective and/or projective model structures even if $\mathscr{M}$ is not nice, see e.g. Section 5.1 in [Hov07].

The second piece of good news is that there are other ways of defining homotopy limits that work very generally (in particular they work in topological spaces), but they are very tedious to set up. They agree (up to isomorphism in the homotopy category) both with our ad-hoc definition of homotopy pushouts and with the elegant definition in this section where it makes sense. The standard reference is [Hir03].

# 4. The model category of cdga's

## 4.1. Preliminaries

We now turn to the model structure on $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ in more detail.

In particular we will prove that we do indeed have a model category, following the original argument in [BG76].

This model structure will later allow us to compare homotopy theories of spaces and algebras In the short run it will for example hel us to prove the uniqueness of minimal models in a very natural way.

**Theorem 4.1.** *Call a map $f$ in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ a*

- weak equivalence *if $f$ is a quasi-isomorphism,*
- fibration *if $f^n$ is surjective for every $n \geq 0$,*
- cofibration *if it has the left lifting property with respect to trivial fibrations.*

*With these definitions $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ is a model category.*

One can show that whenever $\mathscr{C}$ is a model category and $B$ an object of $\mathscr{C}$ then so is the *overcategory* $\mathscr{C}_{/B}$ whose objects are morphisms $C \to B$ in $\mathscr{C}$. By substituting $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ for $\mathscr{C}$ and $\mathbb{Q}$ for $B$ we can express the category of augmented dg algebras as an overcategory and have the following:

**Corollary 4.2.** $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0}$ *is a model category.*

To prepare for the proof of Theorem 4.1 we need to get some understanding of the cofibrations. We will find a few examples and it will turn out these are the only examples we need to understand.

**Lemma 4.3.** *The following are cofibrations:*

*(i) the unit map $\mathbb{Q} \to \mathbb{Q}\langle x\rangle$ where $x$ has arbitrary non-negative degree;*

*(ii) the map $\theta_n : \mathbb{Q}\langle x\rangle \to \mathbb{Q}\langle y, dy\rangle$ defined by $x \mapsto dy$ is a whenever $n = |x| = |y| + 1 \geq 1$;*

*(iii) the map $\theta_0 : \mathbb{Q}\langle x\rangle \to \mathbb{Q}$ sending $x$ to 0 if $|x| = 0$*

*Moreover the composition $\mathbb{Q} \to \mathbb{Q}\langle y, dy\rangle$ is an acyclic cofibration for $|y| \geq 0$.*

*Proof.* (i) The fact that $\mathbb{Q}\langle x\rangle$ is cofibrant follows from the definition of a free algebra: In the diagram

$$\begin{tikzcd} \mathbb{Q} \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ \mathbb{Q}\langle x\rangle \arrow[r] & B \end{tikzcd}$$

we find that the morphism $g : \mathbb{Q}\langle x\rangle \to B$ is uniquely determined by a choice of $g(x) = b \in B$ with $db = 0$. As $f : A \to B$ is an acyclic fibration there is a preimage $[a]$ of $[b]$ in the cohomology of $A$. We have $f(a) = b + db'$. So we choose $a'$ with $f(a') = b'$. Then $a - da'$ is an element of $A$ with differential 0, so it determines the desired lift $\mathbb{Q}\langle x\rangle \to A$.

(ii) A similar argument shows $\theta_n : \mathbb{Q}\langle x\rangle \to \mathbb{Q}\langle y, dy\rangle$ is a cofibration for $n \geq 1$, and then $\mathbb{Q} \to \mathbb{Q}\langle y, dy\rangle$ is a composition of cofibrations.

(iii) We consider the following diagram:

$$\begin{tikzcd} \mathbb{Q}\langle x\rangle \arrow[r] \arrow[d, "\theta_0"] & A \arrow[d, two heads, "f"', "\sim"] \\ \mathbb{Q} \arrow[r] & B \end{tikzcd}$$

By commutativity of the diagram $x \mapsto a \in \ker(f)$. But as $f$ is a quasi-isomorphism this implies $[a] = 0 \in H^0(A)$. As $A^{-1} = 0$ we have $a = 0$ and the map $\mathbb{Q}\langle x\rangle \to A$ factors through $\theta_0$.

Finally since $\mathbb{Q} \to \mathbb{Q}\langle y, dy\rangle$ is a quasi-isomorphism it follows form (i) and (ii) that it is an acyclic cofibration. □

**Remark 4.4.** Note that the last part of the lemma only holds we are working over a field of characteristic zero. In a field of characteristic 2 the element $y^2 \in k\langle y, dy\rangle$ is a nonzero cycle!

## 4.2. Proof of model structure I

*Proof of Theorem 4.1.* The proof is quite non-trivial. There are general schemata for proving categories are model categories. In order not to get sidetracked we will give a quicker direct proof here, but there is some similarity in flavour to the more general argument (known as the *small object argument for cofibrantly generated model categories*).

We will work through Definition 3.4 item by item.

MC 1: It is not hard to show that $\mathsf{cdgA}^{\geq 0}_{\mathbb{Q}}$ has all limits and colimits. Coproducts are given by the tensor product [1] and products by the usual product (equal to the product of underlying chain complexes). In particular the initial object is $k$ (the empty tensor product) and the terminal object is the zero algebra. Coequalizers can be constructed as quotients and equalizers as subalgebras. By Lemma A.33 this shows the existence of all limits and colimits.

[1] Inifinte coproducts are given by infinite tensor products, where $\otimes_i A_i$ has a basis given by expressions $\otimes_i a_i$ with $_i \in A_i$ and all but finitely many $a_i$ equal to 1.

Note in particular that limits in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ can just be constructed as limits of the underyling chain complexes.

MC 2: Clearly being an isomorphism satisfies two-out-of-three, thus so does being a quasi-isomorphism.

MC 3: For weak equivalences and fibrations this is an easy check from the definitions. As regards cofibrations, one can easily verify that *any* class of maps given by the left (or right) lifting property is automatically stable under retracts.

MC 4 (i) is just our definition of cofibrations. MC 4 (ii) is best dealt with after proving the factorization axioms.

## 4.3. Proof of model structure II

*Proof of Theorem 4.1 continued.* For MC 5 we will have to work.

For ease of notation we write $D(i)$ for the dg algebra $\mathbb{Q}\langle y, dy\rangle$ with $y$ of degree $i \geq 0$ and also define $D(-1) := \mathbb{Q}$. We write $S(i)$ for the dg algebra $\mathbb{Q}\langle x\rangle$ with $x$ of degree $i$. The natural map $S(i) \to D(i-1)$ will be denoted $\theta_i$, as in Lemma 4.3. It is given by $x \mapsto dy$ if $i \geq 1$ and $x \mapsto 0$ if $i = 0$.

We recall from Lemmas 3.12 and 3.14 that cofibrations are closed under pushouts, coproducts and transfinite composition.

For MC 5(a) we choose a homomorphism $f : A \to B$ and need to find a factorization $f : A \to A_f \to B$ as an acyclic cofibration followed by a fibration.

Let

$$A_f = A \otimes \bigotimes_{b \in B} D(|b|).$$

where the tensor product is over all the *homogeneous* elements of $B$. Clearly the natural inclusion $A \to A_f$ given by $a \mapsto a \otimes 1$ is a cofibration as a coproduct of cofibrations. It is acyclic as all the $D(|b|) \cong \mathbb{Q}\langle y, dy\rangle$ are quasi-isomorphic to $\mathbb{Q}$.

We define $p : A_f \to B$ by sending $a$ to $f(a)$ and the generator $y$ of $D(|b|)$ to the element $b$ of $B$ indexing the tensor factor. This is clearly surjective.

We turn to MC 5(b). Consider again $f : A \to B$. We have to find a factorization $A \to B_f \to B$ with $A \to B_f$ a cofibration and $B_f \to B$ an acyclic fibration.

$B_f$ will be the colimit of the following sequence of maps.

$$\begin{tikzcd} A \arrow[r, hook, "\beta_1"] \arrow[d, "f"] & B_f(1) \arrow[r, hook, "\beta_2"] \arrow[dl, "\psi_1"] & B_f(2) \arrow[r, hook, "\beta_3"] \arrow[dll, "\psi_2"] & \cdots \\ B \end{tikzcd}$$

Where $\beta : A \to B_f := \operatorname{colim} B_f(n)$ is always going to be a cofibration and our goal is to construct $B_f(i)$ to get closer and closer to an acyclic fibration to $B$.

We let

$$B_f(1) = A \otimes \left( \bigotimes_{b \in B} D(|b|) \right) \otimes \left( \bigotimes_{z \in Z(B)} S(|z|) \right)$$

where $Z(B)$ denotes the cycles in $B$, i.e. all solutions to $db = 0$. We can define $\beta_1$ and $\psi_1$ in the obvious way, parallel to how we defined $A \to A_f \to B$.

By construction $\beta_1$ is a cofibration, $\psi_1$ is a fibration and $H(\psi_1)$ is a surjection.

We collect together all the obstructions to injectivitiy of $H(\psi_1)$, so let $R$ be the set of all pairs $(w, b)$ where $w \in B_f(1)$ with $dw = 0$ and $b \in B$ with $\psi_1(w) = db$.

Then form the following pushout :

$$\begin{tikzcd}
\bigotimes_{(w,b) \in R} S(|w|) \arrow[r] \arrow[d, hook, "\otimes\theta_{|w|}"] & B_f(1) \arrow[d, "\beta_2"] \\
\bigotimes_{(w,b) \in R} D(|b|) \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & B_f(2)
\end{tikzcd}$$

As a pushout of a cofibration $\beta_2$ is a cofibration. By the universal property there is a map $\psi_2 : B_f(2) \to B$.

Moreover the map $\psi_2$ is a homology isomorphism on the image of $\beta_2$ as we have just explicitly killed the kernel of the homology map.

Now we repeat the same procedure, we form the set $R'$ of pairs witnessing the failure of $H(\psi_2)$ to be an injection and form the pushout $B_f(3)$.

We let $B_f := \operatorname{colim} B_f(i)$ and the map $\beta : A \to B_f$ is a transfinite composition of the cofibrations $\beta_i$, thus it is a cofibration.

We claim that the canonical map $\psi : B_f \to B$ is an acyclic fibration. It is degree-wise surjective and surjective on homology since as $\psi_1$ is and we have $\psi_1 = \psi \circ \beta'$.

So consider $w \in B_f$ with $\psi(w) = db$ in $B$. The colimit of the injections $\beta_i$ can be viewed as a union, so $w$ lives in some $B_f(i)$ and $\psi(w) = \psi_i(w)$. But then by construction if $\psi_i(w) = db$ there is a copy of $D(|b|)$ corresponding to the pair $(w, b)$ contributing to $B_f(i + 1)$, in other words there is $v$ with $dv = w$ in $B_f(i + 1)$ and this shows $w = 0 \in H(B_f)$.

Thus $\psi$ is an acyclic fibration and we have constructed the desired factorization.

We conclude with MC 4(ii). We have to show that any acyclic cofibration has the LLP with respect to all fibrations.

We first observe, that the acyclic cofibration $\alpha : A \to A_f$ which we found in our factorization $A \to A_f \to B$ does satisfy the LLP. This follows since $\mathbb{Q} \to D(n)$ satisfies it for any $n$ as can be checked explicitly.

So let $f$ be an arbitrary acyclic cofibration and factor it as $\phi \circ \alpha$ as in MC 5(i). As $f$ and $\alpha$ are weak equivalences so is $\phi$, thus by MC 4(i) we have the lift in the following diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, hook, "f"'] & A_f \arrow[d, two heads, "\phi"] \\ B \arrow[r, "1"'] \arrow[ur, dashed] & B \end{tikzcd}$$

This exhibits $f$ as a retract of $\alpha$, and it follows that it must satisfy the same left lifting property as $\alpha$. $\square$

**Remark 4.5.** The proof should remind you of that of Theorem 2.21!

**Remark 4.6.** The cofibrations $\mathbb{Q}\langle x\rangle \to \mathbb{Q}\langle y, dy\rangle$ from Lemma 4.3 clearly play an important role in this proof. They are *generating cofibrations*, which have the following two crucial properties:

1. Acyclic fibrations are characterized by the RLP with respect to generating cofibrations.
2. All cofibrations can be bulit out of generating cofibrations, as retracts of transfinite compositions of pushouts of coproducts.

The maps $\mathbb{Q} \to \mathbb{Q}\langle y, dy\rangle$ are *generating acyclic cofibrations*.

With the model structure on $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ one can see that $\Omega(1) \otimes A$ from Definition 2.22 is a path object for $A$ in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ since $A \to \Omega(1) \otimes A$ is a weak equivalence by the Künneth theorem.

Moreover $(\partial_0, \partial_1) : \Omega(1) \otimes X \to X \times X$ is onto since $t_0 \otimes x + t_1 \otimes y \mapsto (x, y)$ and it follows from Lemma 4.3 that $A \to \Omega(1) \otimes A$ is a cofibration. So $\Omega(1) \otimes A$ is a path object. So the homotopy we defined earlier is a right homotopy, and all homotopies with cofibrant source may be expressed using this path object.

## 4.4. Revisiting minimal models

We have observed that every object of $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ is fibrant. However, not every object is cofibrant. We have seen some examples in Lemma 4.3. Our next goal is to show that minimal models are cofibrant.

Recall that for a cdga $M$ we define $M(-1)$ to be $k.1$ and $M(n)$ for $n \geq 0$ to be the subalgebra generated by $M^i$ for $0 \leq i \leq n$ and $dM^n$. $M(n, 0)$ is $M(n-1)$ and $M(n, p)$ is the subalgebra generated by $M(n, p-1)$ and all the elements $x \in M^n$ with $dx \in M(n, p-1)$.

Then a semi-free connected cdga $M$ is minimial if it satisfies $M(n) = \bigcup_p M(n, p)$ for all $n$.

**Lemma 4.7.** *Any minimal algebra $M$ is a cofibrant object in* $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$.

*Proof.* We will show that every inclusion $M(n, p-1) \hookrightarrow M(n, p)$ is a cofibration. The result then follows by the Lemma 3.14 as $M(n-1) \to M(n)$ is a transfinite composition of inclusions $M(n, p-1) \to M(n, p)$ and $\mathbb{Q} \to M$ is a transfinite composition of $M(n-1) \to M(n)$

We need to pick an appropriate set of generators for $M$.

Let $G(1, 0) = \emptyset$. Then for each $n, p \geq 1$ extend $G(n, p-1)$ to $G(n, p)$ by adding elements in $M^n$ such that $G(n, p)$ projects to a basis for $M(n, p)^n / M(n, 0)^n$. Let $G(n+1, 0) = \cup_p G(n, p)$. Then $\bigcup_p G(n, p)$ projects to a basis for the indecomposables $IM^n = M^n / M(n, 0)^n$. As $M$ is semi-free it is freely generated by $\bigcup_{n,p} G(n, p)$. The subalgebra $M(n, p)$ is freely generated by $G(n, p)$.

Now recall that $M(n, p)$ is generated over $M(n, p-1)$ by all the elements in $M^n$ whose differential lies in $M(n, p-1)$. Thus we have the following push-out diagram, where $\deg x = n+1$ and $\theta : x \mapsto dy$.

$$\begin{tikzcd} \otimes_I \mathbb{Q}\langle x \rangle \arrow[r] \arrow[d, hook, "\theta"] & M(n, p-1) \arrow[d] \\ \otimes_I \mathbb{Q}\langle y, dy \rangle \arrow[r] & M(n, p) \end{tikzcd} \tag{4.1}$$

where the tensor product is over $I = G(n, p) \setminus G(n, p-1)$. Since the inclusion on the left is a cofibration and push-outs of cofibrations are cofibrations by Lemma 3.12 the claim follows. □

## 4.5. Homotopy groups and uniqueness

For the next results we need to consider cohomology of the indecomposabels of an augmented cdga. These cohomology groups play an important role later and we use a suggestive name.

**Definition 4.8.** Let $A$ be an augmented cdga and recall from Definition 2.14 the indecomposables $IA = \bar{A}/\bar{A}.\bar{A}$ of $A$. Then we define the graded $\mathbb{Q}$-module $\pi(A) = H(IA)$ and call the $\pi^k(A)$ the *homotopy groups* of $A$.

**Lemma 4.9.** *Let $f, g : A \to B$ be homotopic maps in $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$ via $B \otimes \Omega(1)$. Then the induced maps $f_*, g_* : \pi(A) \to \pi(B)$ are equal.*

*Proof.* Applying $\pi$ to the homotopy between $f$ and $g$ we have

$$\pi(A) \to \pi(\Omega(1)\tilde{\otimes}B) \to \pi(B \oplus B)$$

But $I(\Omega(1)\tilde{\otimes}B) = I(\mathbb{Q} \oplus (\Omega(1) \otimes \bar{B}))$ is isomorphic to $\Omega(1) \otimes IB$ as $\omega \otimes b \in \Omega(1) \otimes \bar{B}$ is decomposable exactly if $b$ is decomposable.

Thus we may rewrite the homotopy as

$$HIA \to H(\Omega(1) \otimes IB) \to HIB \oplus HIB$$

and proceed as in Lemma 2.24: Using the Künneth theorem together with $H(\Omega(1)) \cong \mathbb{Q}$ we find $f_* = g_* : \pi(A) \to \pi(B)$. □

**Lemma 4.10.** *A weak equivalence between minimal models is an isomorphism.*

*Proof.* Using the unique augmentation of minimal models we work in $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$. A weak equivalence between minimal models is a homotopy equivalence by Theorem 3.21 and Lemma 4.7.

So let $M$ and $N$ be minimal models with homotopy inverse maps $f : M \to N$ and $g : N \to M$. This means $g \circ f \simeq \mathbf{1}_M$. By Lemma 4.9 this shows $(g \circ f)_*$ is the identity on $\pi(M) = H(IM)$. But $HIM = IM$ as the differential of a minimal model is decomposable. Thus $gf$ induces the identity on indecomposables.

But $M$ (considered without its differential) is a free graded commutative algebra, and by the definitions it is the free graded commutative algebra on its indecomposables. Thus $gf$ induces an isomorphism on all of $M$. The same argument applies to $fg$, and thus $M \cong N$. □

**Theorem 4.11.** *The minimal model of a cdga is unique up to isomorphism.*

*Proof.* Suppose we have a cdga $B$ and two minimal models $f : M \xrightarrow{\sim} B$ and $f' : M' \xrightarrow{\sim} B$. We freely use the statements from Lemma 3.18. Since every object in $\mathsf{cdgA}^{\geq 0}_{\mathbb{Q}}$ is fibrant and minimal models are cofibrant by Lemma 4.7, homotopies form an equivalence relation and composition with $f'$ induces an isomorphism from homotopy classes $[M, M']$ to $[M, B]$. So there is $g : M \to M'$ such that $f' \circ g \simeq f$.

As $f$ and $f'$ are weak equivalences so is $g$ by the two-out-of-three property. As a weak equivalence between minimal models $g$ is an isomorphism by Lemma 4.10. □

# 5. Simplicial sets

## 5.1. Definition and examples

Simplicial sets are a combinatorial model for the homotopy theory of topological spaces. They are a bit technical, but they are extremely useful both in classical algebraic topology and for higher or derived algebra and geometry.

The idea goes back to triangulating topological spaces and considering *simplicial complexes*, which are glued together out of *simplices*, like the point, line segment, triangle, tetrahedron etc.

They have some advantages over general topological spaces. For examples they are completely described by discrete combinatorial data, and it is not difficult to define simplicial homology and simplicial cohomology.

Then Eilenberg and Zilber realized that one could consider a more abstract and more flexible notion, that of *simplicial sets*.

**Definition 5.1.** Let us first define the *simplex category* $\mathbf{\Delta}$ which has objects given by the ordered sets $[n] := (0 < 1 < \cdots < n)$, and which has morphisms given by nondecreasing functions. Then we define a *simplicial object* in a category $\mathscr{C}$ as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$ from the opposite category of $\mathbf{\Delta}$ to $\mathscr{C}$. There is a natural category $\mathscr{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ of simplicial objects in $\mathscr{C}$, with morphisms given by natural transformations.

We can dually define a *cosimplicial object* in $\mathscr{C}$ as a functor from $\mathbf{\Delta}$ to $\mathscr{C}$.

The most important case is when $\mathscr{C}$ is just the category of sets. We denote the category of simplicial sets by $\mathsf{sSet}$.

We write a simplicial set as $A_*$ where $A_n$ is $A([n])$, which we call the set of *n-simplices* or *simplices of degree n*.

Next we need to understand the data coming from morphisms. We notice two families of morphisms in $\mathbf{\Delta}$, corresponding to leaving out respectively repeating the $i$-th term.

**Definition 5.2.** For each $n$ define the $i$-th *face map* $\epsilon_i : [n-1] \to [n]$ to be the injection only leaving out $i \in [n]$, and define the $i$-th *degeneracy map* $\eta_i : [n+1] \to [n]$ to be the surjection mapping both $i$ and $i+1$ to $i$. As the maps are non-decreasing this determines them uniquely.

A straightforward, if tedious, check shows the following identities:

$$\begin{aligned}
\epsilon_j \epsilon_i &= \epsilon_i \epsilon_{j-1} \quad \text{if } i < j \\
\eta_j \eta_i &= \eta_i \eta_{j+1} \quad \text{if } i \leq j \\
\eta_j \epsilon_i &= \begin{cases} \epsilon_i \eta_{j-1} & \text{if } i < j \\ \mathbf{1}_{[n-1]} & \text{if } i = j, j+1 \\ \epsilon_{i-1} \eta_j & \text{if } i > j+1 \end{cases}
\end{aligned}$$

**Proposition 5.3.** *Any map $\alpha : [n] \to [m]$ can be factored essentially uniquely as a composition of degeneracy maps followed by a composition of face maps, i.e. $\alpha = \epsilon_{i_1} \dots \epsilon_{i_s} \eta_{j_1} \dots \eta_{j_t}$ where $i_k$ are non-increasing and $j_s$ are non-decreasing.*

*Proof.* A proof is outlined for example in [Wei95, Lemma 8.1.2]. Mostly you follow your nose. □

Hence it suffices to understand $\partial_i := A(\epsilon_i) : A_n \to A_{n-1}$ and $\sigma_i := A(\eta_i) : A_n \to A_{n+1}$ to complete our understanding of $A$. These maps satisfy the following equations.

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \qquad \text{if } i < j \\ \sigma_i \sigma_j &= \sigma_{j+1} \sigma_i \quad \text{if } i \leq j \\ \partial_i \sigma_j &= \begin{cases} \sigma_{j-1} \partial_i & \text{if } i < j \\ \mathbf{1} & \text{if } i = j, j+1 \\ \sigma_j \partial_{i-1} & \text{if } i > j+1 \end{cases} \end{aligned}$$

So to give a simplicial set it is sufficient to specify for all $n \geq 0$ sets $K_n$ and maps $\partial_i$ and $\sigma_i$ satisfying the above relations.

Similarly to give a morphism of simplicial sets $K_* \to L_*$ (which we defined as a natural transformation of functors $\mathbf{\Delta}^{\text{op}} \to \mathsf{Set}$) it suffices to give $f_n : K_n \to L_n$ for all $n \geq 0$, compatible with all $\partial_i$ and $\sigma_i$.

**Example 5.4.** The singular simplices of a topological space $X$ form a simplicial set $\operatorname{Sing}(X)$. We will go back to this example in the next section.

**Example 5.5.** Given two smplicial sets $K$ and $L$ we can easily define $K \times L$ by $(K \times L)_n = K_n \times L_n$ and applying all face or degeneracy maps diagonally, i.e. $\partial_i(x, y) := (\partial_i(x), \partial_i(y))$. This is the categorical product in $\mathsf{sSet}$.

## 5.2. The standard simplices

**Example 5.6.** We define the *standard n-simplex* $\Delta[n]$ as the functor $\Delta^{\text{op}} \to \mathsf{Set}$ represented by $[n]$.

This means $\Delta[n] = \operatorname{Hom}_{\mathbf{\Delta}}(-, [n])$, so $\Delta[n]_i = \operatorname{Hom}_{\mathbf{\Delta}}([i], [n])$ and $\partial_i = \operatorname{Hom}_{\mathbf{\Delta}}(\epsilon_i, [n])$.

Another way of phrasing this is that $\Delta[n]$ is the image in $\mathsf{sSet}$ of $[n]$ under the contravariant Yoneda embedding.

**Example 5.7.** $\Delta[0]$ is the constant simplicial set with a single simplex in each degree. We also denote this by $*$ as it is the terminal object in $\mathsf{sSet}$.

**Example 5.8.** Let us consider $\Delta[2]$ in a little more detail. Any map $[n] \to [2]$ in $\mathbf{\Delta}$ sends the first $k$ terms to 0, the next $m$ terms to 1 and the remaining $n-k-m$ terms to 2. We may write this as a sequence of zeros, ones and twos. So

$$\begin{aligned}
\Delta[2]_0 &= \{0,1,2\} \\
\Delta[2]_1 &= \{00,01,11,02,12,22\} \\
\Delta[2]_2 &= \{000,001,011,111,002,012,112,022,122,222\} \\
\Delta[2]_3 &= \{0000,0001,0011,\dots\}
\end{aligned}$$

Most elements have repeats and hence occur in the image of some $\sigma_j$. These are called *degenerate simplices*. The only non-degenerate simplices are then 0, 1, 2 in degree zero, 01, 02, 12 in degree one and 012 in degree two. We think of $\Delta[2]$ as consisting of 3 vertices, 3 edges between the vertices and 1 triangular face (plus a lot of degenerate simplices).

Now take the edge 01. We have $\partial_0(01) = 0$ and $\partial_1(01) = 1$ by unravelling the definitions. (We compose the embedding $01 \subset 012$ defining the edge with the maps $\epsilon_0 : 0 \subset 01$ respectively $\epsilon_1 : 1 \subset 01$.) Thus the two faces of 01 are the endpoints of the line segment.

It follows from the Yoneda Lemma (see Remark A.19) that

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[m],\Delta[n]) \cong \mathrm{Hom}_{\mathbf{\Delta}}([m],[n]).$$

Thus the maps from $\Delta[m]$ to $\Delta[n]$ are induced by the maps on the vertices of the simplices.

**Remark 5.9.** I claimed we wouldn't need the Yoneda lemma, yet here we are. Don't worry too much about this point but here is the argument: The Yoneda Lemma says that natural transformation from a representable functor $h^C$, like the representable simplicial set $\Delta[m] = h^{[m]}$, to any functor $F$, in our example $\Delta[n]$, are found by evaluating the second functor on the object $C$ representing the first. In symbols $\mathrm{Nat}(h^C, F) \cong F(C)$.

In our case this gives $\mathrm{Hom}_{\mathsf{sSet}}(\Delta[m],\Delta[n]) \cong \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$.

The same argument using the Yoneda Lemma also shows the following:

**Lemma 5.10.** *For any simplicial set $K$ we have* $\mathrm{Hom}(\Delta[m],K) = K_m$.

Recall here that the set of $m$-simplices of $K$ is just $K_m = K([m]))$.

**Definition 5.11.** For any simplicial set $K$ we define its *$n$-skeleton* to be the subsimplicial set $sk_n(K)$ generated by the simplices of degree $\leq n$. So $sk_n(K)_i = K_i$ if $i \leq n$ and in degree greater than $n$ we only have degeneracies of $i$-simplices for $i \leq n$.

**Example 5.12.** We define the boundary $\partial\Delta[n]$ of the standard $n$-simplex by leaving out the non-degenerate $n$-simplex $s$ corresponding to $\mathbf{1}_{[n]}$ (and its degeneracies).

Explicitly, $\partial\Delta[n]_i$ consists of non-surjective maps $[i] \to [n]$.

Then $\partial\Delta[n] = sk_{n-1}\Delta[n]$.

## 5.3. Simplicial sets and topological spaces

Next we investigate the relation between simplicial sets and topological spaces. We begin with the standard simplices.

Example 5.8 showed the geometric meaning of the standard simplices. Let us formalise this.

Consider the *geometric n-simplices* $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum x_i = 1)$. Picking standard bases for all $\mathbb{R}^n$ we can give coordinates for all the elements of all $\Delta^n$. Then any map $\alpha : [m] \to [n] \in \mathbf{\Delta}$ induces a covariant map $\alpha_* : \Delta^m \to \Delta^n$ by using $\alpha$ on the set of vertices and extending linearly.

Hence the $\Delta^n$ form a cosimplicial topological space in a natural way, i.e. a functor $\mathbf{\Delta} \to \mathsf{Top}$. We write $\Delta^\bullet$ as shorthand for $[n] \mapsto \Delta^n$. In particular the $i$-th face map induces the map $\Delta^{n-1} \to \Delta^n$ given by including $\Delta^{n-1}$ as the $i$-th face of $\Delta^n$, which explains the name.

Since taking the hom-set out of a diagram is contravariant, the level-wise hom set from the cosimplicial space $[n] \mapsto \Delta^n$ into $X$ naturally defines a simplicial set!

**Definition 5.13.** Define Sing : $\mathsf{Top} \to \mathsf{sSet}$ to be the functor $\mathrm{Hom}_{\mathsf{Top}}(\Delta^\bullet, -)$.

This definition is very elegant, but needs some unravelling.

A simplicial set is a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to $\mathsf{Set}$. On objects the functor Sing is given by $[n] \mapsto \mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X)$ and on maps it is defined by sending $\alpha$ to $\alpha^* : f \mapsto f \circ \alpha_*$.

Concretely, the map $\partial_i : \mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n-1}$ is induced by the inclusion of $\Delta^{n-1}$ in $\Delta^n$ as the $i$-th face.

This is exactly the singular simplicial set whose linearization $\mathbb{Z}\,\mathrm{Sing}(X)$ you have met when computing singular homology.

**Definition 5.14.** We define the *realisation* functor $A \to |A|$ from simplicial sets to topological spaces as follows. We will again use $\Delta^\bullet$, the cosimplicial topological space of geometric $n$-simplices. As $A_n$ is just a set we can write $A_n \times \Delta^m$ for the topological space obtained by taking a disjoint union of $\Delta^m$, indexed by $A_n$. Define

$$|A| := \mathrm{coeq}\left( \coprod_{\alpha:[m]\to[k]} \Delta^m \times A_k \rightrightarrows \coprod_n \Delta^n \times A_n \right)$$

where the two arrows send $\Delta^m \times A_k$ to $\Delta^k \times A_k$ and $\Delta^m \times A_m$ by applying $\alpha$ to the first or second factor. This works because $\Delta^\bullet$ is a covariant functor on $\mathbf{\Delta}$ and $A_\bullet$ is a contravariant functor.

Since colimits are functors the construction $|-|$ gives a functor $\mathsf{sSet} \to \mathsf{Top}$.

Unravelling the definition one finds that $|A| = \amalg_n A_n \times \Delta^n/_\sim$ where the equivalence relation identifies $(\alpha^*(x), y) \in A_m \times \Delta^m$ and $(x, \alpha_*(y)) \in A_k \times \Delta^k$ for any $\alpha : [m] \to [k]$. Here $\alpha_*$ is the map defined in the example above, and $\alpha^*$ is the map $A(\alpha) : A_k \to A_m$ that is part of the structure of $A$ as a simplicial set.

This just means we take a copy of $\Delta^n$ for each element of $A_n$ and then we glue the geometric simplices together according to the face and degeneracy maps of $A_*$.

You should verify the following example as a sanity check.

**Example 5.15.** The realisation of $\Delta[n]$ is (canonically isomorphic to) $\Delta^n$.

**Proposition 5.16.** *The functor* $|-|$ *is left adjoint to* Sing.

*Proof.* Remarkably, this follows from Example 5.15 together with unravelling definitions. Using the example and the definition of $\mathrm{Sing}(X)$ we have

$$\mathrm{Hom}_{\mathsf{Top}}(|\Delta[n]|, X) = \mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], \mathrm{Sing}(X)).$$

It follows that $\mathrm{Hom}_{\mathsf{Top}}(A_k \times |\Delta[n]|, X) \cong \mathrm{Hom}_{\mathsf{Set}}(A_k, \mathrm{Sing}(X)_n)$. Now we consider

$$\mathrm{Hom}\left(\mathrm{coeq}\left(\coprod_{\alpha:[m]\to[k]} \Delta^m \times A_k \rightrightarrows \coprod_n \Delta^n \times A_n\right), X\right).$$

We can pull the colimit out of the hom to obtain a limit

$$\mathrm{eq}\left(\prod_{\alpha:[m]\to[k]} \mathrm{Hom}(\Delta^m \times A_k, X) \leftleftarrows \prod_n \mathrm{Hom}(\Delta^n \times A_n, X)\right)$$

which is

$$\mathrm{eq}\left(\prod_{\alpha:[m]\to[k]} \mathrm{Hom}(A_k, \mathrm{Sing}(X)_m) \leftleftarrows \prod_n \mathrm{Hom}(A_n, \mathrm{Sing}(X)_n)\right).$$

This equalizer expresses exactly the compatibility conditions necessary for a collection of maps $A_n \to \mathrm{Sing}(X)_n$ to be a map of simplicial sets. (Unravelling this is an exercise in the formalism of limits.) $\square$

## 5.4. The model category of simplicial sets

There is a model structure on simplicial sets closely related to the one on topological spaces. Define the following classes of maps:

- Weak equivalences are those maps whose realisations are weak homotopy equivalences.
- Cofibrations are inclusions.
- Fibrations are all the maps with the RLP with respect to acyclic cofibrations.

**Theorem 5.17.** *With these classes of weak equivalences, cofibrations and fibrations* sSet *is a model category.*

*Proof.* A concise proof that this is a model structure is in Seciton 17.5 of [MP11]. □

We can make this more concrete by exhibiting sets of *generating cofibrations* $\mathscr{I}$ and *generating acyclic cofibrations* $\mathscr{J}$ as in Remark 4.6.

The meaning of these is that to a map $f$ is a fibration if it has the RLP with respect to all the generating acyclic cofibrations, with no need to test all the other acyclic cofibrations. Similarly $f$ is an acyclic fibration if it has the RLP with respect to all the generating cofibrations.

**Remark 5.18.** If a model category $\mathscr{M}$ has sets (rather than classes) of such generating cofibrations and generating acyclic cofibrations (with some smallness conditions on domain) then we say $\mathscr{M}$ is *cofibrantly generated*. Cofibrantly generated model categories have many good properties, and often constructing generating (acyclic) cofibrations is the best way of showing the model category axioms.

We were already using this perspetive for $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$, and it's equally useful for $\mathsf{sSet}$.

We define the following special simplicial sets:

**Definition 5.19.** We define the *k-th horn* $\Lambda_k[n]$ by leaving out the nondegenerate $n$-simplex $s \in \Delta[n]_n$ and the $k$-th nondegenerate $(n-1)$-simplex, $\partial_k(s)$ from $\Delta[n]$. (Of course we also leave out all their degeneracies.)

Explicitly, $\Lambda_k[n]_i$ consists of maps $[i] \to [n]$ whose image does not include the set $\{1, \ldots, k-1, k+1, \ldots, n\}$.

**Example 5.20.** The name comes from the case $n = 3$, where $\Lambda_k[3]$ looks like a (very crude and polygonal) horn.

**Definition 5.21.** A simplicial set $K$ is called a *Kan complex* if any map $\Lambda_k[n] \to K$ extends to a map $\Delta[n] \to K$.

Informally the Kan condition is expressed as: "Every horn can be filled."

The set of all $\Lambda_k[n] \to \Delta[n]$ is a set of generating acyclic cofibrations, in other words:

**Proposition 5.22.** *A map $K \to L$ in* $\mathsf{sSet}$ *is a fibration if and only if it has the right lifting property with respect to all inclusions $\Lambda_k[n] \to \Delta[n]$.*

*In particular a simplicial set is fibrant if and only if it is Kan.*

*Proof.* This follows from [MP11, Lemma 17.6.1], taking note of [MP11, Definition 15.1.1]. □

With $\partial\Delta[n]$ defined as in Example 5.12 the $\partial\Delta[n] \to \Delta[n]$ form a set of generating acyclic cofibrations.

Here is an important class of examples for Kan complexes:

**Definition 5.23.** A *simplicial group $K$* is a functor $\mathbf{\Delta}^{op} \to \mathsf{Group}$, i.e. a simplicial set such that each $K_n$ is a group and the group operations and simplicial maps are compatible.

Via the forgetful functor $\mathsf{sSet} \to \mathsf{Group}$ any simplicial group has an underlying simplicial set.

**Proposition 5.24.** *Let $K$ be a simplicial group. Then its underlying simplicial set is Kan.*

*Proof.* One can just write down the simplex filling the horn using the group operations. See for example Lemma 8.2.8 in [Wei95]. □

## 5.5. The two model categories of spaces

Let us now compare simplicial sets and topological spaces.

It is unsurprising that $|\partial\Delta[n]|$ is the boundary of $\Delta^n = |\Delta[n]|$ and $|\Lambda_k[n]|$ is the boundary of $\Delta^n$ with the interior of one face removed. Thus they are homeomorphic to $S^{n-1}$ and $D^{n-1}$ respectively.

For the following theorem we consider the category $\mathsf{CGSp}$ of *compactly generated* topological spaces.

$\mathsf{CGSp}$ is a large subcategory of $\mathsf{Top}$ containing almost all the spaces homotopy theorists care about, in particular it contains all CW complexes.

It inherits the classical model structure from $\mathsf{Top}$, and moreover the two model structures are Quillen equivalent! See Section 2.4 in [Hov07] for details.

**Remark 5.25.** Formally $\mathsf{CGSp}$ is the subcategory of topological spaces consisting of spaces which satisfy the following two conditions:

1. the image of any compact Hausdorff space in $X$ is closed (we also say $X$ is weak Hausdorff)

2. a subspace $A$ is closed in $X$ if and only if $A \cap K$ is closed in $K$ for all compact Hausdorff subspaces $K \subset X$.

Feel free to forget these conditions immediately!

The distinction between $\mathsf{CGSp}$ and $\mathsf{Top}$ is not very important in every day life. The reason for considering $\mathsf{CGSp}$ instead of $\mathsf{Top}$ is that it has some nicer properties: in particular the interaction between mapping spaces and products is better behaved in $\mathsf{CGSp}$.

The theory of simplicial sets is justified by the following:

**Theorem 5.26.** *The adjunction $|-| : \mathsf{sSet} \rightleftarrows \mathsf{CGSp} : \mathrm{Sing}$ is a Quillen equivalence.*

*Proof.* For a proof see e.g. [MP11, Theroem 17.5.2]. □

Thus the homotopy theory of spaces can be replaced by the homotopy theory of simplicial sets, which is combinatorial and explicit in nature.

We may define pointed simplicial sets as simplicial sets equipped with a map from $*$ (i.e. equipped with a special 0-simplex). Then there is a pointed analogue of Theorem 5.26 and we have the following Corollary:

**Corollary 5.27.** *Let $K$ be a pointed Kan space. Fix a point of $\partial\Delta[n+1]$. Then $[\partial\Delta[n+1], K] \cong \pi_n(|K|, *)$.*

*Proof.* This follows from the equivalence of homotopy categories in Theorem 5.26 together with the fact that $K$ is fibrant and $|\partial\Delta[n+1]| \cong S^n$ is cofibrant. □

In the remainder of this course we will define the key constructions on sSet rather than Top (or CGSp). Our examples and intuition will still be about topological spaces, and we will use many theorems about topological spaces, but formally we will consider the rational homotopy theory of simplicial sets.

Theorem 5.26 says that this is fine: we are free to switch between simplicial sets and topological spaces.

## 5.6. Cochains

When you first met a simplicial set it was probably the singular simplices of a topological space. Which was then promptly used to compute (co)homology. We will generalize this construction.

To any simplicial set $K$ we can associate a differential graded algebra called its cochain complex.

If $K$ is $\mathrm{Sing}(X)$ for some topological space $X$ this is just the singular cochain complex that is used to compute cohomology.

**Definition 5.28.** Let $K$ be a simplicial set and $R$ a commutative ring. We define the *cochain complex* $C_u^*(K,R)$ of $K$ by $C_u^n(K,R) = \mathrm{Hom}(K_n, R)$ with differential $d : C_u^n(K,R) \to C_u^{n+1}(K,R)$ defined by $df(\sigma) = \sum_{i=0}^{n+1}(-1)^i f(\partial_i\sigma)$.

We define the *normalized cochain complex* $C^*(K,R)$ as the subcomplex of $C_u^*(K,R)$ consisting of functions vanishing on all degenerate simplices.

One can check that $C^*(K,R)$ and $C_u^*(K,R)$ are cochain complexes, i.e. they satisfy $d^2 = 0$.

One can also check that $C^*(K,R)$ and $C_u^*(K,R)$ are dg algebras with product $(f \cup g)(\sigma) = \sum_{i=1}^{n}(-1)^i f(\sigma_{0\ldots i})g(\sigma_{i\ldots n})$.

Here $\sigma_{0\ldots i}$ is the front $i$-face of $\sigma$, obtained by applying $\partial_0$ to the $n$-simplex $\sigma$ $n-i$ times. Similarly $\sigma_{i\ldots n}$ is the back $(n-i)$-face of $\sigma$, obtained by applying $\partial_{n-i+1}\cdots\partial_n$.

You have probably checked this explicitly for $C^*(\mathrm{Sing}\,X, \mathbb{Z})$ in your second course on topology.

**Remark 5.29.** Note that this product is not graded commutative. It only becomes graded commutative once we take cohomology.

**Proposition 5.30.** *The map $C^*(K,R) \to C_u^*(K,R)$ is a quasi-isomorphism.*

*Proof.* This follows e.g. from Theorem 8.3.8 in [Wei95]. □

From now on we will fix $R = \mathbb{Q}$ and suppress it from notation, writing $C^*(K)$ for $C^*(K,\mathbb{Q})$.

# 6. Sullivan's Polynomial de Rham functor

## 6.1. The polynomial de Rham algebra of the standard simplex

We now have all the tools to define and examine the functor from toplogical spaces to cdga's.

We recall the cdga's $\Omega(n)$ from Example 2.4. For each $n$ this is the polynomial de Rham algebras on the $n$-simplex.

$\Omega(n)^p$ is the set of elements of degree $p$.

We now introduce face and degeneracy maps. Let $s_i : \Omega(n) \to \Omega(n+1)$ and $\partial_i : \Omega(n) \to \Omega(n-1)$ be morphisms in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ on generators as follows [1]:

$$s_i t_k = \begin{cases} t_{k+1} & \text{if } i < k \\ t_k + t_{k+1} & \text{if } i = k \\ t_k & \text{if } i > k \end{cases} \qquad \partial_i t_k = \begin{cases} t_{k-1} & \text{if } i < k \\ 0 & \text{if } i = k \\ t_k & \text{if } i > k \end{cases}$$

Note that the $t_k$ on the right hand side and on the left hand side of these equations live in different $\Omega(n)$ and are different elements.

We then extend to $dt_i$ and products by letting $\partial_i$ and $s_i$ commute with them, i.e. $\partial_i(dt_j) = d(\partial_i t_j)$ and $s_i(t_j t_k) = s_i(t_j)s_i(t_k)$.

A straightforward calculation shows that this makes $\Omega(*)$, and each graded piece $\Omega(*)^m$ into an object of $\mathsf{sSet}$. In fact, since $s_i$ and $\partial_i$ commute with differential and multiplication $\Omega(*)$ is a simplicial cdga, an element in $(\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0})^{\Delta^{\mathrm{op}}}$.

To prove, for example, $\partial_i \partial_j = \partial_{j-1}\partial_i$ (assuming $i < j$) it is enough to check the following:

$$\partial_i \partial_j t_k = \left\{ \begin{array}{ll} t_{k-2} & \text{if } j < k \\ 0 & \text{if } j = k \\ t_{k-1} & \text{if } i < k, j > k \\ 0 & \text{if } i = k \\ t_k & \text{if } i > k \end{array} \right\} = \partial_{j-1}\partial_i t_k$$

The other computations are similar.

## 6.2. The PL de Rham functor

In the last section we construted $\Omega(*)$ and found that it lives in both $\mathsf{sSet}$ and in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$. This allows the crucial idea that if we consider the set of maps from an object $X$ to $\Omega(*)$ in one category there is structure "left over" in the other category.

[1] Here we consider the full set $\{t_0, t_1, \ldots, t_n\}$ of generators for $\Omega(n)$. In particular $\Omega(0)$ has one generator $t_0$ and one relaton $t_0 = 1$.

**Definition 6.1.** Define the *polynomial de Rham functor* or *piecewise linear de Rham functor* $A : \mathsf{sSet}^{\mathrm{op}} \to \mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ by $A : K \mapsto \mathrm{Hom}_{\mathsf{sSet}}(K, \Omega(*))$. Define $F : \mathsf{cdgA}_{\mathbb{Q}}^{\geq 0} \to \mathsf{sSet}^{\mathrm{op}}$ by $F : B \mapsto \mathrm{Hom}_{\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}}(B, \Omega(*))$.

In particular we note $A(*) = \Omega(0) = \mathbb{Q}$ and $F(\mathbb{Q}) = *$ as for all $n$ there is only one map from $\mathbb{Q}$ to $\Omega(n)$.

The polynomial de Rham algebra of a topological space $X$ is then just defined as $A(\mathrm{Sing}(X))$ .

We have to check that these are indeed functors having the claimed target.

Concretely, the definition says $F(B)_n = \mathrm{Hom}_{\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}}(B, \Omega(n))$. The face and degeneracy maps on $\Omega(n)$ induce $\partial_i$ and $s_i$ on $F(B)$, and of course they satisfy the correct identities. Each $F_n : B \mapsto F(B)_n$ is a functor, thus so is $F$.

For the first functor we have $A(K)^m = \mathrm{Hom}_{\mathsf{sSet}}(K, \Omega(*)^m)$. The differential, unit and multiplication on $\Omega(*)$ induce a cdga structure on $A(K)$: We define $d_{AK}(f) = d_\Omega \circ f$ to be the differential of $f \in A(K)^m = \mathrm{Hom}(K, \Omega(*)^m)$. Here $d_\Omega$ is the differential $\Omega(*)^m \to \Omega(*)^{m+1}$. The product of $f \in A(K)^m$ and $g \in A(K)^n$ is defined as $\mu_{AK}(f, g) : x \mapsto \mu_\Omega(f(x), g(x))$ where $\mu_\Omega : \Omega(*)^m \otimes \Omega(*)^n \to \Omega(*)^{m+n}$ is the multiplication on $\Omega(*)$. Finally the unit is given by the constant map that sends any $\sigma \in K_n$ to $1 \in \Omega(n)$.

**Theorem 6.2.** *There is an adjunction* $F : \mathsf{cdgA}_{\mathbb{Q}}^{\geq 0} \leftrightarrows \mathsf{sSet}^{op} : A$.

*Proof.* To establish the adjunction we have to show that there is a natural bijection:

$$\mathrm{Hom}_{\mathsf{sSet}}(K, FB) \cong \mathrm{Hom}_{\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}}(B, AK)$$

Given a simplicial map $f : K \to FB$ we define a map $f^\# : B \to AK$ in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ by

$$b \mapsto (x \mapsto f(x)(b)).$$

This makes sense as $f(x) \in \mathrm{Hom}_{\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}}(B, \Omega(*))$, so $f(x)(b) \in \Omega(*)$, and one easily checks that $f^\#$ is a map of cdga's. Conversely, given $g : B \to AK$ we define $g^\flat(x) : b \mapsto g(b)(x)$.

These natural maps are inverse. □

**Remark 6.3.** If we forget degrees, differentials, face maps and all that for a moment what this adjunction boils down to is the well-known natural isomorphism of sets

$$\mathrm{Hom}(A, \mathrm{Hom}(B, C)) \cong \mathrm{Hom}(B, \mathrm{Hom}(A, C))$$

which follows from "interchanging the arguments".

Observe that a choice of basepoint $x : * \to K$ in $\mathsf{sSet}$ becomes an augmentation $Ax : AK \to A(*) = \mathbb{Q}$ under this functor, so we have the following corollary:

**Corollary 6.4.** *There is an adjunction* $F : \mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0} \leftrightarrows \mathsf{sSet}_*^{op} : A$ *between augmented cdga's and pointed simplicial sets.*

We also want to show that this adjunction is compatible with the homotopy structures on simplicial sets and cdga's. But first we will need to analyse $A$.

## 6.3. Useful properties of the de Rham functor

The following immediate corollary of Theorem 6.2 is extremely useful:

**Corollary 6.5.** *The functors $A$ and $F$ send colimits to limits.*

*Proof.* By Lemma A.38 right adjoints preserve limits and left adjoints preserve colimits. Since taking the opposite category exchanges limits and colimits the corollary follows from Theorem 6.2. □

For the next two results we have to work a bit more. First we need to understand simplicial homotopies.

**Proposition 6.6.** *For any simplicial set $K$ a cylinder object is given by $K \times \Delta[1]$.*

*Proof.* This looks straightforward, clearly the map $K \times \{0, 1\} \to K \times \Delta[1]$ is a cofibration so it remains to check that $K$ is weakly equivalent to $K \times \Delta[1]$. This is surprisingly hard! It follows from the fact that $|-| : \mathsf{sSet} \to \mathsf{CGSp}$ commutes with finite products. Under mild assumptions this is proved in [May67, Theorem 14.3]. In full generality it is considered in [GZ12, Section III.3]. □

**Remark 6.7.** One can unravel this definition, see e.g. [Wei95, Theorem 8.3.12], and find an explicit (if unpleasant) combinatorial characterization of simplicial homotopies, which is often used as the definition.

**Lemma 6.8.** *The simplicial set $\Omega(*)^p$ is contractible Kan complex for every $p$, i.e. the identity map on $\Omega(*)^p$ is homoty equivalent to a constant map.*

*Proof.* If we are trying to find a homotopy from $\mathbf{1}$ to the constant map 0 then it is enough to find $h : \Omega(n)^p \to \Omega(n+1)^p$ satisfying

$$\begin{cases} \partial_0 h = \mathbf{1} & \\ \partial_{i+1} h = h\partial_i & \text{if } n > 0 \\ \partial_1 h = 0 & \text{if } n = 0 \\ s_{i+1} h = h s_i & \end{cases}$$

The map $h$ is sometimes called an *extra degeneracy* (and wirtten $s_{-1}$). To compare this with the definition of simplicial homotopy as in [Wei95, Section 8.3.11] define $h_i = s_0^i h \partial_0^i$ and check all the conditions!

Then let $t_1, \dots, t_n$ be the free generators of $\Omega(n, *)$ and write $T = t_1 + \cdots + t_n$.

Then define

$$\begin{aligned} h(1) &= T^2 \\ h(t_i) &= T.t_{i+1} \\ h(dt_i) &= T.dt_{i+1} - dT.t_{i+1} \end{aligned}$$

where the $t_i$ on the left hand side are understood in $\Omega(n)$, and those on the right in $\Omega(n+1)$.

We extend multiplicatively to monomials and then by addition to all of $\Omega(n)$. It preserves $\Omega(*)^p$. It is tedious but straightforward to check this defines a homotopy. (To get started note that $T = 1 - t_0$, so by definition $\partial_0 h(1) = \partial_0(1 - t_0).\partial_0(1 - t_0) = 1$.)

The fact that $\Omega(*)^p$ satisfies the Kan condition follows from Proposition 5.24 since it is a group under addition. □

**Lemma 6.9.** *The functor $A : \mathsf{sSet}^{op} \to \mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ sends injections to surjections.*

*Proof.* The lemma asserts that for an inclusion $X \subset Y$ of simplicial sets we can extend any map $X \to \Omega(*)^q$ to $Y$. In other words, we are looking for the diagonal lift in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & \Omega(*)^q \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

But this is nothing but the LLP for the cofibration $X \subset Y$ with respect to the acyclic fibration $\Omega(*)^p \to *$, using Lemma 6.8! □

**Lemma 6.10.** *Consider a pushout square*

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, hook] & M \arrow[d, "g"] \\ L \arrow[r, "h"] & N \end{tikzcd}$$

*in* $\mathsf{sSet}$ *where i is a cofibration. Then there is a natural Mayer-Vietoris exact sequence of $\mathbb{Q}$-modules:*

$$\cdots \to H^n AN \xrightarrow{(h^*,g^*)} H^n AL \oplus H^n AM \xrightarrow{i^*-j^*} H^n AK \to H^{n+1}AN \to \cdots$$

*Proof.* By Corollary 6.5 we obtain a pullback square of dg algebras with $Ai : AL \to AK$ a surjection by Lemma 6.9.

Unravelling the pullback we have a short exact sequence

$$0 \to AN \xrightarrow{(Ah,Ag)} AL \oplus AM \xrightarrow{Ai-Aj} AK \to 0$$

and the associated long exact sequence in homology is what we need. □

## 6.4. Poincaré Lemma and Stokes' Theorem

We have constructed $A$ without checking what kind of information about the simplicial set $K$ it contains. We will now prove that $K$ gives the same cohomology as the normalised cochain complex$C^*(K)$ from Definition 5.28.

The first step, like for the usual de Rham complex, is to prove a Poincaré lemma.

**Lemma 6.11** (Poincaré Lemma)**.** *$\Omega(n) \cong \mathbb{Q}$ for each $n$.*

*Proof.* For $n = 1$ we have already checked this: $\Omega(1) = \mathbb{Q}\langle t, dt\rangle$ has 1-dimensional cohomology concentrated in degree 0, as any $t^i dt$ is the differential of $\frac{t^{i+1}}{i+1}$.

For $n > 1$ it is easy to check that $\Omega(n) = \Omega(1)^{\otimes n}$, and so the Künneth theorem (Lemma 2.25) tells us that $H^*(\Omega(n)) \cong \mathbb{Q}^{\otimes n} \cong \mathbb{Q}$. □

Next we construct a map $\rho$ from $AX$ to $C^*X$ which will induce the isomorphism on cohomology. We will obtain this map $\rho$ as formal integration of polynomial forms. Given an element $s = s(t_1, \dots, t_p)\, dt_1 \dots dt_p \in \Omega(p)^p$ and a simplex $\Delta^p = \{t_i \mid 0 \le i \le p, 0 \le t_i, \sum t_i = 1\}$ we first define

$$\int s = \begin{cases} \int_{\Delta^p} s(t_1, \dots, t_p)\, dt_1 \dots dt_p & \text{if } p > 0 \\ s & \text{if } p = 0 \end{cases}$$

as a real integral. Since $s$ is a polynomial so is its integral and we can define $\int s$ over any field of characteristic 0, in particular over the rationals.

We can define a *total differential* $\partial = \sum(-1)^i \partial_i : \Omega(n) \to \Omega(n-1)$. This allows us to state a version of Stokes' theorem.

**Lemma 6.12.** *Let $s \in \Omega(p)^{p-1}$. Then $\int ds = \int \partial s$.*

*Proof.* Unravelling the definition $\int \partial s$ is defined as $\int_{\partial\Delta^p} s$. Hence this follows by inspecting the usual proof for Stokes' theorem and observing it still holds for rational polynomials on a simplex. □

## 6.5. Differential forms and cochains

Next we define $\rho : AK \to C^*K$ by observing that integration of differential forms gives a natural functional on simplices. So let $w \in A^pK$, explicitly $w$ is a function $K \to \Omega(*)^p$. Then we have to use $w$ to send any $n$-simplex $x \in K_n$ to a number. We define $\langle \rho(w), x\rangle = \int w(x) \in \mathbb{Q}$.

**Lemma 6.13.** *$\rho$ is a cochain map into $C^*K$ respecting the unit.*

*Proof.* Note first that $\rho(e_{AK}) = e_{C^*K}$ by the definition of $\int$ in degree 0.

Next, for a degenerate simplex $sx$ we find $\langle \rho(w), sx\rangle = \int w(sx) = \int sw(x) = 0$ since $w(x) \in \Omega(p-1)^p = 0$, so $\rho$ really is a map into normalised cochains.

Finally consider $\langle \rho(dw), x\rangle = \int dw(x) = \int \partial w(x)$ by Lemma 6.12. Then we have $\int \partial w(x) = \int w(\partial x)$ since $w \in A^*(X) = \mathrm{Hom}_{\mathsf{sSet}}(X, \Omega)$ commutes with the total differential.

Next $\int w(\partial x) = \langle \rho(w), \partial x\rangle$ is $\langle d\rho(w), x\rangle$ by the definition of the differential on cochains. Thus $\rho$ is a cochain map. □

**Theorem 6.14.** *Let $K$ be an object of* sSet*. Then $H(\rho) : H(AK) \cong H(C^*K)$ is an isomorphism of graded vector spaces.*

*Proof (finite-dimensional case).* To show that $\rho$ is a isomorphism one can use an induction along the $n$-skeleton of $K$ (see Definition 5.11).

We give here the proof for the case $K$ is finite-dimensional, i.e. it is equal to its $n$-skeleton for some $n$. The general result will follow from our proof of Theorem 6.16 below.

We need three ingredients.

1. $HA(\Delta[n]) \cong \mathbb{Q} \cong HC^*(\Delta[n])$.

2. There are isomorphism $HA(\amalg_j K_j) \cong \prod_j HA(K_j)$ and $HC^*(\amalg_j K_j) \cong \prod_j HC^*(K_j)$.

3. Given a push-out square in $\mathsf{sSet}$ there are associated long exact Mayer-Vietoris sequence for $HA$ and $HC^*$.

where all the constructions are natural with respect to $\rho$.

We have already given proofs of (1) to (3) for $A$.

(1) is an immediate consequence of Lemma 6.11.

(2) follows since $A$ sends coproducts to products by Corollary 6.5.

(3) is the statement of Lemma 6.10.

Similar proofs work for $C^*$ and the naturality follows.

Now we show how these statements together give the theorem.

The isomorphism $HA(K_0) \cong HC^*(K_0)$ follows from (1) and (2). Then we write $sk_nK$ as a colimit

$$\begin{tikzcd}
\amalg_{K_n^{nondeg.}} \partial\Delta[n] \arrow[r] \arrow[d, "i"] & sk_{n-1}K \arrow[d, "p"] \\
\amalg_{K_n^{nondeg.}} \Delta[n] \arrow[r] & sk_nK
\end{tikzcd}$$

$HA$ and $HC^*$ send this colimit diagram to long exact sequences by (3), and by naturality $\rho$ gives a map between the long exact sequences. Now $\partial\Delta[n]$ is nothing but the $(n-1)$-skeleton of $\Delta[n]$, so by induction assumption together with (1) all the comparison maps except for $HA(sk_nK) \to HC^*(sk_nK)$ are isomorphisms. Then we conclude by the five lemma. □

**Remark 6.15.** This proof can be extended to arbitrary simplicial sets by writing $K = \cup K_{\leq n}$. This needs an additional piece of homological algebra (the limit $\lim^1$), so I skip it here. Details can be found in [BG76, §14].

Note that $\rho$ is not multiplicative! This should not be a surprise since $A(K)$ is graded commutative but $C^*(K)$ is not! However, we have the following theorem, whose proof will be the subject of a bonus section a few weeks from now.

**Theorem 6.16.** *For any simplicial set $K$ there is a natural zig-zag of quasi-isomorphisms of dg algebras between $A(K)$ and $C^*(K)$. In particular there is a natural isomorphism of cohomology rings.*

## 6.6. Homotopy theory of the fundamental adjunction

**Lemma 6.17.** *The functors $F$ and $A$ take cofibrations to fibrations, i.e. the adjuncton $F$ : $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}} \leftrightarrows \mathsf{sSet}^{op}_{*} : A^{op}$ is left Quillen.*

*Proof.* It suffices to show that $F$ takes cofibrations to fibrations and trivial cofibrations to trivial fibrations. (Recall that the cofibrations in $\mathsf{sSet}^{\mathrm{op}}$ are exactly the fibrations in $\mathsf{sSet}$, see Remark 3.13.)

So let $i : B \to C$ be a cofibration in $\mathsf{cdgA}^{\geq 0}_{\mathbb{Q}}$. To check that $i^* : FC \to FB$ is a fibration it suffices to check it has the RLP with respect to the generating cofibrations $u : \Lambda[n]_k \to \Delta[n]$ of $\mathsf{sSet}$. But applying the adjunction from Theorem 6.2 this follows if we can establish the LLP for $B \to C$ with respect to the maps $u^* : A\Delta[n] \to A\Lambda[n]_k$. As $B \to C$ is a cofibration by assumption it suffices to check that these maps are trivial fibrations.

By definition of the model structure on $\mathsf{cdgA}^{\geq 0}_{\mathbb{Q}}$ we need $u^*$ to be a surjective quasi-isomorphism. The fact it is a quasi-isomorphism follows as Theorem 6.14 applied to $\Lambda[n]_k$ and $\Delta[n]$ shows these spaces with contractible realization have cohomology isomorphic to $\mathbb{Q}$. (Note that here we do not use the fact that $H(\rho)$ is an algebra map: The algebra structure on $\mathbb{Q}$ is unique!)

For surjectivity we need to check that a polynomial differential form can be extended from $\Lambda[n]_k$ to $\Delta[n]$. This follows from Lemma 6.9.

To show that $F$ sends trivial cofibrations to trivial fibrations we repeat the same analysis, except now we only need to check that $A\Delta[n] \to A(\partial\Delta[n])$ is surjective, which is again Lemma 6.9. □

Since $F \dashv A$ is a Quillen adjunction it follows that there are adjunctions of homotopy categories

$$LF : \mathrm{Ho}(\mathsf{cdgA}^{\geq 0}_{\mathbb{Q}}) \leftrightarrows \mathrm{Ho}(\mathsf{sSet}^{\mathrm{op}}) : RA^{\mathrm{op}}$$

and

$$LF : \mathrm{Ho}(\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}) \leftrightarrows \mathrm{Ho}(\mathsf{sSet}^{\mathrm{op}}_{*}) : RA^{\mathrm{op}}.$$

As all simplicial sets are cofibrant $RA(K) = A(K)$, but $LF(A) = F(QA)$ for a cofibrant replacement of $A$.

In analogy to Definition 2.17 we make the following definition:

**Definition 6.18.** A *minimal model* for a simplicial set $K$ is any minimal model $e_K : MK \xrightarrow{\sim} AK$ for the polynomial de Rham algebra of $K$.

A *minimal model* for a topological space $X$ is a minimal model for the singular simplicial set of $X$

Minimal models are cofibrant cdga's, but as opposed to the cofibrant replacement functor they are quite manageable.

As any homologically connected cdga has a minimal model it is easy to see that we have an adjunction: $F : \mathrm{Ho}(\mathsf{MinMod}) \leftrightarrows \mathrm{Ho}(\mathsf{sSet}^{\mathrm{op}}_{0}) : M$ between the category of minimal models with homotopy classes of algebra homomorphisms between them and the homtopy category of connected simplicial sets. The functor $M$ is obtained by sending $K$ to a minimal model of $AK$. (It's an exercise to see this can be lifted to a functor.)

## 6.7. Homotopy groups I

The adjunction gives us the tools to understand homotopy groups of $FB$ in terms of $\pi^* B$. This will become a very powerful computational once we have proved our main theorem (using the contents of this section as a crucial ingredient).

We recall from Definiton 4.8 the *homotopy groups* of an augmented cdga given by $\pi^n(B) = H^n(IB)$ where $IB = \bar{B}/\bar{B}.\bar{B}$ are the indecomposables in $B$.

**Theorem 6.19.** *Let $B$ be a cofibrant cdga. If $B$ is homologically connected then $\pi_0 FB = *$. For $n \geq 1$ there is a natural bijection of sets $\pi_n(FB) \cong Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$ .*

*If $n \geq 2$ this induces an isomorphism of groups $\pi_n(FB) \cong Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$.*

*There is also an isomorphism of group if $n = 1$ and $B(1) = B(1, 1)$ in the notation of Section 2.4.*

It is worth stopping for a moment to consider the meaning of the theorem. Homotopy groups in general are very hard to compute, but the homotopy groups of $FB$ can be easily read off from the indecomposables of $B$. Of course the space $FB$ has been defined in a very abstract way, the key to making our theorem useful will be to show that the rational homotopy groups of $X$ agree with those of $FAX$.

While the idea for this proof is simple, we need a number of technical ingredients. First we define $V(n)$ in $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$ to have a single generator in dimension $n$ and trivial multiplication and differential. (If $n$ is odd $V(n) \cong \mathbb{Q}\langle x_n \rangle$, but if $n$ is even $V(n)$ is not semi-free.)

**Lemma 6.20.** *Let $B \in \mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$ be cofibrant. Then homotopy is an equivalence relation on $\mathrm{Hom}_{\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}}(B, V(n))$ and we have a natural bijection $[B, V(n)] \cong Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$.*

*Sketch of Proof.* The first statement follows from Lemma 3.18 as all objects in $\mathsf{cdgA}^{\geq 0}_{\mathbb{Q}}$ are fibrant.

The natural map $\phi_B : [B, V(n)] \to Hom_{\mathbb{Q}}(\pi^n B, \pi^n V(n)) = Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$ exists since homotopic maps become identical on applying $\pi^n$. We have shown this in Lemma 4.9.

We observe that $\phi_B$ is surjective (by unravelling the definitions) so it is left to show that $\phi_B$ is injective. Since $V(n)$ has trivial multiplication the maps in $[B, V(n)]$ factor through $B/\bar{B}.\bar{B}$ and we can assume that $B$ has trivial multiplication. Now assume $f_* = g_* : \pi^n B \to \pi^n V(n)$. As $\pi^i V(n) = 0$ for $i \neq n$ we get $f_* = g_* : \pi^* B \to \pi^* V(n)$. We can deduce that the maps induced by $f$ and $g$ from $\bar{B}$ to $\bar{V}(n)$ are homotopic in $\mathsf{dgMod}^{\geq 0}_{\mathbb{Q}}$. Unravelling definitions this means there is a cochain homotopy, i.e. a map $D : \bar{B} \to \bar{V}(n)[-1]$ with $d_V D + D d_B = f - g$. Using the augmentation and the fact that multiplication is trivial we extend this to a map $D : B \to V(n)[-1]$.

We will define the homotopy $H : B \to \Omega(1) \tilde{\otimes} V(n)$ as follows.

$$H(x) = dt_1 \otimes Dx + (1 - t_1) \otimes fx + t_1 \otimes gx$$

As $B$ has trivial multiplication it is easy to check that this is an algebra map and by our definition of $D$ it commutes with the differential. Finally, $\partial_0 H = g$ and $\partial_1 H = f$. Thus $f \simeq g$ in $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$ and $\phi_B$ is injective. □

**Lemma 6.21.** *Given a push-out square in* $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, hook, "i"] & C \arrow[d, "k"] \\ D \arrow[r, "h"] & L \end{tikzcd}$$

*where i is a cofibration, we have a long exact sequence*

$$\pi^0 B \xrightarrow{(i_*, j_*)} \pi^0 C \oplus \pi^0 D \xrightarrow{k_* - h_*} \pi^0 L \longrightarrow \pi^1 B \xrightarrow{(i_*, j_*)} \cdots$$

*Proof.* One first needs to show that applying $I$ sends the push-out square in $\mathsf{cdgA}^{\geq 0}_{\mathbb{Q}}$ to push-out in chain complexes. This follows by unravelling the definitions: as $L$ is generated by $C$ and $D$, the indecomposables in $L$ are given by the sum of indecomposables in $C$ and in $D$.

Now a push-out of an injection of chain complexes gives rise to a short exact sequence of chain complexes and thus to a long exact sequence on homology. So all we have to do is prove that $Ii$ is an injection on $IB^n$ for every $n$.

We will prove this by showing that $Ii$ induces a surjection $Hom_{\mathbb{Q}}(ID^n, \mathbb{Q}) \to Hom_{\mathbb{Q}}(IB^n, \mathbb{Q})$ for every $n$. Introduce the algebra $U(n)$ with generators in degree $n$ and $n - 1$, trivial multiplication and zero cohomology. Then by unravelling definitions we have a natural bijection $\mathrm{Hom}_{\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}}(B, U(n)) \cong Hom_{\mathbb{Q}}(IB^n, \mathbb{Q})$.

Since $i$ is a cofibration and the augmentation $\epsilon : U(n) \to \mathbb{Q}$ is a trivial fibration, we can apply the left lifting property to show that the map $i^* : \mathrm{Hom}_{\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}}(D, U(n)) \to \mathrm{Hom}_{\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}}(B, U(n))$ is a surjection. Thus so is $\mathrm{Hom}_{\mathbb{Q}}(ID^n, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(IB^n, \mathbb{Q})$ and this completes the proof. □

## 6.8. Homotopy groups II

*Proof of Theorem 6.19.* The proof makes use of the Quillen adjunction $LF : \mathrm{Ho}\,\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}} \rightleftarrows \mathrm{Ho}\,\mathsf{sSet}^{\mathrm{op}}_* : RA$ from Lemma 6.17 which induces an adjunction on homotopy categories by Theorem 3.41.

First note that if $B$ is homologically connected it has a minimal model $MB$ by Theorem 2.21. By inspection $MB$ has a single augmentation, so $[MB, A*] \cong [MB, \mathbb{Q}] \cong *$. By the Quillen adjunction it follows that $[*, FMB] \cong *$, but this is exactly $\pi_0(FMB) \cong \pi_0(FB)$ as $F$ sends the weak equivalence between cofibrant cdga's to a weak homotopy equivalence.

For $n \geq 1$ we use the adjunction $F \dashv A$ and the de Rham algebra of the sphere to get the first two bijections of

$$\pi_n FB = [S^n, FB] \cong [B, AS^n] \cong [B, V(n)] \cong Hom_{\mathbb{Q}}(\pi^n B, \mathbb{Q}) \tag{6.1}$$

where $V(n)$, defined as above, is clearly quasi-isomorphic to $AS^n$. The last bijection is then Lemma 6.20.

For the group structure we first simplify the problem. Assume without loss of generality that $B$ is minimal, and we recall the notation $B = \cup B(n)$ from Section 2.4. By inspecting the proof of Lemma 4.7 we have cofibrations $B(n) \hookrightarrow B$ and $B(n-1) \hookrightarrow B(n)$.

We apply Lemma 6.21 to the push-out diagram for $B \hookleftarrow B(n) \to \mathbb{Q}$. As the quotient $B//B(n)$ of $B$ by the ideal generated by $B(n)$ has no generators in degree less than equal to $n$ the long exact sequence gives an isomorphism $\pi^n B \cong \pi^n B(n)$.

Next take the pushout diagram $B(n) \hookleftarrow B(n-1) \to \mathbb{Q}$ and we obtain $\pi^n(B(n) \cong \pi^n(B(n))//B(n-1))$.

So it is enough to prove the result for $B' := B(n)//B(n-1)$. But this is just a tensor product of free cdga's on one generator with trivial differential, $\mathbb{Q}\langle x \rangle$. This is clear if $n \geq 2$ and if $n = 1$ this follows from the assumption $B(1) = B(1,1)$, that is, there are no nontrivial differentials on $B(1)$.

But $B' = \otimes \mathbb{Q}\langle x \rangle$ has a *comultiplication* $\Delta : B' \to B' \otimes B'$, given by $x \mapsto (x \otimes 1) + (1 \otimes x)$ on each generator. There is also a *counit* $B' \to \mathbb{Q}$ sending all generators to 0. Thus $B'$ is a coalgebra, it is easy to see that it satisfies the dual conditions to the associativity and unit axioms.

By naturality this structure is compatible with all the bijections in Equation 6.1. A comultiplication and counit on the source induce a multiplication and unit on the hom space, e.g.

$$[B', AS^n] \times [B', AS^n] \cong [B' \otimes B', AS^n] \xrightarrow{\Delta^*} [B', AS^n]$$

Similarly, the contravariant functor $F$ sends the coalgebra structure on $B'$ to a monoid structure on $FB'$ since we have $F(A \otimes B) \cong FA \times FB$. (We say that $F$ is *monoidal* if it has this last property.)

Thus all the terms of Equation 6.1 are *monoids*. Recall that a monoid is just a set equipped with an associative multiplication with unit. It is just like a group except it may not have inverses.

We just have to check that the multiplication on $[S^n, FB']$ and $Hom_{\mathbb{Q}}(\pi^n B', \mathbb{Q})$ is the expected one.

For the homotopy groups of $FB'$ the Eckmann-Hilton argument implies that the multiplication on any (simplicial or topological) monoid induces the usual multiplication of homotopy groups.

Finally for the chain complex of indecomposables we have $I(B' \otimes B') = (IB' \otimes \mathbb{Q}) \oplus (\mathbb{Q} \otimes IB') \cong IB' \oplus IB'$ and the comultiplication induces $x \mapsto (x, x)$ and this induces the usual additive structure $Hom_{\mathbb{Q}}(\pi^n B', \mathbb{Q}) \oplus Hom_{\mathbb{Q}}(\pi^n B', \mathbb{Q}) \to Hom_{\mathbb{Q}}(\pi^n B', \mathbb{Q})$. □

**Corollary 6.22.** *Let $A$ be the de Rham algebra of the sphere $S^n$. The homotopy groups of $\pi_k(FAS^n, 1)$ are 0 unless $k = n$ or $n$ is even and $k = 2n - 1$, in which case $\pi_k(FAS^n) \cong \mathbb{Q}$.*

As we mentioned at the beginning of this section: We will see later that $\pi_k(FAS^n) \cong \pi_k(S^n) \otimes \mathbb{Q}$. Thus Corollary 6.22 determines the rational homotopy groups of spheres.

*Proof.* We computed the minimal model in Example 2.18. The setting was slighly different, but computation we did to find the minimal model for the smooth de Rham algebra also holds for the polynomial de Rham algebra $AS^n$.

Let us consider the more interesting case that $n$ is even. Then we have a minimal model $\mathbb{Q}\langle x_n, y_{2n-1} \mid dy = x^2\rangle$ and the only indecomposables are $x_n$ and $y_{2n-1}$, showing $\pi^i(MS^n)$ is 1-dimensional if $i$ is $n$ or $2n-1$ and 0 for other positive $i$.

The result then follows from Theorem 6.19. □

# 7. Spectral Sequences

We will need to consider rational equivalences of spaces, so we should learn to compute rational cohomology. In this section we introduce a very powerful tool: Spectral sequences. Spectral sequences have a reputation for being intimidating, but in many cases they are just an incredibly useful (and quite fun) computational machine.

The first question we want to answer is as follows: Given a fibration $F \to E \to B$ of topological spaces, can we compute the cohomology of $E$ in terms of that of $F$ and $B$? The answer is given by the Leray-Serre spectral sequence.

We may also want to compute the cohomology of $F$ in terms of $E$ and $B$. This is achieved by the Eilenberg-Moore spectral sequence.

There are many resources on spectral sequences. A brief and precise introduction is in Chapter 5 of [Wei95]. Many details can be found in the book [McC01]. The best quick-and-dirty introduction is given in a note written by Ravi Vakil [Vak]. Another very nice explanation in particular for the spectral sequence of a double complex is in [BT82].

## 7.1. Introducing spectral sequences

We work with cochain complexes of $R$-modules for a commutative ring $R$, but feel free to let $R = \mathbb{Q}$.

Many uses of spectral sequence come from the following important situation:

**Definition 7.1.** A *double complex* $C^{**}$ is a collection of objects $C^{ij}$ together with differentials $d_h^{ij} : C^{ij} \to C^{i+1,j}$ and $d_v^{ij} : C^{ij} \to C^{i,j+1}$ which satisfy $d_h^2 = d_v^2 = 0$ and $d_h d_v = -d_v d_h$.

**Definition 7.2.** The *total complex* $\operatorname{Tot} C$ is defined as the complex $(D^*, d)$ where $D^n = \oplus_{i+j=n} C^{ij}$ and $d = d_h + d_v$.

**Example 7.3.** This looks familiar from our definition of the total tensor product. We can first define a double complex $(A \otimes B)_{pq} = A_p \otimes B_q$ with horizontal differential $d^h = d^A \otimes \mathbf{1}$ and vertical differential $d^v = (-1)^p \mathbf{1} \otimes d^B$. Then $(A \otimes B)_* = \operatorname{Tot}^{\oplus}((A \otimes B)_{**})$.

**Remark 7.4.** Sometimes we want to consider the direct product total complex $\operatorname{Tot}^{\prod} C$ instead, which has $(\operatorname{Tot}^{\prod} C)^n = \prod_{i+j=n} C^{ij}$ and the same differential. In many cases of interest the double complex is bounded below in both degrees and the two constructions agree.

Our aim now is to compute cohomology of the total complex. Let us assume our double complex is concentrated in the first quadrant, i.e. $C^{ij} = 0$ if $i < 0$ or $j < 0$.

Consider an element $[x]$ in $H^n(\operatorname{Tot} C)$. We can represent it as $x = \sum_{i=p}^n x_i$ where $x_i \in C^{i,n-i}$. (Here $p$ is minimal such that $x_p \neq 0$.) Now if $x$ is to be a cocycle, we need $d_v x_p = 0$, and $d_h x_p = d_v x_{p+1}$, next $d_v x_{p+1} = d_h x_{p+2}$ and so on.

It is useful to introduce a *filtration* on $C$. We are going to define filtrations properly soon, here we just define subcomplexes $F^n C = \bigoplus_{j \geq n} C^{j*}$. So $x \in F^p C$ and $d_h(x_p) \in F^{p+1}C$.

We want to approximate $x$ by only considering the first few $x_i$. In the quotient complex $F^p C / F^{p+1} C$ we have $[d_h x_p] = 0$, so $[x_p]$ represens a cocycle. If we want to improve our approximation we consider $F^p C / F^{p+2} C$. If we find any $x_{p+1}$ with $d_h x_p = d_v x_{p+1}$ then $x_p + x_{p+1}$ is a cocycle in $F^p C / F^{p+2} C$. In other words $x_p$ is a cocycle for $d_v$ and $d_h x_p$ is a coboundary with respect to $d_v$. More generally assume we have a cycle $[x] = [x_p + \cdots + x_{p+r-1}] \in F^p C / F^{p+r} C$ that we want to lift to a class in $F^p C / F^{p+r+1} C$. This is possible if $d_h x_{p+r-1}$ can be written as some kind of coboundary.

The idea is that $x_p \mapsto d_h x_{p+r-1}$ may be turned into a kind of differential map $d_r$ from $C^{pq}$ to $C^{p+r,q-r+1}$, or rather between subquotients of these spaces. (Here $q = n - p$.) We approximate the cohomology of $C$ by repeatedly taking cohomology with respect to these maps $d_r$ for larger and larger $r$.

In our example looking at a double complex, this means we first compute homology with respect to $d^v$ and then with respect to $d^h$, and then with respect to some new differential of degree $(2, -1)$.

Hopefully this helps explain where the definition in the next section comes from.

## 7.2. Definition

**Definition 7.5.** A *(cohomology) spectral sequence* starting at $E_a$ is the following data:

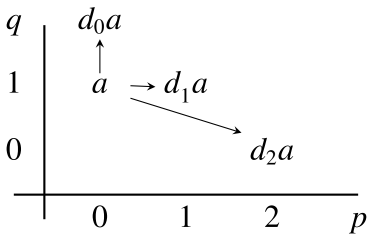
- A collection $E_r^{pq}$ of $R$-modules, where $r \geq a$ and $p, q \in \mathbb{Z}$,
- a collection of morphisms $d_r = d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ satisfying $(d_r)^2 = 0$,
- isomorphisms $E_{r+1}^{pq} \cong \ker d_r^{pq} / \operatorname{Im} d_r^{p-r,q+r-1}$.

The spectral sequence is usually denoted $E_r^{**}$.

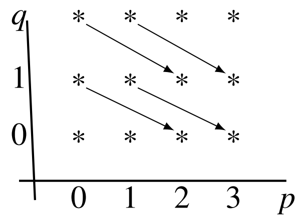
Here is the first page of a spectral sequence:

$$\begin{array}{c|cccc}
q & E_0^{02} & E_0^{12} & E_0^{22} & E_0^{32} \\
 & \uparrow & \uparrow & \uparrow & \uparrow \\
1 & E_0^{01} & E_0^{11} & E_0^{21} & E_0^{31} \\
 & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & E_0^{00} & E_0^{10} & E_0^{20} & E_0^{30} \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

Here is a picture of the first few differentials, slightly abusing notation:



This is a schematic of $E_2$ with differentials:



**Example 7.6.** Let $C^{**}$ be a first quadrant double complex. Then there as a spectral sequence with

$$^{I}E_1^{pq} = H_v^q(C^{p*}) \text{ and } {}^{I}E_2^{pq} = H_h^p H_v^q(C).$$

Here $H_v^q(C)$ denotes the $q$-th cohomology of $C^{p*}$ with respect to $d_v$. One checks that $H_v^q(C)$ is a complex with a differential induced by $d_h$, and the $p$-th cohomology group of this complex is ${}^{I}E_2^{pq}$.

Changing the role of $d_h$ and $d_v$ there is a second spectral with.

$$^{II}E_1^{pq} = H_h^q({}^tC) \text{ and } {}^{II}E_2^{pq} = H_v^p H_h^q({}^tC).$$

where we write ${}^tC$ for the complex defined by ${}^tC^{pq} = C^{qp}$. [1] (If we do not swap the rows and columns than we obtain a different kind of spectral sequence where $d^r$ has degree $(1-r, r)$. But everything works the same way.)

We typically want to investigate what happens as $r$ becomes large.

**Definition 7.7.** We define $E_\infty^{pq} := E_r^{pq}$ whenever $E_r^{pq}$ becomes eventually stable, i.e. for large enough $r$ we have $E_r^{pq} \cong E_{r+1}^{pq} \cong \dots$.

We say a spectral sequences *degenerates* at $E_r$ if all $d_{s \geq r}$ are 0. In particular $E_\infty = E_r$.

The next definition is a bit rough and will be refined later.

**Definition 7.8.** Let $E_r^{pq}$ be spectral sequence. If there is some graded $R$-module $H$ with $H^n \cong \oplus_{p+q=n} E_\infty^{pq}$ we say that $E_r^{pq}$ *converges* to $H$, written $E_r^{pq} \Rightarrow H$.

[1]Of course changing row and column changes the direction of the differentials, so from the point of view of ${}^tC$ you could say that we again take first $H_v$ and then $H_h$.

**Example 7.9.** The spectral sequences from Example 7.6 becomes eventually stable for degree reasons, i.e. for every $p, q$ we have $d_s^{pq} = 0$ for large enough $s$ because the differential leaves the first quadrant.

Moreover, both spectral sequences converge to $H(\operatorname{Tot} C)$, i.e. $H^n(\operatorname{Tot} C) = \oplus_{p+q=n} E_\infty^{pq}$.

We will see later why this statement is true.

A spectral sequence is a tool for computing something complicated, e.g. the cohomology of a total complex, in terms of simpler objects, e.g. the vertical and horizontal cohomology.

## 7.3. An example

Let us consider our first example. We will use the spectral sequences from Example 7.6 to compute something you have known for a long time.

Consider the following map between exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[u, "f"] & B' \arrow[r] \arrow[u, "g"] & C' \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd}$$

Then the snake lemma says that there is a long exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0$$

I claim that this follows from two spectral sequences!

Our map of exact sequences is equivalent to a double complex. (To stay consistent with our definition we need to replace $g$ by $-g$, but this does not affect the kernel or cokernel.)

So we consider the two spectral sequences.

If we take the horizontal differential first we have ${}^{II}E_2 = H^*(0, d_v) = 0$, so the spectral sequences converges to 0.

If we take the vertical differential first we first obtain $E_1 = H^*(C, d_v)$ which looks as follows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h \arrow[r] & 0 \\
0 \arrow[r] & \ker f \arrow[r] & \ker g \arrow[r] & \ker h \arrow[r] & 0
\end{tikzcd}$$

where the horizontal morphisms are induced by $d_h$. Taking cohomology with respect to $d_h$ we will obtain another collection of 6 modules. But as we consider the next differential we find the following term ${}^{I}E_2$:

$$\begin{tikzcd}
0 \arrow[drr] & X \arrow[drr, "\alpha"] & ? \arrow[drr] & ? & 0 \\
0 & ? & ? & Y & 0
\end{tikzcd}$$

Because of the degrees that the entries live in, there can only be one nonzero map, from $X$ to $Y$.

On all the subsequent pages of the spectral sequence $d_r$ shifts the row by at least 2, so there can be no more nonzero maps.

But we know the spectral sequence ${}^{I}E$ converges to 0 since the sister spectral sequence ${}^{II}E$ does.

The only possibility is that all ? terms are 0 and the map $\alpha : X \to Y$ is an isomorphism.

It is easy to see that $\alpha^{-1}$ : coker(ker $g \to$ ker $h$) $\cong$ ker(coker $f \to$ coker $g$) completes the desired long exact sequence.

## 7.4. Filtrations and convergence

We formalise our previous work by considering the spectral sequences arising from filtrations.

**Definition 7.10.** A *(decreasing) filtration* on a cochain complex $C$ is a collection of subobjects $F^nC$ of $C$ satisfying $F^{n+1}C \subset F^nC$.

There are also increasing filtrations, typically denoted $F_nC$, which can be defined completely analogously.

**Example 7.11.** Let $C$ be a cochain complex. There are two important filtrations we can always define. The *stupid filtration* is defined by.

$$(\sigma^{\geq n}C)^m = \begin{cases} C^m & \text{if } m \geq n \\ 0 & \text{if } m < n \end{cases}$$

The *canonical filtration* on $C$ is an increasing filtration defined by

$$(\tau_{\leq n}C)^m = \begin{cases} 0 & \text{if } m > n \\ Z^m & \text{if } m = n \\ C^m & \text{if } m < n \end{cases}$$

**Definition 7.12.** The *associated graded object* or just the *associated graded* of $(C, F)$ is defined as the graded object which is $Gr_F^pC = F^pC/F^{p+1}C$ in degree $p$.

**Example 7.13.** The associated graded of the stupid filtration is $Gr_\sigma^p(C) := \sigma^{\geq p}C/\sigma^{\geq p+1}C \cong C^p$.

The associated graded of the canonical filtration is $Gr_p^\tau(C) := \tau_{\leq p}C/\tau_{\leq p-1}C \simeq H^p(C)[-p]$.

**Example 7.14.** A double complex may be filtered by columns, i.e. $F^n(C) = \oplus_{p\geq n}C^{p*}$.

Then the associated graded is just the direct sum of the columns considered with the vertical differential. The horizontal differential is now 0. The cohomology of the associated graded is the cohomology with respect to $d_v$.

Our aim now is to associate to a filtered complex $C$ a spectral sequence which computes cohomology of the filtered complex starting with the cohomology of the associated graded complex.

First we define more carefully what it means for a spectral sequence to converge. Asking for the cohomology of $C$ to be a direct sum of the $E_\infty$ terms is too much to ask in general. But the filtration on $C$ induces a filtration on cohomology.

**Definition 7.15.** A spectral sequence $E_r$ *weakly converges* to $H^*$ if there is a filtration on every $H^n$ such that $E_\infty^{pq} \cong \mathrm{Gr}^p H^{p+q}$. We often write this as $E_a^{pq} \Rightarrow H^*$

This is unfortunately not a very strong definition, because a general filtration can be quite pathological. We might not have $\cup_n F^n = C$ or $\cap_n F^n = 0$.

**Definition 7.16.** We say a filtration $F$ on a cochain complex $C$ is *bounded* if for each $n$ there are $s < t$ such that $C^n = F^s C^n \supset F^t C^n = 0$.

**Definition 7.17.** Wesay a spectral sequence is *bounded* if for each fixed $n$ there are only finitely many terms $p, q$ with $p + q = n$ and $E_a^{pq} \neq 0$.

We will only be concerned with bounded spectral sequences.

**Example 7.18.** If a spectral sequence is concentrated in the first quadrant then it is certainly bounded.

**Definition 7.19.** We say a bounded spectral sequence $E_r$ *converges* to $H^*$ if the spectral sequence weakly converges to $H^*$ for some bounded filtration on $H^*$.

**Remark 7.20.** To make a confusing subject even more confusing the nomenclature is not quite standardised. People say a spectral sequence *weakly converges*, *converges* or *strongly converges* to $H$ or say that it *approaches* $H$ or *abuts* to $H$.

Partly this language is used to distinguish subtle differences in convergence for spectral sequences that are not bounded, but sometimes people use the same words for different things and different words for the same thing! Always watch out for the definitions in the reference you are consulting.

## 7.5. The spectral sequence of a filtration

We are now ready to state the main theorem of this chapter.

**Theorem 7.21.** *For every filtered cochain complex $(C, F)$ there is a spectral sequence with $E_1^{pq} = H^{p+q}(Gr_F^p(C))$.*

*If $F$ is bounded then this spectral sequence is bounded and converges to $H^*C$.*

*Not a proof.* I'm not going to prove this, but I will define the spectral sequence. First we define the first page: $E_0^{pq} = F^pC^{p+q}/F^{p+1}C^{p+q}$.

We see that this shows the spectral sequence is bounded if the filtration is bounded. $F$ naturally induces a flitraton on $H^*$ and it's easy to see this is also bounded.

The crucial definition is the next one:

$$A_r^{pq} := \{x \in F^pC^{p+q} \mid dx \in F^{p+r}C^{p+q+1}\}$$

These are approximate cycles, and as $r$ becomes larger and larger they approximate the actual cycles. Now $Z_r^{pq}$ is the image of $A_r^{pq}$ in $E_0^{pq} = F^pC^{p+q}/F^{p+1}$, and $B_{r+1}^{p+r,q-r+1}$ is the image of the differential $dA_r^{pq}$ in $F^{p+r}C^{p+q+1}/F^{p+r+1}$. Now we can define $E_r^{pq} = Z_r/B_r$ and there is a differential induced by the differential $d$ of $C$. To be precise, pick $c$ representing an element in $E_r^{pq}$. So $dc \in F^{p+r}C^{p+q+1}$ and we need to check it represents an element of $Z^{p+r,q+r-1}$ and that $d$ factors through $B^{pq}$.

The next step is to show that $E_{r+1}$ is the homology of $E_r$. Then we need to worry about convergence, i.e. we must identify $E_\infty$ with $\operatorname{Gr} H^*$. This is arguably just book-keeping (although it is a lot of it). Details can be found in Section 5.5 of [Wei95] or Section 2.2 of [McC01]. □

**Example 7.22.** Let $C$ be a first quadrant double complex. The filtration by rows and by columns give rise to the spectral seqences from Example 7.6 and Theorem 7.21 implies the convergence result in Example 7.9.

We are also interested in maps of spectral sequences. But since we care about the behaviour of spectral sequences only as $r$ becomes large, we can be a bit lax:

**Definition 7.23.** A map between spectral sequences $E$ and $E'$ is a collection of maps $f_r^{pq} : E_r^{pq} \to E_r'^{pq}$, for $r \geq b$ for some fixed $b$, that commute with the $d_r$.

**Theorem 7.24.** *A map $f : C \to D$ of filtered complexes induces a map $f_r$ of spectral sequences compatible with the induced map on cohomology. Assume the filtrations are bounded and there is some $r$ such that $f_r^{pq}$ is an isomorphism for all $p$ and $q$. Then $H^*(f) : H^*(C) \to H^*(D)$ is an isomorphism.*

*Proof.* See [Wei95, Theorem 5.5.11]. □

## 7.6. The Leray-Serre spectral sequence

We now turn to topological examples. To simplify matters we fix a ground field $k$ of coefficients and $H^*(X)$ shall always denote cohomology of $X$ with coefficients in $k$, computed as cohomology of the complex $C^*(X)$ of normalized singular cochains on $X$ with coefficients in $k$.

One of the most important spectral sequences in topology is the Leray-Serre spectral sequence for the cohomology of a fibration. The proof is somewhat delicate, I will only give the statement and the general idea.

**Definition 7.25.** A *spectral sequence of algebras* is a spectral sequence $E_*^{*,*}$ equipped with a bigraded multiplication for each $r$, i.e. there are maps $E_r^{p,q} \otimes E_r^{s,t} \to E_r^{p+s,q+t}$ for all $p, q, s, t$, and such that each $d_r$ is a (graded) derivation, so that the multiplication on sheet $r$ induces the multiplication on sheet $r + 1$.

A spectral sequence of algebras converges to an algebra $A$ if

1. $E_r^{p,q} \Rightarrow A$
2. the filtration on $A$ is compatible with the product, and thus there is an induced multiplication on $\mathrm{Gr}(A)$
3. the product on $E_\infty$ agrees with the product on $\mathrm{Gr}(A)$.

**Theorem 7.26** (Leray-Serre Spectral Sequence)**.** *Let $f : E \to B$ be a fibration in* $\mathsf{Top}$ *such that $B$ is simply connected and the fiber $F$ is connected. Then there is a spectral sequence of algebras converging to $H^*(E; k)$ with*

$$E_2^{p,q} \cong H^p(B) \otimes H^q(F)) \Rightarrow H^*(E)$$

*As an algebra $E_2$ is the tensor product of the cohomology algebras with the cup produc.*

Explicitly the product on $E_2$ is given by $(u \otimes v) \cdot_2 (u' \otimes v') = (-1)^{|v||u'|}(u \cup u') \otimes (v \cup v')$.

*Sketch of proof.* This is highly non-trivial, in particular the mulitplication is subtle. A standard reference is Section 5 in [McC01]. The idea is to filter $B$ by dimension. So one reduces to the case where the base $B$ is a CW complex and then filters by skeleta, such that $B^n$ consists of cells up to dimension $n$. Then the preimages $f^{-1}(B^n)$ provide a filtration on $E$, and this induces a filtration on the cochains $C^*E$. As the filtration is bounded it converges to the cohomology of $C^*E$ by Theorem 7.21.

It remains to determine the $E_2$-term and work out the multiplicative structure. This is the technical heart of the matter. □

**Example 7.27.** The projection $S^2 \times S^1 \to S^2$ is a fibration with fiber $S^1$, as is the Hopf fibration $S^1 \to S^3 \to S^2$. The two spectral sequences have the same objects in $E_2$:

$$\begin{tikzcd}
q & & & \\
1 & H^0(S^2) \otimes H^1(S^1) \arrow[rrd, "?"] & 0 & H^2(S^2) \otimes H^1(S^1) \\
0 & H^0(S^2) \otimes H^0(S^1) & 0 & H^2(S^2) \otimes H^0(S^1) \\
 & 0 & 1 & 2 & p
\end{tikzcd}$$

But the differential $d^2 : E_2^{01} \cong E_2^{20}$ is 0 for the product and is an isomorphism for the Hopf fibration.

Note that we can deduce this because we know the cohomology of the total space already! Working out the differential from the topology is often difficult.

**Example 7.28.** By construction $\mathbb{C}P^n$ is a quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $\mathbb{C}^*$. Restricting to the unit sphere in $\mathbb{C}^{n+1}$ we obtain a fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. We have the following $E_2$-page.

$$\begin{tikzcd}
H^0(\mathbb{C}P^n) \arrow[drr] & H^1(\mathbb{C}P^n) \arrow[drr] & H^2(\mathbb{C}P^n) & \dots & H^{2n}(\mathbb{C}P^n) \\
H^0(\mathbb{C}P^n) & H^1(\mathbb{C}P^n) & H^2(\mathbb{C}P^n) & \dots & H^{2n}(\mathbb{C}P^n)
\end{tikzcd}$$

Where we used that $H^1(S^1) \cong k$ Of course we don't know most of the $H^i(\mathbb{C}P^n)$!

But everything is concentrated in two rows, and that means that the differential $d_2$ is the only one that does not vanish for degree reasons.

We know $E_2^{1,0}$ is 1-dimensional generated by some generator $x$ of $H^1(S^1)$. We know $H^1(S^{2n+1}) = 0$, so $d_2$ cannot kill $x$ and there is some $a \in H^2(\mathbb{C}P^n)$ with $a = d_2(x)$. Moreover $E_\infty^{0,1} = 0$, and thus $H^1(\mathbb{C}P^n)$, is also zero In $E_2^{2,1}$ we find the element $a.x$, product of the generators of $H^2(\mathbb{C}P^n)$ and $H^1(S^1)$.

If $n > 1$ then we also know $E_\infty^{2,1} \oplus E_\infty^{3,0} = 0$. This means $H^3(\mathbb{C}P^n)$ must vanish and $a.x$ maps to a nonzero element in $H^4(\mathbb{C}P^n)$. But by the Leibniz rule $a.x \mapsto d_2(a).x + a.d_2(x) = a^2$. So $a^2$ is not zero. We repeat the same computation for $a^2.x$, and this goes on until we reach $H^{2n}(\mathbb{C}P^n) \otimes H^1(S^1)$, which must be nonzero, as we know that $H^{2n+1}(S^{2n+1}) \cong k$. So $a^{n+1} = d_2(a^n.x) = 0$.

We rewrite our diagram, just writing generators.

$$\begin{tikzcd}
x \arrow[drr] & 0 & a.x \arrow[drr] & 0 & a^2.x \arrow[drr, dotted] & \dots & a^{n-1}.x \arrow[drr] & 0 & a^n.x \\
1 & 0 & a & 0 & a^2 & \dots & a^{n-1} & 0 & a^n
\end{tikzcd}$$

We find that $H^*(\mathbb{C}P^n)$ is the graded algebra $k[a]/(a^{n+1})$ with $a$ in degree 2.

## 7.7. The bar construction

While the Leray-Serre spectral sequence computes the cohomology of the total space of a fibration, the Eilenberg-Moore spectral sequence compute the cohomology of a fiber product (for example the fiber of a fibration).

Consider the following homotopy pullback square in $\mathsf{Top}$:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & E \arrow[d, two heads, "f"] \\
X \arrow[r] & B
\end{tikzcd}$$

The Eilenberg Moore spectral sequence is a tool to compute $H^*(Y, k)$ in terms of the cohomologies of the other spaces. To deduce this we have to dig a bit deeper.

Cohomology comes fom singular cochains, so we consider the following square of dg algebras of normalized cochains with coefficiens in some field $k$:

$$\begin{array}{ccc} C^*(Y) & \longleftarrow & C^*(E) \\ \uparrow & & \uparrow \\ C^*(X) & \longleftarrow & C^*(B) \end{array}$$

We would like to compute $C^*(Y)$ (up to quasi-isomorphism) in terms of $C^*(E)$, $C^*(X)$ and $C^*(B)$.

This is achieved by the following construction.

We first observe that any homomorphism of dg algebras $\phi : A \to C$ turns $C$ into a *left dg A-module*. I.e. $C$ is a dg-module over $k$ and there is an action map $A \otimes_k C \to C$ which is a morphism of dg modules over $k$ satisfying the usual assocaitivity and unital axioms for a module. The action in this case is defined by $a \otimes c \mapsto \phi(a).c$. Similarly we can define a *right dg A-module* structure by $(c \otimes a) \mapsto c.\phi(a)$

**Definition 7.29.** Let $A$ be an augmented dg algebra and let $\bar{A}$ be the kernel of the augmentation map. Let $N$ be a left dg $A$-module and $M$ a right dg $A$-module.

Recall that the shifted complex $A[1]$ is defined by $A[1]^i = A^{i+1}$ for all $i$ and $d_{A[1]} = -d_A$.

We define the *two-sided bar construction* $\mathrm{Bar}(M, A, N)$ as follows. The underlying graded $k$-module is graded complex $\oplus_{k\geq 0} M \otimes (\bar{A}[1])^{\otimes k} \otimes N$ and the differential is $d = d_{int} + d_{ext}$ obtained by adding the internal differential $d_{int}$ induced by $d_A$, $d_M$ and $d_N$, and an external differential $d_{ext} : M \otimes \bar{A}^{\otimes n} \otimes N \to M \otimes \bar{A}^{\otimes n-1} \otimes N$ induced by the multiplication on $A$ and the action on $M$ and $N$.

We will make this definition more explicit below.

We can visualise $\mathrm{Bar}(M, A, N)$ as

$$M \otimes N \leftarrow M \otimes \bar{A}[1] \otimes N \leftarrow M \otimes \bar{A}[1] \otimes \bar{A}[1] \otimes N \leftarrow \cdots$$

The unit and augmentation of $A$ together give a decomposition $A = \bar{A} \oplus \mathbb{Q}$, thus any $a \in A$ has a canonical image $[a] \in \bar{A}$.

To describe the differentials we will write $m[a_1|\dots|a_k]n$ instead of $m \otimes [a_1] \otimes \cdots \otimes [a_k] \otimes n$ for a homogeneous element in $\mathrm{Bar}(M, A, N)$.

The grading shift for $\bar{A}$ means that the degree of $m[a_1|\dots|a_k]n$ is $|m| + |a_1| + \cdots + |a_k| + |n| - k$. The last $k$ is a sum of $k$ shifts by 1.

The internal differential is given by

$$\begin{aligned} d_{int} : \quad m[a_1|\dots|a_n]n \quad & \mapsto dm[a_1|\dots|a_k]n \\ & +(-1)^{|m|} \sum (-1)^{|a_1|+\cdots+|a_{i-1}|-i} m[a_1|\dots|da_i|\dots|a_k]n \\ & +(-1)^{|m|+|n|+|a_1|+\cdots+|a_k|-k} m[a_1|\dots|a_k]dn \end{aligned}$$

and the external differential is defined by

$$\begin{aligned} d_{ext}: \quad m[a_1|\dots|a_k]n & \mapsto (-1)^{|m|} m a_1[a_2|\dots|a_k]n \\ & +(-1)^{|m|} \sum_i (-1)^{|a_1|+\dots+|a_i|-i} m[a_1|\dots|a_i a_{i+1}|\dots|a_k]n \\ & +(-1)^{|m|+|a_1|+\dots+|a_{k-1}|-k} m[a_1|\dots|a_k]a_k n \end{aligned}$$

One can check that $d_{int}^2 = d_{ext}^2 = d_{int}d_{ext} + d_{ext}d_{int}$, thus the bar construction is indeed a chain complex.

**Remark 7.30.** You are welcome to ignore the signs in this course as we will only use the bar construction as a black box. They do follow from the usual sign rules when remembering that the shift by 1 is itself an operator of degree 1.

**Remark 7.31.** Where does this definition comes from? $\mathrm{Bar}(M, A, N)$ is a model for the derived tensor product $M \otimes_A^L N$. To be precise $\mathrm{Bar}(A, A, N)$ is a cofibrant replacement of $N$ in the category of left dg $A$-modules, and then $\mathrm{Bar}(M, A, N) \simeq M \otimes_A \mathrm{Bar}(A, A, N)$.

But we would like something stronger be true: That for dg algebras $A, C, D$ with homomorphisms $A \to C$ and $A \to D$ the bar construction $\mathrm{Bar}(C, A, D)$ should be a model for the homotopy pushout $C \amalg_A^L D$. This is unfortunately not true in the category of dg algebras. (It's not even clear how the bar construction would be an algebra!)

This will be true in the category of commutative dg algebras. Of course $C^*(Y)$ is not graded commutative. But we can use this when we replace $C^*$ by the polynomial de Rham functor $A$.

## 7.8. Differential Tor and the algebraic Eilenberg Moore spectral sequence

The bar construction looks pretty intimidating, but we can compute its cohomology using the algebraic Eilenberg-Moore spectral sequence.

For simplicity we will assume that all our dg algebras and dg modules are concentrated in non-negative degrees.

**Definition 7.32.** Let $A$ be a dg algebra and $M$ and $N$ a right respectively left dg $A$-module. Then define the *differential Tor* $\mathrm{Tor}_A^i(M, N)$ as $H^i(\mathrm{Bar}(M, A, N))$.

There is a bi-grading on $\mathrm{Tor}_A^i$, i.e. we can write $\mathrm{Tor}_A^i = \oplus_{p+q=i} \mathrm{Tor}_A^{p,q}$ where $p$ denotes the piece $M \otimes \bar{A}^{\otimes -p} \otimes N$ of the bar construction. We call $p$ the *internal degree* and $q = i - p$ the *homological degree*. Note that $p$ will always be nonpositive while $q$ will by assumption be nonnegative.

**Remark 7.33.** If $A$, $M$ and $N$ are concentrated in degree 0 we have redefined the familiar Tor functor. It follows from the fact that $\mathrm{Bar}(A, A, N)$ is a free resolution of $N$ that the two definitions agree and $\mathrm{Tor}_A^i = \mathrm{Tor}_A^{i,0}$ in this case.

**Example 7.34.** Let $A$ be equal to some field $k$. Then for dg $k$-modules $M$ and $N$ we have $\mathrm{Tor}_k^*(M,N) \simeq H^*(M \otimes N)$.

**Example 7.35.** The part of internal degree 0 is just the usual tensor product: $\mathrm{Tor}_A^{0,i}(M,N) = H^i(M \otimes_A N)$.

**Theorem 7.36.** *Let $M$ and $N$ be a right and a left dg module over a dg algebra $A$ and assume moreover that $A$ is simply connected in the sense that $A^0 = k$ and $A^1 = 0$. Then there is a spectral sequence with $E_2^{pq} = \mathrm{Tor}_{HA}^{p,q}(HM, HN)$ that converges to $\mathrm{Tor}_A(M,N)$.*

*Sketch of proof.* This follows from filtering the bar construction by the homological degree of $A$, $M$ and $N$. Then we get the $E_2$-term given by the bar construction on cohomology. See Theorem 7.6 and Corollary 7.9 in [McC01].

Note that $E_2^{pq}$ will be zero unless $p \leq 0$ and $q \geq 0$. This is called a second-quadrant spectral sequence.

The assumption that $A^1 = 0$ is needed to make sure the filtration is bounded. If we have a nonzero element $a \in A^1$ then we have contributions in total degree zero from infinitely many $E^{-q,q}$. □

## 7.9. The topological Eilenberg-Moore spectral sequence

**Theorem 7.37.** *Assume the square of cochain complexes*

$$\begin{array}{ccc} C^*(Y) & \longleftarrow & C^*(E) \\ \uparrow & & \uparrow \\ C^*(X) & \longleftarrow & C^*(B) \end{array}$$

*comes from a pullback square where $E \to B$ is a fibration and $B$ is simply connected. Then there is a natural quasi-isomorphism $\mathrm{Bar}(C^*(E), C^*(B), C^*(X) \simeq C^*(Y)$ of dg modules.*

*Moreover, there is a natural structure of graded algebra on $\mathrm{Tor}_{C^*(B)}(C^*(E), C^*(X))$ and the map*

$$\mathrm{Tor}_{C^*(B)}(C^*(E), C^*(X)) \to H^*(Y)$$

*is an algebra isomorphism.*

*Proof.* This is Theorem 7.14 in [McC01]. (Note that McCleary's condition on local coefficients is automatic for simply connected spaces.) For the algebra structure see [McC01, Proposition 7.17]. □

**Theorem 7.38.** *In the homotopy pullback square above with $B$ simply connected there is a spectral sequence of algebras*

$$E_2 = \mathrm{Tor}_{H^*B}(H^*X, H^*E) \Rightarrow H^*(Y)$$

*that converges strongly.*

*Proof.* This follows from Theorem 7.36 together with Theorem 7.37. For the algebra structure see Corollary 7.18 in [McC01]. □

To make things more accessible I describe an example. We will however only use the Eilenberg-Moore spectral sequence as a black box: The fact that it exists will allow us to prove theorems, even if we never compute it for a concrete fibre product!

**Example 7.39.** Consider the loop space $\Omega S^n$ of a sphere with $n \geq 2$. We know this is the homotopy pullback of $* \leftarrow S^n \rightarrow *$, so by Theorem 7.38 there is a spectral sequence:

$$\mathrm{Tor}_{H^*(S^n)}(k, k) \Rightarrow H^*(\Omega S^n, k)$$

We use the bar construction and note that the augmentation ideal of $A = H^*(S^n)$ is just $k[-n]$. Thus the piece $k \otimes \bar{A}[1]^{\otimes -p} \otimes k$ of the bar construction is just a copy of $k$ in bidegree $(p, -np)$. (Notel that $p < 0$.)

It is easy to see there is no room for differentials and $H^k(\Omega S^n, k) \cong k$ iff $k = -p(n-1)$ and 0 otherwise.

You may guess now that the algebra structure is just the polynomial algebra on a generator in degree $n - 1$. This is correct over $k$.

Note that the computation up to here remains correct if $k$ is no longer a field. But over $\mathbb{Z}$ the multiplication is given by a *divided power algebra* with multiplication $x^{[p]}.x^{[q]} = \binom{p+q}{p} x^{[p+q]}$.

# 8. Some homotopy theory

## 8.1. Fibrations

Recall from Example 3.5 that fibration of topological spaces is a continuous map $f : X \to Y$ which has the right lifting property with respect to all inclusion $D \times \{0\} \to D \times I$ for $D$ a CW complex.

**Example 8.1.** We say that $f : X \to Y$ is a *fibre bundle* with fibre $F$ if there is some open cover $\{U_i\}$ of $Y$ such that for each $U_i$ and $E_i := f^{-1}(U_i)$ there is a homeomorphism $h_i : E_i \cong U_i \times F$ such that $pr_1 \circ h_i \cong f|_{E_i}$. We may express this as saying $X$ is locally a product. Any fibre bundle is a fibration as long as $Y$ is nice enough (for example paracompact). For a proof see Section 5.4 of [May99].

**Theorem 8.2.** *Let $f : X \to Y$ be a fibration. Fix points $x \in X$ and $y \in Y$ with $y = f(x)$ and let $F = f^{-1}(\{y\})$ be the fibre over $y$.*

*Then there is a long exact sequence of homotopy groups (and pointed sets)*

$$\cdots \pi_2(X, x) \to \pi_2(Y, y) \to \pi_1(F, x) \to \pi_1(X, x) \to \pi_1(Y, y) \to \pi_0(F) \to \pi_0(X) \to \pi_0(Y)$$

*Proof.* See Section 8.6 and 9.3 of [May99], or any other book on homotopy theory. □

We note that the homotopy type of the fiber $F$ does not depend on the choice of $y$ if $Y$ is path-connected.

Since an arbitrary map $f : X \to Y$ of topological spaces may be replaced by a fibration $\tilde{f} : \tilde{X} \to Y$ with $\tilde{X} \simeq X$ we always have a long exact sequence as above where $F$ has to be replayed by the fiber of $\tilde{f}$, i.e. the homotopy fiber of $f$, see Definition 3.47.

As in Example 3.48 the homotopy fiber of $y : * \to Y$ is the *loop space* of $Y$, written $\Omega_y(Y)$. It has a natural basepoint given by the constant path $c_y$.

**Corollary 8.3.** *For any $i \geq 0$ we have a group homomorphism $\pi_i(\Omega_y Y, c_y) \cong \pi_{i+1}(Y, y)$*

*Proof.* This follows from the long exact sequence in Theorem 8.2 applied to the path fibration over $Y$. The fact that the bijection is a group homomorphism for $i = 0$ is immediate from the definitions. □

## 8.2. Eilenberg-MacLane spaces

**Definition 8.4.** For any abelian group $A$ and integer $n \geq 1$ as well as for any group $A$ if $n = 1$ we call a pointed space $K(A, n)$ with the property that $\pi_i(K(A, n), *) \cong A$ if $i = n$ and trivial otherwise an *Eilenberg-MacLane space*.

**Example 8.5.** 1. $S^1$ is a $K(\mathbb{Z}, 1)$.

2. $S^1_{\mathbb{Q}}$ as in Example 1.9 is a $K(\mathbb{Q}, 1)$.

3. Infinite real projective space $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$. This can be shown using the fibration exact sequence from Theorem 8.2 together with writing $\mathbb{R}P^n$ as the quotient of the contractible space $S^\infty$ by $\mathbb{Z}/2$.

4. Infinite complex projective space $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$. This can be shown by writing $\mathbb{C}P^n$ as the quotient of $S^\infty$ by $S^1$.

**Theorem 8.6.** *For any abelian group A and integer $n \geq 1$ as well as for any group A if $n = 1$ a space $K(A, n)$ exists, unique up to homotopy equivalence.*

*Moreover the space of homotopy classes of pointed maps $[K(A, n), K(B, n)]$ is in natural bijection with the group homomorphisms from A to B.*

*Instead of a proof.* This is a guided exercise in Chapter 15 of [May99]. □

It follows immediately from Corollary 8.3 that $K(A, n) \simeq \Omega K(A, n + 1)$.

**Remark 8.7.** One of the most remarkable properties of Eilenberg-MacLane spaces is the following: For any CW complex $X$ we have $[X, K(A, n)] \cong H^n(X, A)$.

## 8.3. Cohomology of Eilenberg-MacLane Spaces

We will have to know the rational cohomology of $K(\mathbb{Q}, n)$ for the main construction. We will proceed by induction and start with:

**Lemma 8.8.** $K(\mathbb{Q}, 1) = S^1_{\mathbb{Q}}$.

*Proof.* See Example 1.9. □

For our inductive step we will consider the path fibration $\Omega K(\mathbb{Q}, n) \to \mathscr{P}K(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$, where we note that $\mathscr{P}K(\mathbb{Q}, n) \cong *$ implies that $\Omega K(\mathbb{Q}, n) \cong K(\mathbb{Q}, n-1)$ by considering the long exact sequences of homotopy groups.

**Lemma 8.9.** *The cohomology ring of $K(\mathbb{Q}, n)$ over $\mathbb{Q}$ is $\mathbb{Q}\langle x\rangle$ with* $\deg x = n$.

*Proof.* The result is clear for $K(\mathbb{Q}, 1)$ by the previous lemma. We will use the Leray-Serre spectral sequence for the inductive step, see Theorem 7.26. We assume the result holds for $K(\mathbb{Q}, 2n-1)$ and consider the Leray-Serre spectral sequence for $K(\mathbb{Q}, 2n-1) \to * \to K(\mathbb{Q}, 2n)$. Clearly the fibre is connected and the base is simply connected.

We assume that the only rational cohomology of $K(\mathbb{Q}, 2n - 1)$ is $\mathbb{Q}$ in dimension 0 and $\mathbb{Q}\alpha$ in dimension $2n - 1$. So there are only two nonzero rows, and only the differential $d_{2n}$ can be nontrivial. But the total space has trivial cohomology, so $d_{2n} : E_{2n}^{0,2n-1} \to E_{2n}^{2n,0}$ must be an isomorphism, $E_{2n}^{2n,0} = H^{2n}(K(\mathbb{Q}, 2n), \mathbb{Q})$ is generated by $\beta = d_{2n}\alpha$. But then $E_{2n}^{2n,2n-1} = \mathbb{Q}\alpha\beta$ and there is another isomorphism $d_{2n} : \alpha\beta \mapsto \beta^2$ since $d_{2n}\beta = 0$. Now proceed by induction.

$$
\begin{array}{c|ccccccc}
2n-1 & \mathbb{Q}\alpha & & \mathbb{Q}\alpha\beta & & \cdots & \\
 & & \searrow^{d_{2n}} & & \searrow^{d_{2n}} & & \\
0 & \mathbb{Q} & & \mathbb{Q}\beta & & \mathbb{Q}\beta^2 & \cdots \\
\hline
 & 0 & & 2n & & 4n &
\end{array}
$$

Next assume that the rational cohomology of $K(\mathbb{Q}, 2n)$ is the polynomial algebra in one generator of dimension $2n$. By the same argument as above $d_{2n} : E_{2n+1}^{0,2n} \to E_{2n+1}^{2n+1,0}$ is an isomorphism, say $d_{2n+1} : \beta \mapsto \alpha$. Then $d_{2n+1} : E_{2n+1}^{0,4n} \to E_{2n+1}^{2n+1,2n}$ is given by $\beta^2 \mapsto \alpha\beta + \beta\alpha = 2\alpha\beta$, and this is an isomorphism on $\mathbb{Q}$. So $E_{2n+1}^{4n+2,0} = 0$ since otherwise it would live to $E^\infty$, contradicting the triviality of the total space. Again the proof finishes by induction.

$$
\begin{array}{c|ccccc}
4n & \mathbb{Q}\beta^2 & & \vdots & & \vdots \\
 & & \searrow^{d_{2n+1}} & & & \\
2n & \mathbb{Q}\beta & & \mathbb{Q}\alpha\beta & & 0 \\
 & & \searrow^{d_{2n+1}} & & \searrow^{0} & \\
0 & \mathbb{Q} & & \mathbb{Q}\alpha & & 0 \\
\hline
 & 0 & & 2n+1 & & 4n+2
\end{array}
$$

□

## 8.4. Nilpotent spaces

One often restricts attention to simply connected spaces in rational homotopy theory. However, this is not necessary. There are genuine difficulities dealing with arbitrary fundamental groups, so we have to impose some condition.

The condition depends not just on the fundamental group, but also on its action on the higher homotopy groups.

One way of defining this action is to use the universal covering space $\tilde{X}$ and then $\pi_1(X)$ acts by deck transformations on $\tilde{X}$, which induces an action on $\pi_n(\tilde{X}) \cong \pi_n(X)$.

**Definition 8.10.** A connected space $X$ is *nilpotent* if its fundamental group is nilpotent and acts nilpotently on all higher homotopy groups.

Recall from algebra that a group $G$ is *nilpotent* if the lower central series terminates. That is, inductively defining $G_{i+1} = [G, G_i]$ starting with $G_0 = G$ we reach $G_n = \{e\}$ after finitely many steps. Nilpotent groups are not too far from being abelian, they can be obtained as iterated central extensions of abelian groups.

**Example 8.11.** Consider the group $N_n(\mathbb{Z})$ of strictly upper triangular integer matrices, i.e. $n \times n$-matrices with integer coefficients, ones on the diagonal and zeros below the diagonal. This group is nilpotent for all $n$.

A group action on an abelian group $A$ is called *nilpotent* if, the series defined by letting $A_{i+1}$ be the submodule generated by $\{ga - a \mid g \in G, a \in A_i\}$ with $A_0 = A$, terminates with $A_m = \{0\}$ for some $m$.

**Example 8.12.** 1. Any simply connected space is trivially nilpotent.

2. The circle $S^1$ is nilpotent as $\pi_1(S^1, 1)$ is abelian and all higher homotopy groups are 0.

3. The wedge sum of two circles $S^1 \vee S^1$ is not nilpotent as the commutator subgroup of $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$ is an infinity generated free group! And things don't get better from there.

4. $\mathbb{R}P^2$ is not a nilpotent space! The fundamental group is abelian and thus nilpotent. But its action on the universal cover $S^2$ is given by the antipodal map, wich has degree $-1$ for an even-dimensional sphere. Thus with $A_0 = \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}$ we have $A_1$ generated by $-2\mathbb{Z} \subset \mathbb{Z}$ and $A_1$ and every successive $A_n$ is isomorphic to $\mathbb{Z}$.

## 8.5. Postnikov towers

There is an equivalent way of describing nilpotent spaces in terms of *Postnikov towers*. This is the description we will use in proving our main theorem.

The proof of the equivalence is quite non-trivial and beyond the scope of this course. I think this goes back to [BK72], but a very nice presentation is in Section 3 of [MP11].) If you don't want to take this result on trust you can check the proof (that's an excellent long exercise in algebraic topology), or just use Theorem 8.19 below as your *definition* of a nilpotent space.

We work in the pointed categories $\mathsf{sSet}_*$ and $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$ of pointed spaces and augmented cdgas respectively.

**Definition 8.13.** A *Postnikov tower* for a pointed connected space $X$ is given by an inverse system of pointed spaces $\ldots \to X_n \to X_{n-1} \to \ldots \to X_1 \to X_0$ (called the *sections* of the Postnikov tower) with compatible maps $\phi_n : X \to X_n$ such that

- $X$ is the limit of the $X_n$,
- $X_0 = *$,
- every map $X_n \to X_{n-1}$ is a fibration,
- $(\phi_n)_* : \pi_i(X) \cong \pi_i(X_n)$ if $i \leq n$
- $\pi_i(X_n) \cong 0$ if $i > n$.

It follows from the long exact sequence of homotopy groups that the homotopy fiber of $X_n \to X_{n-1}$ is a $K(\pi_n(X), n)$. (In fact, this is equivalent to the conditions on homotopy groups.)

**Remark 8.14.** Postnikov towers are probably much less familiar to you than cell complexes, but they can be thought of as the dual notion: A CW complex consists of cofibrations $X_{n-1} \to X_n$ inducing isomorphisms in a range of cohomology groups and $X = \operatorname{colim} X_n$.

This interpretation relies on the idea that cohomology is dual to homotopy. It helps to view homotopy groups as $[S^n, X]$ and cohomology as $[X, K(\mathbb{Z}, n)]$.

So why are $S^n$ and $K(\mathbb{Z}, n)$ dual? We have $S^{n+1} = \Sigma S^n$, where the suspension is the homotopy colimit of the diagram $* \leftarrow S^n \rightarrow *$. On the other hand $K(\mathbb{Z}, n-1) = \Omega K(\mathbb{Z}, n)$ where the loop space is the homotopy limit of the diagram $* \rightarrow K(\mathbb{Z}, n) \leftarrow *$.

**Remark 8.15.** The limit of an inverse system of CW complexes is not necessarily a CW complex, but we may always replace it by a CW complex up to weak equivalence.

Postnikov towers exist for all connected CW complexes, but they are particularly useful if the maps $X_n \rightarrow X_{n-1}$ take a nice form.

**Definition 8.16.** A map $f : X \rightarrow Y$ is a *principal fibration* if there exists an abelian group $A$, a positive integer $i \geq 2$ and a map $k_i : Y \rightarrow K(A, i)$ such that $f$ is the pull-back of the path fibration $\mathscr{P}K(A, i) \rightarrow K(A, i)$ along $k_i$. The map $k_i$ is called the *k-invariant*.

**Remark 8.17.**

1. More concisely one can say that $X$ is the homotopy fibre of the $k$-invariant.
2. By the long exact sequence for a fibration (Theorem 8.2) the fiber of a principal fibration is $\Omega K(A, i) \cong K(A, i-1)$.
3. It is interesting to note that with the description $H^n(Y, A) = [Y, K(A, n)]$ the $k$-invariant is nothing but a cohomology class in $H^n(Y, A)$. Thus we can construct principal fibrations over $Y$ by writing down cohomology classes!

**Definition 8.18.** A *principal refinement* for a Postnikov tower $\{X_n\}$ consists of factorizations $X_{n+1} = Y_{n,i_n} \rightarrow Y_{n,i_n-1} \rightarrow \ldots \rightarrow Y_{n,1} \rightarrow Y_{n,0} = X_n$ such that each $Y_{n,j+1} \rightarrow Y_{n,j}$ is a principal fibration induced by a $k$-invariant $k_{n,j} : Y_{n,j} \rightarrow K(A_j, n+2)$:.

**Theorem 8.19.** *A simplicial set X is nilpotent if and only if its Postnikov tower has a principal refinement up to homotopy.*

*Proof.* This is Theorem 3.2.2 in [MP11]. The proof takes up the rest of section 3 in loc. cit.!

A hint as to why these concepts have anything to do with each other: If $X \rightarrow Y$ is a principal fibration then the action of $\pi_1(Y)$ on the homotopy groups of the fiber must be trivial as it is pulled back from the action of $\pi_1(K(A, n))$ on the total space, which is trivial.

A clever argument using a surprising spectral sequence allows one to show that this is the only obstacle to finding principal fibrations. The key is to construct the class in $H^n(Y, A)$ that is represented by the $k$-invariant. $\square$

# 9. The Quillen-Sullivan equivalence

## 9.1. Finiteness conditions

To consider an equivalence of suitable rational spaces and dg algebras we need to make some assumptions on finiteness.

Recall that a connected topological space $X$ is rational if $H_n(X,\mathbb{Z})$ is a $\mathbb{Q}$-vector space for every $n \geq 1$.

**Definition 9.1.** A nilpotent connected rational space $X$ is *of finite* $\mathbb{Q}$*−type* if each map $X_n \to *$ as in Definition 8.13 factors (up to homotopy) into a finite composition of principal fibrations with fibre $K(\mathbb{Q}, i)$, $1 \leq i \leq n$.

It follows from this definition that all the homotopy groups (in degree at least 2) are finite-dimensional $\mathbb{Q}$-vector spaces, and in the finite lower central series of $G = \pi_1(X)$ all the quotients $G_i/G_{i+1}$ are finite-dimensional $\mathbb{Q}$-vector spaces.

Of course all these definitions work equally well for simplicial sets. A connected rational nilpotent simplicial set of finite $\mathbb{Q}$-type is a simplicial set whose realization is a nilpotent connected rational space of finite $\mathbb{Q}$-type.

**Definition 9.2.** A homologically connected cofibrant algebra $B \in \mathrm{Ob}(\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}})$ is *of finite* $\mathbb{Q}$*-type* if its minimal model $M$ is finite-dimensional in each degree.

One can check that this definition is satisfied iff $\pi^n B$ is finite-dimensional over $\mathbb{Q}$ for all $n$.

## 9.2. Statement of the Main Theorem

We can now introduce our protagonists.

**Definition 9.3.** Let $\mathbb{Q}\mathsf{Nil}^{\mathrm{ft}}_*$ denote the full subcategory of $\mathsf{sSet}_*$ whose objects are pointed connected, nilpotent, rational Kan complexes of finite $\mathbb{Q}$-type. We define $\mathrm{Ho}(\mathbb{Q}\mathsf{Nil}^{\mathrm{ft}}_*)$ as the category whose objects are those of $\mathbb{Q}\mathsf{Nil}^{\mathrm{ft}}_*$ and whose morphisms are homotopy classes of maps.

**Definition 9.4.** Let $\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}}$ denote the full subcategory of $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$ whose objects are cofibrant homologically connected algebras of finite $\mathbb{Q}$-type. We define the $\mathrm{Ho}(\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}})$ as the category whose objects are those of $\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}}$ and whose morphisms are homotopy classes of maps.

Note that $\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$ and $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}$ are not model categories, but we have a notion of homotopy from the ambient category and can talk about their homotopy categories. (In fact the cylinder and path objects $K\times\mathrm{Sing}(I)$ and $\Omega(1)\tilde{\otimes}B$ are in $\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$, respectively $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}$, if $K$, respectively $B$, are.)

We are now ready to state the main theorem.

**Theorem 9.5.** *The Quillen adjunction $F : \mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0} \rightleftarrows \mathsf{sSet}_*^{op} : A$ induces an equivalence of categories*

$$LF : \mathrm{Ho}(\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}) \cong \mathrm{Ho}(\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}})^{op} : A.$$

*The same is true for the adjunction of unpointed categories.*

This completes the promised algebraic characterisation of rational homotopy theory.

**Corollary 9.6.** *There is an equivalence between homotopy classes of rational nilpotent connected CW-complexes of finite $\mathbb{Q}$-type and isomorphism classes of minimal models of finite $\mathbb{Q}$-type.*

*Proof.* By the Quillen equivalence from Theorem 5.26 we may interchange simplicial sets and topological spaces. Theorem 9.5 identifies homotopy classes of objects in $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}$ and $\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$. Every object in $\mathrm{Ho}(\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}})$ may be uniquely represented by its minimal model by Theorems 2.21 and 4.11. □

## 9.3. Strategy of proof

We restrict ourselves to the proof of the pointed version of the theorem, the unpointed version then follows easily. The proof proceeds by "induction up a Postnikov tower".

We want to show that on the category $\mathrm{Ho}(\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}})$ the unit of the adjunction $X \mapsto LFA(X)$ is a an isomorphism in the homotopy category, so it is a weak equivalence in the ordinary category. The desired property is first established for rational Eilenberg-Mac Lane spaces, then it is shown to be preserved under the pull-backs which are used to construct the Postnikov tower of an arbitrary connected nilpotent rational space. The last step is to show that if the property holds for all Postnikov sections of a space, it holds for the space itself.

The dual procedure can be used in the other direction to show we have quasi-isomorphisms $B \to ALF(X)$.

The construction dual to the Postnikov tower of a space is the decomposition of a minimal model into successive cofibrations.

**Example 9.7.** We consider an example for the duality of Postnikov tower and decomposition of the minimal model. We showed in Example 2.18 that the rational 2-sphere $S^2_{\mathbb{Q}}$ has minimal model $M = \mathbb{Q}\langle x, y \mid dy = x^2\rangle$ where $\deg x = 2, \deg y = 3$. So the decisive step in the construction leads from $\mathbb{Q}\langle x\rangle$ to $M$. We have the following push-out square (compare diagram 4.1).

$$\begin{tikzcd}
\mathbb{Q}\langle w\rangle \arrow[r, "f"] \arrow[d, hook, "\theta"] & \mathbb{Q}\langle x\rangle \arrow[d] \\
\mathbb{Q}\langle y, dy\rangle \arrow[r] & M \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

Where $\deg w = 4$ and we have the maps $f : w \mapsto x^2$ and $\theta : w \mapsto dy$. This gives $M$ as above. Sometimes $f$ is called a *Hirsch extension*.

Dually we have the first nontrivial Postnikov section $(S^2_{\mathbb{Q}})_2 = K(\mathbb{Q}, 2)$, whose minimal model is $\mathbb{Q}\langle x\rangle$, and dually to the map $\theta : \mathbb{Q}\langle w\rangle \to \mathbb{Q}\langle y, dy\rangle$ we have the path fibration $\pi : \mathscr{P}K(\mathbb{Q}, 4) \to K(\mathbb{Q}, 4)$.

We get the following pull-back, where $k$ is the second $k$-invariant of the Postnikov tower for $S^2_{\mathbb{Q}}$.

$$\begin{tikzcd}
S^2_{\mathbb{Q}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{P}K(\mathbb{Q}, 4) \arrow[d, two heads, "\pi"] \\
(S^2_{\mathbb{Q}})_2 \arrow[r, "k"] & K(\mathbb{Q}, 4)
\end{tikzcd}$$

## 9.4. Base case

Let us now begin working towards the proof of the main theorem. As the functor $F$ needs to be derived we will cofibrantly replace $AX$ by its minimal model $e_X : MX \to AX$. (As $X$ is connected $AX$ will be homologically connected.)

From the units $\epsilon_X, \eta_B$ of our adjunction $F : \mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}} \leftrightarrow (\mathsf{sSet}_*)^{\mathrm{op}} : A$ we construct maps $\phi : X \to FMX$ and $\psi_B : B \to MFB$ as follows:

First $\phi_X$ is the composite

$$X \xrightarrow{\epsilon_X} FAX \xrightarrow{Fe_X} FMX$$

and $\psi_B$ is any lift in the following diagram.

$$\begin{tikzcd}
 & MFB \arrow[d, "e_{FB}"] \\
B \arrow[r, "\eta_B"] \arrow[ur, dashed, "\psi_B"] & AFB
\end{tikzcd}$$

By Lemma 3.18 the lift exists as $B$ is cofibrant. $A \dashv LF$ provides an adjunction between the homotopy categorie of connected pointed spaces and homologically connected objects in $\mathsf{cdgA}^{\geq 0}_{/\mathbb{Q}}$. The maps $\phi_X$ and $\psi_B$ are (co)units of the adjunction on homotopy categories.

**Lemma 9.8.** *If $X = K(\mathbb{Q}, n)$ then $MX$ is in $\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}}$ and $\phi_X : X \to FMX$ is an isomorphism in* $\mathrm{Ho}(\mathbb{Q}\mathsf{Nil}^{\mathrm{ft}}_*)$

*Proof.* It follows from Lemma 8.9 that $AX$ has a minimal model $\mathbb{Q}\langle x_n \rangle$, the weak equivalence is given by mapping $x_n$ to a generator of the de Rham cohomology ring. This proves the first part of the lemma.

Next it follows from Theorem 6.19 that for $i \geq 1$

$$\pi_i(FMX) \cong Hom_{\mathbb{Q}}(\pi^i MX, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

So we know that $FMX$ is an Eilenberg-MacLane space. We still have to show that the map $\phi_X$ is a weak equivalence. But we know that $M\phi_X$ has a right inverse in the homotopy category as it comes from an adjunction and satisfies $\psi_{MX} \circ M\phi_X \simeq \mathbf{1}_{MX}$. This shows $H(\phi_X)$ has a right inverse and $\phi_X$ is surjective on cohomology.

But the cohomology rings of $X$ and $FMX$ are generated by a single generator in degree $n$, so the map on cohomology is determined by what happens in degree $n$. Any surjection $\mathbb{Q} \to \mathbb{Q}$ is an isomorphism, so we have an endomorphism of $X$ inducing an isomorphism on cohomology. By the universal coefficient theorem together with Hurewicz' Theorem 9.9 this is also an isomorphism on $\pi_n$. By Theorem 8.6 this must be a homotopy equivalence between $X$ and $FMX$. □

**Theorem 9.9** (Hurewicz' Theorem)**.** *There is a natural homomorphism $h_n : \pi_n(X, *) \to H_n(X, \mathbb{Z})$ which is an isomorphism if $\pi_i(X, *) = *$ for i $< n$ if $\pi_n(X, *)$ abelian.*

*Proof.* A nice proof using the Leray-Serre spectral sequence can be found in [BT82, Theorem 17.21]. □

The proof of the dual version of Lemma 9.8 follows similar ideas and can be found in [BG76, §10.3].

**Lemma 9.10.** *If $B = \mathbb{Q}\langle x \rangle$ with $|x| \geq 1$ then $FB$ is in $\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$ and $\psi_B : B \to AFB$ is an isomorphism in* $\mathrm{Ho}(\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}})$.

## 9.5. Pullback squares

We will now analyze what the functor $A$ does to a pull-back square. We recall that it is easy to see $A$ takes colimits to limits, but the other direction only holds with serious restrictions and is one of the hardest inputs in the main theorem.

We first recall the bar construction from Definition 7.29 and quote the following lemma that we hinted at in Remark 7.31.

**Lemma 9.11.** *Let $C \leftarrow A \to D$ be a diagram in* $\mathsf{cdgA}$ *with $A$ and $C$ cofibrant. Then* $\mathrm{Bar}(C, A, D)$ *with its natural product structure is a model for the homotopy pushout $C \otimes_A^L D$ of the diagram.*

*Proof.* This is Corollary 4.6 in [Ols16] (set $A = k$ in the statement of the corollary). There is some notation to unravel, but the proof is elementary (at the level of this course). □

**Remark 9.12.** The product on $\mathrm{Bar}(C, A, D)$ is given by the so-called *shuffle product* defined by $c[a_1|\dots|a_p]d \otimes c'[a_{p+1}|\dots|a_{p+q}]d' \mapsto \sum_{\sigma \in Sh(p,q)} \pm cc'[a_{\sigma(1)}|\dots|a_{\sigma(p+q)})]dd'$ where $Sh(p, q) \subset S_{p+q}$ denotes the set of all shuffles, i.e. permutations preserving the relative order of the first $p$ elements and the last $q$ elements.

The same product is used for the Eilenberg-Moore spectral sequence.

**Lemma 9.13.** *Let $E$ be the pullback of $Y \to X \leftarrow Z$ in $\mathsf{sSet}$, where $X$ is simply connected and all objects are connected of finite $\mathbb{Q}$-type. Then the natural map $AY \otimes^L_{AX} AZ \to AE$ is a weak equivalence.*[1]

*Proof.* We recall first that if we apply the singular cochain functor $C^*$ instead of $A$ then we have
$$\mathrm{Bar}(C^*(Y), C^*(X), C^*(Z)) \simeq C^*(E)$$
by Theorem 7.37.

We replace all polynomial de Rham algebra cofibrantly by their minimal models.

Moreover we know by Theorem 6.16 that there is a zig-zag of natural quasi-isomorphisms $\tau : M \leftrightarrow C^*$ . So we have the following situation:

$$\begin{tikzcd}
 & MX \arrow[rr] \arrow[dl, "\tau_X"] \arrow[dd] & & MY \arrow[dl, "\tau_Y"] \arrow[dd] \\
C^*X \arrow[rr] \arrow[dd] & & C^*Y \arrow[dd] & \\
 & MZ \arrow[rr] \arrow[dl, "\tau_Z"] & & ME \arrow[dl, "\tau_E"] \\
C^*Z \arrow[rr] & & C^*E &
\end{tikzcd}$$

Here all $\tau_{(-)}$ are zig-zags of quasi-isomorphisms and the front square satisfies $\mathrm{Bar}(C^*(Y), C^*(X), C^*(Z)) \simeq C^*(E)$. Next we claim that $\tau$ induces a zig-zag of weak equivalences
$$\mathrm{Bar}(C^*(Y), C^*(X), C^*(Z)) \simeq \mathrm{Bar}(MY, MX, ME)$$
By functoriality of Bar there are certainly zig-zags of morphisms and we only have to compare cohomology.

But both sides of the equation are computed by the algebraic Eilenberg-Moore spectral sequence from Theorem 7.36. In fact, the maps induced by $\tau$ are compatible with the filtrations and we obtain a zig-zag of maps of spectral sequences. But on the $E_2$-term $\mathrm{Tor}_{HX}(HY, HZ)$

[1] The same result holds for pointed spaces and the homotopy pushout of augmented cdga's.

and $\mathrm{Tor}_{HMX}(HMY, HMZ)$ are isomorphic by Theorem 6.16 , and by Theorem 7.24 this shows the desired equivalence of bar constructions.

Together with Lemma 9.11 and the fact that $AE \simeq C^*E$ we have proven the theorem as

$$\begin{aligned} AY \otimes^L_{AX} AZ \simeq MY \otimes^L_{MX} MZ &\simeq \mathrm{Bar}(MY, MX, ME) \\ &\simeq \mathrm{Bar}(C^*(Y), C^*(X), C^*(Z)) \simeq C^*(E) \simeq AE \end{aligned}$$

. □

**Corollary 9.14.** *For connected spaces of finite $\mathbb{Q}$-type we have $A(X \times Y) \simeq A(X) \otimes A(Y)$.*

**Remark 9.15.** It's also possible to avoid talking about homotopy pushouts of algebras here and just consider various cohomology computations. This is the path taken in [BG76].

## 9.6. Inductive step for algebras

The inductive step for algebras is a little easier, so we start here.

**Lemma 9.16.** *Consider the following push-out square in $\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}}$ where $|x| \geq 2$:*

$$\begin{tikzcd} \mathbb{Q}\langle x\rangle \arrow[r] \arrow[d, hook, "\theta"] & B \arrow[d] \\ \mathbb{Q}\langle y, dy\rangle \arrow[r] & N \end{tikzcd}$$

*Here $\theta$ is the cofibration defined earlier. If $FB \in \mathrm{Ob}(\mathbb{Q}\mathsf{Nil}^{\mathrm{ft}}_*)$ and $\epsilon_B : B \simeq AFB$ then $FN \in \mathrm{Ob}(\mathbb{Q}\mathsf{Nil}^{\mathrm{ft}}_*)$ and $\epsilon_N : N \simeq AFN$.*

*Proof.* We carry the identity maps on our diagram across the adjunction to obtain

$$\begin{tikzcd} AF\mathbb{Q}\langle x\rangle \arrow[rrr] \arrow[ddd, "AF(\theta)"'] & & & AFB \arrow[ddd] \\ & \mathbb{Q}\langle x\rangle \arrow[r] \arrow[d] \arrow[ul, "\epsilon_x"'] & B \arrow[d] \arrow[ur, "\epsilon_B"] & \\ & \mathbb{Q}\langle y, dy\rangle \arrow[r] \arrow[dl, "\epsilon_y"] & N \arrow[dr, "\epsilon_N"'] & \\ AF\mathbb{Q}\langle y, dy\rangle \arrow[rrr] & & & AFN \end{tikzcd}$$

and by Lemma 9.10 and our assumption we have weak equivalences $\epsilon_x$, $\epsilon_y$ and $\epsilon_B$. It will follows that $\epsilon_N$ is a weak equivalence if we can show that both the inner and outer square are homotopy colimits.

But the inner square is a homotopy colimit by assumption and it follows that $FN$ is a homotopy pullback. Since $F\mathbb{Q}\langle x\rangle$ is simply connected we can apply Lemma 9.13 to deduce that $AFN$ is a homotopy pushout and we have shown $N \simeq AFN$.

To see that $FN \in \mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$ we observe that it is the homotopy fiber of $FB \to F\mathbb{Q}\langle x\rangle$, so this is immediate from Definition 9.1. □

## 9.7. Inductive step for spaces

Now we perform the second part ouf our inductive step.

**Lemma 9.17.** *Consider the following pullback square in $\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$ where $n \geq 2$.*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & P \arrow[d, two heads, "\pi"] \\ Y \arrow[r, "k"] & K(\mathbb{Q}, n) \end{tikzcd}$$

*Here $P = \mathscr{P}K(\mathbb{Q}, n) \simeq *$ and $\pi$ is the usual path fibration.*

*If $MY$ is in $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}$ and $\phi_Y : Y \simeq FMY$ then $MX$ is in $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}$ and $\phi_X : X \simeq FMX$.*

*Proof.* We want to apply $A$ to the given square and then use Lemma 9.13 and properties of $AK(\mathbb{Q}, n), AP$ and $AY$ to deduce properties of $AX$. The image under $A$ is not in general cofibrant, so we need to find suitable cofibrant replacements for all objects.

$$\begin{tikzcd} \mathbb{Q}\langle x\rangle \arrow[rrr] \arrow[dr, "m"] \arrow[ddd, "\theta"'] & & & S \arrow[dl, "s"'] \arrow[ddd] \\ & AK(\mathbb{Q}, n) \arrow[r, "Ak"] \arrow[d] & AY \arrow[d] & \\ & AP \arrow[r] & AX & \\ \mathbb{Q}\langle y, dy\rangle \arrow[rrr] \arrow[ur, "n"] & & & R \arrow[ul, "r"'] \end{tikzcd}$$

In this diagram $\theta : x \mapsto dy$ as before, all objects in the outer square are cofibrant and all the diagonal arrows are weak equivalences.

We construct it as follows. Since we know the minimal model for $K(\mathbb{Q}, n)$ and since $P$ is contractible we get the quadrilateral on the left. We then factor $Ak \circ m : \mathbb{Q}\langle x\rangle \to AY$ as a cofibration followed by a weak equivalence. Finally let $R$ be the push-out of the outer square. Since $\theta$ is a cofibration, so is its pushout $S \to R$ and thus $R$ is cofibrant. The map $r$ exists by the universal property of a push-out.

As $\theta$ is a cofibration the outer square is also a homotopy pushout. The inner square is a homotopy pushout by Lemma 9.13. It follows from Corollary 3.51 that $r$ is a weak equivalence since $m$, $n$ and $s$ are.

So we have $R \simeq AX \simeq MX$ and to show $MX$ is in $\mathrm{Ob}(\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}})$ we show that $R \in \mathrm{Ob}(\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}})$. It suffices to show that all the homotopy groups of $R$ are finite-dimensional over $\mathbb{Q}$ and this follows from Lemma 6.21, using that $S \simeq MY$ and $\mathbb{Q}\langle x\rangle$ are in $\mathrm{Ob}(\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}})$.

Next we form the adjoint of our diagram.

$$\begin{tikzcd}
F\mathbb{Q}\langle x\rangle & & & FS \arrow[lll] \\
& K(\mathbb{Q},n) \arrow[ul, "m^\#"'] & Y \arrow[l] \arrow[ur, "s^\#"'] & \\
& \mathscr{P} \arrow[u] \arrow[dl, "n^\#"'] & X \arrow[l] \arrow[u] \arrow[dr, "r^\#"] & \\
F\mathbb{Q}\langle y, dy\rangle \arrow[uuu, "\theta^\#"] & & & FR \arrow[lll] \arrow[uuu]
\end{tikzcd}$$

We claim that $\phi_X : X \to FMX$ is a weak equivalence in $\mathsf{sSet}_*$ if and only if $r^\# : X \to FR$ is. To show this recall that $r^\# = Fr \circ \epsilon_X$ and $\phi_X = Fe_X \circ \epsilon_X$. Moreover as $R$ is cofibrant there is a weak equivalence $f : R \to MX$ compatible with the maps $r$ and $e_X$, and by Lemma 3.35 this induces a weak equivalence $Ff : FMX \to FR$. Thus we have $Ff \circ \phi_X = r^\#$ and the claim follows from 2-out-of-3.

Moreover $s^\#$ is a weak equivalence since $\phi_Y$ is (by assumption), $n^\#$ is trivially a weak equivalence, and $m^\# = \phi_{K(\mathbb{Q},n)}$ is a weak equivalence by Lemma 9.8.

By assumption $X$ is a homotopy pullback, and as $R$ is a homotopy pushout and $LF$ sends homotopy colimits to homotopy limits we see that $FR$ is also a homotopy pullback. So by Corollary 3.51 we have $FR \simeq X$, completing the proof. □

## 9.8. Conclusion

We need one more lemma and then we are ready to conclude the proof of the main theorem.

**Lemma 9.18.** *Let $\{X_n\}$ be the Postnikov tower for a space $X$. Then the induced map $H^i(X_n, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ are isomorphisms for $i \leq n$.*

*Partial proof.* Let $Y$ be the homotopy fiber of $X \to X_n$. By construction of the Postnikov tower $\pi_i(Y) = 0$ for $i \leq n$, thus by Theorem 9.9 and the universal coefficient theorem $H^i(Y) = 0$ in the same range. We consider the Leray-Serre spectral sequence and the vanishing of $H^i$ of the fiber for $i \leq n$ implies that $H^i(X) \cong H^i(X_n)$ in the same range.

Note that we have only stated the Leray-Serre spectral sequence for a simply connected base. If the base is not simply connected there is still a convergent spectral sequence, it's just harder to state the $E_2$ term. But the same vanishing result applies. □

In fact the same result holds for any other system of coefficients with the same proof. We just need the general statement of the Leray-Serre spectral sequence.

*Proof of Theorem 9.5.* We begin with $X \in \mathrm{Ob}(\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}})$. We have to show $MX \in \mathrm{Ob}(\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}})$ and $\phi_X : X \simeq FMX$.

Consider $X_n$, the $n$-th Postnikov section of $X$. Its homotopy type can be obtained in finitely many steps as in Lemma 9.17). So by Lemma 9.17 and Lemma 9.8 the assertion holds for $X_n$.

By Lemma 9.18 we have $H^i(X_n) \cong H^i(X)$ for $i \leq n$ via the induced map.

We now consider the construction of minimal models for $AX_n$ and $AX$ as in Theorem 2.21. For $i < n$ we see that $MX_n(i) \to AX_n \to AX$ is minimal and induces an isomorphism in cohomology up to degree $i + 1$, thus it can play the role of $MX(i)$ and we can build $MX$ by adding generators in higher degrees. As for any $i$ there is $X_n$ with $n > i$ this shows $MX$ is of finite $\mathbb{Q}$-type as in Definition 9.2 and then $MX$ is in $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}$

Lemma 9.18 also shows that $H^iMX_n \to H^iMX$ is an isomorphism for $i \leq n$.

Now consider the tower of principal fibrations for $X$:

$$X \cong \lim\left(X^0 \twoheadleftarrow X^1 \twoheadleftarrow X^2 \twoheadleftarrow \cdots\right)$$

Note that we have changed notation. We have factored each $X_n \to X_{n-1}$ into principal fibrations with fiber $K(\mathbb{Q}, n)$. For each $n$ there is an $i(n)$ such that $X^{i(n)}$ is $X_n$, but there may be many more distinct $X^i$. Here we are using that $X$ is of finite type.

We apply $A$ to get a diiagram of algebras and replace it by a sequence of cofibrations $N^0 \hookrightarrow N^1 \hookrightarrow N^2 \hookrightarrow \cdots$ such that each $N^{i+1}$ is obtained from $N^i$ by pushout along $\theta : \mathbb{Q}\langle x\rangle \to \mathbb{Q}\langle y, dy\rangle$ as in the proof of Lemma 9.17. Note that $N^i$ is a cofibrant cdga quasi-isomorphic to $MX^i$ but will not usually be minimal.

We write $N$ for the colimit of the $N^i$ which may be viewed as $\cup N^i$. One can check that for each $n$ and all large enough $i$ we have $H^nN \cong H^nN^i$. There are maps $N^i \to MX^i \to MX$ which give a canonical map $N \to MX$. This is a quasi-isomorphism since for each $p$ we can find a large enough $i$ such that

$$H^pN \cong H^pN^i \cong H^pMX^i \cong H^pMX$$

So $N \simeq MX$.

Consider the tower $FN^0 \twoheadleftarrow FN^1 \twoheadleftarrow FN^2 \cdots$. All the fibrations are principal with fibers of the form $K(\mathbb{Q}, n)$ so this is a refinement of the Postnikov tower for $\lim FN^i$. As $F$ takes colimits to limit we rewrite the limit as $FN \simeq FMX$.

Using the defining properties of Postnikov towers as well as Lemma 9.17 we see that the natural maps induce isomorphisms

$$\pi_pX \cong \pi_pX^i \cong \pi_pFMX^i \cong \pi_pFN^i \cong \pi_pFN \cong \pi_pFMX$$

for any large enough $i$. This proves that $\phi_X : X \to FMX$ is a weak equivalence.

Dually we have to show that for $B$ in $\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}$ we have $FB \in \mathrm{Ob}(\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}})$ and $\psi_B : B \simeq MFB$. We may assume $B$ is minimal and write $B$ as a colimit of $B_0 \hookrightarrow B_1 \hookrightarrow B_2 \hookrightarrow \cdots$. Here each $B_i \to B_{i+1}$ is a cofibration obtained as a pushout along $\theta : \mathbb{Q}\langle x\rangle \to \mathbb{Q}\langle y, dy\rangle$, so this is a refinement of the canonical filtration $B(0) \hookrightarrow B(1) \hookrightarrow B(2) \cdots$ of minimal models. We are using that $B$ is of finite type.

We apply $F$ and find that each $FB_{i+1} \to FB_i$ is a principal fibration and by Lemma 9.10 the fiber is a $K(\mathbb{Q}, n)$. Thus we have a refined Postnikov tower and $FB = \lim FB_n$ is in $\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$.

We use our notation as above and find that $FB_{i(p)}$ is the $p$-th Postnikov section of $FB$. Then for $i > i(p)$ we have isomorphisms $H^n FB \cong H^n FB_i$ by Lemma 9.18 and thus $H^n MFB \cong H^n MFB_i$. Together with $H^n B \cong H^n B_i$ and $H^n B_i \cong H^n MFB_i$ from Lemma 9.16 this shows that $\psi_B$ is an quasi-isomorphism.

So we have shown that our adjunction restricts to

$$F : \mathrm{Ho}(\mathsf{cdgA}_{/\mathbb{Q}}^{\geq 0,\mathrm{cft}}) \rightleftarrows \mathrm{Ho}(\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}})^{\mathrm{op}} : M$$

and both units of this adjunction are isomorphisms. This is the desired equivalence of categories. □

# 10. Rationalization and Applications

## 10.1. Rationalization

We still need to relate the homotopy groups of $X$ to the homotopy groups of $FMX$, which we know how to compute.

We recall from the introduction that a *rational equivance* is a map $f : X \to Y$ such that $f_* : H_*(X, \mathbb{Q}) \cong H_*(Y, \mathbb{Q})$ or equivalently $f^* : H^*(Y, \mathbb{Q}) \cong H^*(X, \mathbb{Q})$. (For a proof of the equivalence see Proposition 3.3.11 in [MP11].)

A *rationalization* of a space $X$ is a rational space $X_{\mathbb{Q}}$ together with a rational equivalence $q : X \to X_{\mathbb{Q}}$.

**Theorem 10.1.** *For any connected nilpotent space of finite type $X$ there exists a rationalization $\phi : X \to X_{\mathbb{Q}}$. Moreover, $\phi_*$ induces isomorphisms $\pi_i(X) \otimes \mathbb{Q} \cong \pi_i(X_{\mathbb{Q}})$ for all $i \geq 1$.*

Here $\pi_1(X) \otimes \mathbb{Q}$ is the *Mal'cev completion* of $\pi_1(X)$, it as a uniquely divisible nilpotent group canonically associated to $\pi_1(X)$. It can be defined as the fundamental group of the rationalization of $K(\pi_1(X), 1)$.

*Sketch of proof.* The strategy of proof is similar to the proof of the main theorem. There are quite a few technical details to worry about, but the outline is as follows:

We first show by explicit computation that $K(A \otimes \mathbb{Q}, n)$ is the rationalization of $K(A, n)$ for any abelian group $A$, see [MP11, Theorem 5.2.8]. We did the case of $K(\mathbb{Z}, 1)_{\mathbb{Q}} = K(\mathbb{Q}, 1)$ in the introduction.

Then we consider the Postnikov tower for $X$ and inductively form the rationalizations, see [MP11, Theorem 5.3.2].

Then the desired statement about homotopy groups can be deduced from the long exact sequence of homotopy groups, see [MP11, Theorem 6.1.2]. □

The existence of rationalizations implies that we may describe $\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$ either as the category of connected nilpotent rational spaces of finite type up to homotopy, or as the category of conected nilpotent spaces of finite type up to rational equivalence.

## 10.2. Rational homotopy groups

**Corollary 10.2.** *The unit $\phi_X : X \to FMX$ of our adjunction is a rationalisation of $X$ for every connected nilpotent space $X$ of finite type.*

*Proof.* Combining Theorem 10.1 with our adjunction we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\phi_X"] & X_{\mathbb{Q}} \arrow[d, "\phi_{X_{\mathbb{Q}}}"] \\ FMX \arrow[r, "FM(\phi)"] & FMX_{\mathbb{Q}} \end{tikzcd}$$

If $X$ is connected nilpotent then a cofibrant replacement of $AX$ is in $(\mathsf{cdgA}^{\geq 0,\mathrm{cft}}_{/\mathbb{Q}})$ and thus by Theorem 9.5 we know that $FMX$ is rational.

Now $\phi_i$ is a rational equivalence by Theorem 10.1, $\phi_{X_{\mathbb{Q}}}$ is a rational equivalence by Theorem 9.5 and $FM(\phi)$ is a rational equivalence as $M(\phi)$ is a quasi-isomorphism since $H^*(X, \mathbb{Q}) \cong H^*(X_{\mathbb{Q}}, \mathbb{Q})$.

This shows that the last morphism $e_X$ is also a rational equivalence and $FMX$ is equivalent to the rationalization $X_{\mathbb{Q}}$. □

**Corollary 10.3.** *For a nilpotent connected CW-complex $X$ of finite type with minimal model $M$ we have $\pi_n(X) \otimes \mathbb{Q} \cong \mathrm{Hom}(\pi^n M, \mathbb{Q})$ if $n \geq 2$.*

*Proof.* By Corollary 10.2 and Theorem 10.1 it suffices to compute $\pi_i(FMX)$, which we have done in Theorem 6.19. □

**Remark 10.4.** The isomorphism also holds if $n = 1$ if $\pi_1(X)$ is abelian. In fact there is a refinement of the corollary that also describes nilpotent fundamental groups.

The filtration $M(1, n)$ of the minimal model of $X$ induces a filtration on $\pi^1 M$, and the associated graded vector spaces are dual to the rationalizations of the associated graded of the lower central series of $\pi_1(X)$!

This is Theorem 12.7 in [BG76].

We can now compute a large number of rational homotopy groups. One potential issue is that $\pi_i(X) \otimes \mathbb{Q} \cong \mathbb{Q}$ could mean that $\pi_i(X)$ is $\mathbb{Q}$ or $\mathbb{Z}$. Rational homotopy theory has no way of telling those possibilites apart! The following classic theorem of Serre comes to the rescue:

**Theorem 10.5.** *Let $X$ be simply connecetd. Then all $\pi_i(X)$ are finitely generated if and only if all $H_i(X, \mathbb{Z})$ are finitely generated.*

*Proof.* See Theorem 4.5.4 in [MP11]. The proof uses Postinkov towers and the Leray-Serre spectral sequence. □

Thus most spaces we meet, like finite CW complexes, have finitely generated homotopy groups.

**Example 10.6.** On the example sheet we computed the homotopy groups for the minimal model $\mathbb{Q}\langle x, z \mid dz = x^5\rangle$ where $|x| = 2$. It follows from our Leray-Serre computations that this is quasi-isomorphic to the de Rham algebra of $\mathbb{C}P^4$. It follows that $\pi_i(\mathbb{C}P^4)$ has a factor of $\mathbb{Z}$ if $i = 2$ and $i = 9$ and is torsion otherwise.

## 10.3. Suspensions

We have already seen several examples of computing homotopy groups using the machinery of rational homotopy theory. This worked particularly well if the product structure of the de Rham complex is easy, for example for wedges of spheres.

It turns out that wedges of spheres are ubiquitous in rational homotopy theory!

Let $X$ be a pointed space. Recall that the homotopy pushout $* \leftarrow X \rightarrow *$ is called the *(reduced) suspension* of $X$. This is a fundamental construction in homotopy theory, for example it gives us the spheres just starting from a pair of points: $S^{n+1} = \Sigma S^n = \Sigma^{n+1} S^0$.

In general suspending a space is a way of simplifying it while keeping some of its properties (those properties are the subject of *stable homotopy theory*). In rational homotopy theory the simplification happens fast:

**Theorem 10.7.** *Let $X$ be a pointed space of finite type.*[1] *Then the rational homotopy type of $\Sigma X$ is the same as that of a wedge of spheres, one copy of $S^{i+1}$ for each basis element of $H^i(X)$ with $i > 0$.*

*Proof.* We compute a minimal model for $\Sigma X$. Let $M$ be a minimal model for $X$. The homotopy pushout $\Sigma X$ is sent by $A$ to the homotopy pullback $\mathbb{Q} \to M \leftarrow \mathbb{Q}$. Inspired by the construction of the suspension we are going to rewrite the diagram as follows:

$$\begin{tikzcd} M & M\tilde{\otimes}\Omega(1) \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & M \\ \mathbb{Q} \arrow[u] & & \mathbb{Q} \arrow[u] \end{tikzcd}$$

The map $\epsilon_i$ denotes evaluation of $\Omega(1)$ at $i$ (and the identity on $M = \mathbb{Q} \oplus \bar{M}$). Thus $M\tilde{\otimes}\Omega(1) := \mathbb{Q} \oplus (\bar{M} \otimes \Omega(1))$ is quasi-isomorphic to $M$ and $\epsilon_0$ and $\epsilon_1$ are acyclic fibrations. The homotopy limit of the new diagram is the same as that of $\mathbb{Q} \to M \leftarrow \mathbb{Q}$, as it is objectwise weak equivalent to $\mathbb{Q} \to M = M = M \leftarrow \mathbb{Q}$

We can compute the homotopy limit by computing two homotopy pullbacks. First, as $\epsilon_0$ is a fibration the pullback of the left part of the diagram is a homotopy pullback and takes the value $\mathbb{Q} \oplus (\bar{B} \otimes \ker \epsilon_0)$. This still surjects to $M$ via $\epsilon_1$ so the pullback of the new diagram $\mathbb{Q}\oplus(\bar{B}\otimes\ker \epsilon_0) \to M \leftarrow \mathbb{Q}$ is a homotopy pullback and a homotopy limit of the whole diagram. We can write it as $\mathbb{Q} \oplus (\bar{M} \otimes J)$ where $J$ is the ideal $\ker(\epsilon_0) \cap \ker \epsilon_1$ of $\Omega(1)$. It is generated by $t(1-t)$ and $dt$.

But a computation shows that the inclusion $\mathbb{Q}.dt \to J$ is a quasi-isomorphism and induces a quasi-isoorphism of algebras $\mathbb{Q} \oplus \bar{M}.dt \simeq \mathbb{Q} \oplus (\bar{M} \otimes J)$. The product on $\bar{M}.dt$ is zero since $dt.dt = 0$!

Now pick a subspace $H$ of $\bar{M} \otimes dt$ that maps isomorphically to the cohomology of $\bar{M}.dt$, which is the reduced cohomology of $X$, shifted in degree by 1, equipped with the zero product.

[1] $X$ need not be nilpotent, only $\Sigma X$, which is automatic.

The map
$$\mathbb{Q} \oplus H \to \mathbb{Q} \oplus (\bar{M} \otimes J) \simeq A(\Sigma X)$$
is a quasi-isomorphism and $H.H = 0$.

Thus $A(\Sigma X)$ is quasi-isomorphic to its cohomology algebra. We say it is *formal*. But the cohomology algebra with zero products is also quasi-isomorphic to the de Rham algebra of a wedge of spheres. Thus $X$ and the wedge of spheres given in the statement of the theorem have the same minimal model. Thus their rationalizations are equivalent by Corollary 10.2 and the spaces are rationally equivalent. □

## 10.4. Bousfield localization

There is also a model-theoretic point of view of rationalization. We take a known model category $\mathscr{M}$, in our example sSet, and try and define a new model category by adding new class $E$ of weak equivalences, in our example all the maps $f : X \to Y$ such that $f_* : H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$ is an isomorphism.

We keep the same cofibrations as before, and then the lifting properties tell us that we have additional conditions for fibrations.

**Definition 10.8.** If the model category described above exists it is called the *left Bousfield localization* of $\mathscr{M}$ at $E$, sometimes denoted $L_E\mathscr{M}$.

It turns out that under good conditions the fibrant objects can be described very nicely.

**Definition 10.9.** An object $Y$ in $\mathscr{M}$ is called *$E$-local*, if it is fibrant in $\mathscr{M}$ and any map $X \to X'$ in $E$ induces a bijection $[X', Y] \cong [X, Y]$.

Recall that rational cohomology is classified by homotopy classes of maps to rational Eilenberg-MacLane spaces. It follows that the $K(\mathbb{Q}, n)$ are $\mathbb{Q}$-local, and then one can deduce that nilpotent rational spaces are $\mathbb{Q}$-local.

In good cases the $E$-local objects in $\mathscr{M}$ are exactly the fibrant objects in $L_E\mathscr{M}$.

The identity map $\mathbf{1} : \mathscr{M} \to L_E\mathscr{M}$ obviously has left and right adjoints, but if we consider the model structures it is also left Quillen: It preserves cofibrations (as they are the same) and acyclic cofibrations (as we have added more weak equivalences).

We denote a fibrant replacement of $X$ in $L_E\mathscr{M}$ by $L_EX$. The Quillen adjunction gives us $[X, Y]_{L_E\mathscr{M}} \cong [X, L_EY]_{\mathscr{M}}$. If now $X \to X'$ is in $E$ we have
$$[X', L_EY]_{\mathscr{M}} \cong [X,' Y]_{L_E\mathscr{M}} \cong [X, Y]_{L_E\mathscr{M}} \cong [X, L_EY]_{\mathscr{M}}$$
It follows that if $L_EY \simeq Y$, i.e. if $Y$ is fibrant in $L_E\mathscr{M}$, then $Y$ is $E$-local.

The converse holds whenever the model is *left proper*.

**Theorem 10.10.** *The left Bousfield localization of simplicial sets at rational equivalences exists and the fibrant objects are exactly the local objects.*

*Proof.* See [MP11] Section 19.3. □

The construction of the Bousfield localization did not need the restricition to nilpotent spaces and is thus more general than what we were doing before. However, in exchange we give up our computational control. The example of $\mathbb{R}P^2$ shows already that a rational equivalence (like $\mathbb{R}P^2 \to *$) need not induce isomorphisms on rational homotopy groups.

**Remark 10.11.** We can also Bousfield localise at other homology theories. To begin with we can take homology with coefficients in $\mathbb{F}_p$ or $\mathbb{Z}_{(p)}$. Here $\mathbb{Z}_{(p)}$ denotes the set of integers where all numbers not divisible by $p$ have been inverted. Localization at $H^*(-, \mathbb{F}_p)$ is usually called *completion at p*, while $H^*(-, \mathbb{Z}_{(p)})$ is called *localization at p*.

But this is only the beginning, we can Bousfield localize many model categories at even more classes of maps. In fact the model category of spaces itself arises as a Bousfield localization of the model category of $\infty$-categories!

# A. Basic category theory

I will give a rapid fire overview of category theory. The focus is on definitions and examples, with a few results thrown in, but no proofs. If you have never seen the concept before you will probably have to stop these videos some times and read the script carefully and maybe google some things. That's okay.

If you have met a few concepts here and there this should be nice refresher putting everything we need together in a systematic way

If you are comfortable with categories up to limits and adjunctions you can skip this bit.

## A.1. Basics

### A.1.1. Categories and Functors

**Definition A.1.** A *category* $\mathscr{C}$ consists of the following data:

- a class of *objects* $\mathrm{Ob}(\mathscr{C})$,
- for every pair of objects $X, Y \in \mathrm{Ob}(\mathscr{C})$ a class of *morphisms* $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ (also called arrows),
- for every object $X$ a distinguished morphism $\mathbf{1}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$, the *identity*
- for every three objects $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ a *composition* $\circ : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$,

such that

- compositon is associative: $(f \circ g) \circ h = f \circ (g \circ h)$,
- the identity is an identity for composition: $\mathbf{1}_Y \circ f = f = f \circ \mathbf{1}_X$ for $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Given $f$ in $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ we call $X$ the *source* and $Y$ the *target* of $f$.

**Example A.2.**

1. Sets and functions form a category we denote by $\mathsf{Set}$. (Since we want to consider the category of all sets and want to avoid paradoxa we referred to a class of objects in our definition.)
2. Topological spaces and continuous maps form a category $\mathsf{Top}$. It is easy to consider the subcategory of CW complexes or path connected spaces etc.

3. There is also a category $\mathsf{Top}_*$ whose objects are pointed topological spaces $(X, x_0)$ and whose morphisms are base-point preserving maps, i.e. $f : (X, x_0) \to (Y, y_0)$ is given by $f : X \to Y$ with $f(x_0) = y_0$.

   This is an example of an *undercategory*: Given any category $\mathscr{C}$ with an object $C$ there is a category whose objects are arrows $f : C \to D$ in $\mathscr{C}$, and whose morphisms are maps $g : D \to D'$ making the obvious triangle commute: $g \circ f = f' : C \to D1$. $\mathsf{Top}_*$ is the category of topological spaces under the one point space.

4. In algebra we find many further categories: Groups and homomorphisms form the category $\mathsf{Group}$, vector spaces over $k$ and linear maps form $\mathsf{Vect}_k$, abelian groups, rings, fields, etc. all form categories

5. There is a category with one object and one morphism (the identity of the object). In general a category is called *discrete* if the identities are the only morphisms. Every set $I$ can be considered as a discrete category $\mathbf{I}$ with $\mathrm{Ob}(\mathbf{I}) = I$.

6. For every category $\mathscr{C}$ there is an *opposite category* $\mathscr{C}^{op}$ with the same objects, $\mathrm{Hom}_{\mathscr{C}^{op}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$ and $f \circ_{\mathscr{C}^{op}} g := g \circ_{\mathscr{C}} f$. Thus we obtain the opposite category $\mathscr{C}^{op}$ from $\mathscr{C}$ by turning around all arrows.

We will often abuse notation and write $C \in \mathscr{C}$ as a shortcut for "$C$ is an object of $\mathscr{C}$".

**Definition A.3.** A morphism $f : C \to D$ is called *isomorphism*, if there is $g : D \to C$ such that $g \circ f = \mathbf{1}_C$ and $f \circ g = \mathbf{1}_D$.

Homeomorphisms and (group/ring/vector space) isomorphisms are examples.
In all categories we consider isomorphic object as equivalent and (almost) interchangeable.

**Remark A.4.** If the objects and morphisms of a category form sets we call it a *small category*. If there may be a class of objects but the morphisms between any two pair of objects form a set we say the category is *locally small*.

Many categories we are interested in, like $\mathsf{Top}$, $\mathsf{Set}$ and $\mathsf{Group}$ are not small, but locally small.

**Example A.5.** A small category in which there is at most one morphism between any two objects and in which any isomorphism is an identity is called a *partial order*. Then the composition is uniquely determined by the morphisms (as there is only one function into a set with one element).

An example is the category $\mathbb{N}$ whose objects are the natural numbers and where there is a morphism $i \to j$ if and only if $i \leq j$.

An important motivation for the study of category theory is the observation that mathematical objects are often better understood through the morphisms between them. The same principle holds for categories.

**Definition A.6.** A *functor* $F$ between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of the following data:

- a map that associates to any $X \in \mathrm{Ob}(\mathscr{C})$ an object $F(X) \in \mathrm{Ob}(\mathscr{D})$.
- for each pair of objects $X, Y \in \mathrm{Ob}(\mathscr{C})$ a map from $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ to $\mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ which we write as $f \mapsto F(f)$,

such that

- $F$ is compatible with composition: $F(f \circ g) = F(f) \circ F(g)$,
- $F$ preserves the identities: $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$.

**Example A.7.**

1. For every category $\mathscr{C}$ there is an identity functor $\mathbf{1}_{\mathscr{C}}$ that does nothing on objects and morphisms.
2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $D$ an object of $\mathscr{D}$. Then there is a constant functor $c_D : \mathscr{C} \to \mathscr{D}$ that sends every object of $\mathscr{C}$ to $D$ and any morphism of $\mathscr{C}$ to $\mathbf{1}_D$.
3. A family of topological spaces $(X_i)_{i \in I}$ is nothing but a functor from $I$, considered as a discrete category, to $\mathsf{Top}$.
4. From every category whose objects have an underlying set e.g. $\mathsf{Top}, \mathsf{Group}, \mathsf{Vect}_k$) there is a *forgetful functor* to $\mathsf{Set}$, that forgets all additional structure.
5. Algebraic Topology is in no small part the study of functors from topological spaces to algebraic categories.

   The homotopy groups are functors $\pi_n : \mathsf{Top}_* \to \mathsf{Group}$ associating to any pointed topological space $(X, x_0)$ the homotopy group $\pi_n(X, x_0)$ and to any map $f : X \to Y$ the induced map $f_*$.

   Similary homology groups are functors $H_n : \mathsf{Top} \to \mathsf{Ab}$.

   Cohomology groups are functors $H^n : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Ab}$. Note that these functors turns around the direction of arrows, which is why we write it as a functor from the opposite category. We also call such functors *contravariant*.

It is easy to see that functors can be composed, so there is a *category of categories* whose objects are (small) categories and whose morphisms are functors.

### A.1.2. Natural Transformations

Remarkably, there are not just maps between categories (the functors) but also maps between maps betwen categories.

**Definition A.8.** Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors. A *natural transformation* $\alpha$ from $F$ to $G$ consists of maps $\alpha_C : FC \to GC$ for every $C \in \mathscr{C}$ such that for every map $f : C \to C'$ in $\mathscr{C}$ there is a commutative diagram:

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[d, "\alpha_C"] & FC' \arrow[d, "\alpha_{C'}"] \\ GC \arrow[r, "Gf"] & GC' \end{tikzcd}$$

**Remark A.9.** You might think that it is easier to write $\alpha_{C'} \circ Ff = Gf \circ \alpha_C$ instead of drawing the commutative diagram.

The commutative diagram has the advantage that it keeps track of all the objects as well as the morphisms between them. More importantly, in category theory, algebraic topology and homological algebra there is often a plethora of maps whose compositions we want to compare, and it is much easier to keep track if one arrange them all in a beautiful diagram.

**Example A.10.**

1. There is a functor $D : \mathsf{Vect}_k \to \mathsf{Vect}_k$ that takes every vector space to its double dual $V \mapsto (V^*)^*$. Then for every vector space there is a map $\iota : V \to DV$ that sends $v \in V$ to the functional $\alpha \mapsto \alpha(v)$. This map is natural, meaning it is compatible with linear maps. In other words, $\iota$ is a natural transformation from the identity functor $\mathbf{1}_{\mathsf{Vect}}$ to the double dual $D$.

2. For any functor $F : \mathscr{C} \to \mathscr{D}$ there is the identity natural transformation $\mathbf{1}_F$ defined by $(\mathbf{1}_F)_C = \mathbf{1}_{FC}$ for every $C \in \mathscr{C}$.

3. Fix two categories $I$ and $\mathscr{C}$, where we may think of $I$ as being somehow small.

   We will consider a functor $F : I \to \mathscr{C}$ as a *diagram* in $\mathscr{C}$, given by objects $F(i)$ together with arrows $F(f) : F(i) \to F(j)$ for every morphism $f : i \to j$ in $I$.

   Any object $C$ of $\mathscr{C}$ determines a constant functor $c_C : I \to \mathscr{C}$ that sends any $i$ to $C$ and any $f : i \to j$ to $\mathbf{1}_C$.

   Then natural transformation from $c$ to another functor $F : I \to \mathscr{C}$ is given by maps $\alpha_i : C \to F(i)$ for every $i \in I$ such that $F(f) \circ \alpha_i = \alpha_j$ for every $f : i \to j$.

   We call a natural transformation from a constant diagram to $F$ a *cone* over $F$. We think of $C$ as the tip of the cone, and there are arrows going to all the vertices of the diagram, making all the triangles commute.

$$
\begin{tikzcd}
& C \arrow[dl, dashed] \arrow[dd, dashed] \arrow[dr, dashed] & \\
F(i) \arrow[rr] \arrow[dr] & & F(k) \\
& F(j) \arrow[ur] &
\end{tikzcd}
$$

4. For every $n \geq 1$ the Hurewicz homomorphism $h_n : \pi_n(X, *) \to H_n(X, \mathbb{Z})$ from homotopy to homologoy of path connected spaces is a natural transformation. (To be precise it is a natural transformation from $\pi_n$ to the composition of homology with the functor forgetting basepoints. If $n = 1$ we also have to compose with the inclusion functor from abelian groups to all groups.)

5. For every topological space $X$ we have a functor which takes the underlying set of $X$ and equips it with the discrete topology, write this as $X^\delta$. Then the identity map from $X^\delta$ to $X$ is continuous. In fact it is a natural transformation from the discretization functor to the identity functor $X^\delta \mapsto X$.

Natural transformations may be composed and form the morphism in the *category of functors* $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ between two categories.

**Definition A.11.** A natural tranformation $\alpha$ such that all $\alpha_C$ are isomorphisms is an isomorphism in the category of functors and is called a *natural isomorphism*.

### A.1.3. Equivalences

**Definition A.12.** Two categories are *equivalent* if there are functor $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that $F \circ G$ is naturally isomorphic to $\mathbf{1}_{\mathscr{D}}$ and $G \circ F$ is naturally isomorphic to $\mathbf{1}_{\mathscr{C}}$.

We can give a more concrete description, for which we need some definitions.

**Definition A.13.** functor $F : \mathscr{C} \to \mathscr{D}$ is *full* if it induces surjections on all hom sets, i.e. every $g : FC \to FC'$ in $\mathscr{D}$ is $F(f)$ for some $f : C \to C'$.

The functor $F$ is *faithful* if it induces injections on all hom sets, i.e. $F(f) = F(f')$ only if $f = f'$.

$F$ is *fully faithful* if it is both full and faithful.

$F$ is *essentially surjective* if every object in $\mathscr{D}$ is isomorphic to some object $FC$ in the image of $F$.

Then one can prove that $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective. (The "if" direction needs the axiom of choice.)

**Example A.14.** 1. Let $k$ be a field. There is an equivalenc of categories from finite-dimensional $k$-vector spaces to its opposite category, given by $V \mapsto V^*$ on objects.

2. Let $\mathsf{Mat}$ be the category whose objects are non-negative integers and whose morphisms from $m$ to $n$ are $(m \times n)$-matrices. Composition is given by matrix multiplication.

   Then there is a natural functor from $\mathsf{Mat}$ to the category of finite-dimensional $\mathbb{R}$-vector spaces, given by $n \mapsto \mathbb{R}^n$ on objects. This is an equivalence of categories.

### A.1.4. Opposite categories

We recall the following Example A.2.6:

**Definition A.15.** Let $\mathscr{C}$ be any category. Then its *opposite category* $\mathscr{C}^{\mathrm{op}}$ is defined to have the same objects as $\mathscr{C}$ but $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(C, D) := \mathrm{Hom}_{\mathscr{C}}(D, C)$ and $f \circ_{\mathscr{C}^{\mathrm{op}}} g := g \circ_{\mathscr{C}} f$.

In words $\mathscr{C}$ is obtained by turning around all the arrows in $\mathscr{C}$.
Clearly any functor $F : \mathscr{C} \to \mathscr{D}$ induces an opposite functor $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$.
Many natural functors, like cohomology, turn around the order of arrows, i.e. cohomology is a functor $\mathsf{Top}^{\mathrm{op}} \to \mathsf{Ab}$.

**Definition A.16.** We call a functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ a *contravariant functor* from $\mathscr{C} \to \mathscr{D}$.

By using the opposite of categories and functors, we can dualize all the definitions and results in category theory.
Moreover, whenever we prove a statement about a category $\mathscr{C}$ then the *dual statement* holds for its opposite category.
This is a very powerful idea, which we will come back to soon.

### A.1.5. The hom functor

Forming the hom sets in a category is actually functorial. Let us explain what this means.
Let $\mathscr{C}$ be a locally small category, i.e. the morphisms between any two objects form a set (rather than a proper class). Let $C$ be an object of $\mathscr{C}$.

**Definition A.17.** The *hom-functor*, denoted $h_C : \mathscr{C} \to \mathsf{Set}$, sends any object $D$ to $\mathrm{Hom}_{\mathscr{C}}(C, D)$ and any morphism $f : D \to D'$ to the map $f_* : \mathrm{Hom}_{\mathscr{C}}(C, D)$ to $\mathrm{Hom}_{\mathscr{C}}(C, D')$ defined by $g \mapsto f \circ g$.

We can of course also put the object $C$ in the second place of Hom. Then our functor will be contravariant and turn around the order of arrows. We obtain $h^C : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$ which is defined by $D \mapsto \mathrm{Hom}_{\mathscr{C}}(D, C)$ and $f \mapsto f^*$, where $f^*(g) = g \circ f$.
For another level of abstraction, $h_{(-)}$ defines a functor from $\mathscr{C}^{\mathrm{op}}$ to the category of functors $\mathrm{Fun}(\mathscr{C}, \mathsf{Set})$. This is a fully faithful functor that is called the *Yoneda embedding*. Any functor naturally isomorphic to $h_C$ is called *representable*.

**Example A.18.** The forgetful funtor $U : \mathsf{Group} \to \mathsf{Set}$ is representable by the group of integers.

Unravelling our definition this means that there for every group $G$ there is an isomorphism $\mathrm{Hom}_{\mathsf{Group}}(\mathbb{Z}, G) \cong U(G)$, and these isomorphisms are compatible with group homomorphisms.

But this just says that the set of morphisms from $\mathbb{Z}$ to $G$ is exactly the set of elements of $G$, the isomorphism is given by sending $f : \mathbb{Z} \to G$ to $f(1) \in G$.

**Remark A.19.** A key result in category theory is the *Yoneda lemma*. It states that natural transformations from $h^C$ to some other functor $F : \mathscr{C} \to \mathsf{Set}$ are in natural bijection with $F(C)$. It's not hard, but very consequential. (Although we won't need it.)

## A.2. Universal constructions

### A.2.1. Limits

Category theory allows us to unify many constructions in mathematics, in particular those characterised by *universal properties*.

**Definition A.20.** Let $I$ be a small category and $\mathscr{C}$ any category. A *diagram of shape $I$ in $\mathscr{C}$* is just a functor $I \to \mathscr{C}$.

We will often write $F_i$ for the objects $F(i)$ for $i \in I$.

**Definition A.21.** A *limit* of the diagram $F : I \to \mathscr{C}$ is an object $L$ of $\mathscr{C}$ together with a natural transformation $\alpha : c_L \Rightarrow F$ that is *universal* in the sense that any natural transformation from a constant functor $c_C$ to $F$ factors uniquely through $c_L$.

In other words, $L$ and $\alpha$ have the property that whenever we have $C$ in the following diagram there is exactly one dashed arrow $C \to L$ making the diagram commute.

$$\begin{tikzcd}
C \arrow[r, dashed] \arrow[d] \arrow[ddr] \arrow[drr] & L \arrow[dl, "\alpha_i"'] \arrow[dd, "\alpha_j"] \arrow[dr, "\alpha_k"] & \\
F(i) \arrow[rr, dotted] \arrow[dr, dotted] & & F(k) \\
& F(j) \arrow[ur, dotted] &
\end{tikzcd}$$

This universal property (like all universal property) ensures that if there are two limits $L$ and $L'$ there is a unique isomorphism between them. We thus also speak of *the limit* and denote it by $\lim_I F$ or $\lim F_i$.

**Remark A.22.** Note that the limit need not exist! If we can form arbitrary (small) limits in a category $\mathscr{C}$ we say that $\mathscr{C}$ *has all small limits*.

Let us make this more concrete.

**Definition A.23.** Let $I$ a set considered as a discrete category. The limit of $F : I \to \mathscr{C}$ is called the product of the $F(i)$, often written $\prod_{i \in I} F_i$.

Thus $\prod_i F_i$ has the property that there are natural maps $\pi_j : \prod_i F_i \to F_j$ for all $j$ (called *projection*) and whenever we are given maps $\beta_j : C \to F_j$ for all $j$ we obtain a map $\beta : C \to \prod_i F_i$ such that $\beta_j = \pi_j \circ \beta$.

This recovers the familiar product of sets, topological spaces, abelian groups etc.

We consider a special case:

**Definition A.24.** Let $I$ be the empty set considered as a discrete category without objects! The limit of the unique functor $I \to \mathscr{C}$ is called the *terminal* object of $\mathscr{C}$, often written $*$. It has the property that for every $C \in \mathscr{C}$ there is a unique morphism $C \to *$.

The terminal object in $\mathsf{Set}$ is the set with 1 Element.

**Definition A.25.** Let $I$ be the category with two objects and two arrows in the same direction $\bullet \rightrightarrows \bullet$. The limit of $F : I \to \mathscr{C}$ is called *equalizer*.

**Definition A.26.** Let $I$ be the category with three objects $\bullet \to \bullet \leftarrow \bullet$. The limit of $F : I \to \mathscr{C}$ is called *pullback*.

**Example A.27.**

1. The terminal object in $\mathsf{Groups}$ is the group with 1 element.
2. The terminal object in $\mathsf{Top}$ is the topological space with 1 point.
3. In the diagram $\bullet \to \bullet \leftarrow \bullet$ that defines pull-backs the middle object is terminal.
4. If a pull-back diagram in $\mathsf{Set}$ or $\mathsf{Top}$ takes the form $* \to Y \xleftarrow{f} X$ then the pull-back is the fiber of $f$ (equipped with the subspace topology in the case of $\mathsf{Top}$).
5. If a pull-back diagram takes the form $X \to * \leftarrow Y$, i.e. the middle object goes to the terminal object of $\mathscr{C}$, then the limit is the product $X \times Y$.
6. In the category $\mathsf{Groups}$ there is a unique map from $*$ to any group $H$ and the pullback of the diagram $* \to H \xleftarrow{f} G$ is nothing but the kernel of $f$.
7. The equalizer of two maps $f, g : A \to B$ in $\mathsf{Set}$ is exactly the subset of $A$ given by all elements $a$ with $f(a) = g(a)$, this explains the name.

### A.2.2. Colimits

We now apply the idea of *dualizing* categorical notions by turning around all the arrows to the previous section.

So we change the orientation of all the arrows in the definition of a limit. This gives the dual notion of a limit, called the colimit.

**Definition A.28.** A *colimit* of the diagram $F : I \to \mathscr{C}$, denoted by $\operatorname{colim}_I F$, is an object $D$ of $\mathscr{C}$ together with a natural transformation $\alpha : F \Rightarrow c_D$ that is *universal*, in the sense that any natural transformation from $F$ to a constant functor $c_C$ factors uniquely through $c_D$.

The corresponding diagram looks like this:

$$\begin{tikzcd}
C & & D \arrow[ll, dashed] & \\
F(i) \arrow[u] \arrow[urr] \arrow[rrr, dotted] \arrow[drr, dotted] & & & F(k) \arrow[ulll] \arrow[ul] \\
& & F(j) \arrow[uull] \arrow[uu] \arrow[ur, dotted] &
\end{tikzcd}$$

**Remark A.29.** To make the duality of limit and colimit more precise we can observe that $(D, \alpha)$ is a colimit of the diagram $F : I \to \mathscr{C}$ exactly if $(D, \alpha^{\mathrm{op}})$ is a limit of the diagram $F^{\mathrm{op}} : I^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$. Here $\alpha^{\mathrm{op}} : c_D^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$ is the natural transformation corresponding to $\alpha : F \Rightarrow c_D$ under the correspondence of morphisms in $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$.

**Definition A.30.** The colimit over a discrete category is called the *coproduct* or sum.

The colimit of the empty diagram is called the *initial object*.

The colimit of the diagram $\bullet \leftarrow \bullet \rightarrow \bullet$ is called *pushout*.

The colimit of a diagram of shape $\bullet \leftleftarrows \bullet$ is called *coequalizer*.

**Example A.31.**

1. In $\mathsf{Set}$ and $\mathsf{Top}$ ithe coproduct is given by the disjoint union.
2. In $\mathsf{Group}$ the coproduct is given by the free product of groups.
3. In $\mathsf{Vect}$ the product and coproduct of two vector spaces $V$ and $W$ agree, both are given by $V \oplus W$. (This holds for all finite products and coproducts in $\mathsf{Vect}$, but it is no longer true for infinite products and coproducts!)
4. The initial object in $\mathsf{Set}$ is given by the empty set.
5. The group with one object is both initial and terminal.
6. The pushout of the diagram $0 \leftarrow V \rightarrow W$ of vector spaces is the quotient space $W/V$.
7. The coequalizer of two maps $f, g : A \to B$ in $\mathsf{Set}$ is given by the quotient of $B$ by the relation generated by $f(a) \sim g(a)$ for all $a \in A$.

From the definition of limit and colimits it is not hard to obtain the following extremely useful result:

**Lemma A.32.** *Let $F : I \to \mathscr{C}$ and $G : J \to \mathscr{C}$ be diagrams. Then we have natural isomorphisms*

$$\mathrm{Hom}_{\mathscr{C}}(C, \lim_I F_i) \cong \lim_I \mathrm{Hom}_{\mathscr{C}}(C, F_i)$$

*and*

$$\mathrm{Hom}_{\mathscr{C}}(\operatorname{colim}_J G_i, C) \cong \lim_J \mathrm{Hom}_{\mathscr{C}}(G_j, C)$$

### A.2.3. Existence of (co)limits

We say a category $\mathscr{C}$ *has all small limits* if every diagram $I \to \mathscr{C}$ has a limit. Similarly we say $\mathscr{C}$ has all small colimits if every diagram $I \to \mathscr{C}$ has a colimit.

This may seem extremely difficult to check, but in fact one can build any limit from just two types of limit:

Recall that an equalizer is a limit for a diagram of the shape $\bullet \rightrightarrows \bullet$ and a product is a diagram whose shape is a discrete category.

We say a category $\mathscr{C}$ has all equalizers if any equalizer diagram has a limit, and similarly for products (and other shapes of diagrams).

**Lemma A.33.** *A category $\mathscr{C}$ has all limits if and only if it has all products and equalizers. It has all colimits if and only if it has all coproducts and coequalizers.*

### A.2.4. Adjunctions

It is rare that categories are equivalent, but a weaker notion is extremely fruitful.

**Definition A.34.** We say $F : \mathscr{C} \to \mathscr{D}$ is *left adjoint* to $G : \mathscr{D} \to \mathscr{C}$, in symbols $F \dashv G$ if for all $C \in \mathscr{C}$ and $D \in \mathscr{D}$ there are natural isomorphisms

$$\phi_{C,D} : \mathrm{Hom}_{\mathscr{C}}(C, GD) \cong \mathrm{Hom}_{\mathscr{D}}(FC, D)$$

Here naturality means that for every map $C \to C'$ in $\mathscr{C}$ the natural diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(C', GD) & \xrightarrow{\phi_{C',D}} & \mathrm{Hom}_{\mathscr{D}}(FC', D) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle Ff^*} \\ \mathrm{Hom}_{\mathscr{C}}(C, GD) & \xrightarrow{\phi_{C,D}} & \mathrm{Hom}_{\mathscr{D}}(FC, D) \end{array}$$

and a similar diagram commutes for $g : D \to D'$ in $\mathscr{D}$.

If $\mathscr{C}$ and $\mathscr{D}$ are locally small we can also phrase naturality as saying that the two functors $\mathrm{Hom}_{\mathscr{C}}(-, G(-))$ and $\mathrm{Hom}_{\mathscr{D}}(F(-), -)$ from $\mathscr{C}^{\mathrm{op}} \times \mathscr{D}$ to $\mathsf{Set}$ are naturally isomorphic.

**Example A.35.** 1. Throughout algebra there are adjunctions between free and forgetful functors. For example the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ has a left adjoint given by taking a set $X$ to the free group with set of $X$ as set of generators.

2. The forgetful functor $\mathsf{Top} \to \mathsf{Set}$ has a left adjoint given by equipping any set with the discrete topology. It also has a right adjoint given by equipping any set with the indiscrete topology.

Left and right adjoints are naturally dual: If $F : \mathscr{C} \to \mathscr{D}$ is left adjoint to $G$, then $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$ is right adjoint to $G^{\mathrm{op}}$.

Let $F \dashv G : \mathscr{C} \rightleftarrows \mathscr{D}$ and $C \in \mathscr{C}$. By the adjunction the identity map $\mathbf{1}_{FC} : FC \to FC$ corresponds to a map $\epsilon_C : C \to GFC$. By naturality in the definition of an adjunction the $\epsilon$ assemble into a natural transformation $\epsilon : \mathbf{1}_{\mathscr{C}} \Rightarrow GF$. This is called the *unit* of the adjunction.

Simlarly there is a natural transformation $\eta : FG \Rightarrow \mathbf{1}_{\mathscr{D}}$, called the *counit* of the adjunction.

**Lemma A.36.** *Let $F \dashv G$. Then unit and counit satisfy the following identities of natural transformations: For every $C \in \mathscr{C}$ we have*

$$\eta_{FC} \circ F(\epsilon_C) = \mathbf{1}_{FC}$$

*and for every $D \in \mathscr{D}$ we have*

$$G(\eta_C) \circ \epsilon_{GD} = \mathbf{1}_{GD}.$$

Put a little differently, we have the following identities of natural transformations: $G\eta \circ \epsilon_G = \mathbf{1}_G$ and $\eta_F \circ F\epsilon = \mathbf{1}_F$.

In fact, adjoints may be equivalently characterized by the existence of unit and counit.

**Remark A.37.** An adjunction induces an equivalence of categories if and only if unit and counit are natural isomorphisms.

One can also show that adjoints are given by a universal property and are thus unique up to unique natural isomorphism.

Adjoints are closely related to limts:

**Lemma A.38.** *Let $F$ be a left adjoint. Then $F$ preserves colimits, i.e. whenever $(D, \alpha)$ is a colimit of a diagram $G : I \to \mathscr{C}$ then $(FD, F\alpha)$ is a colimit for $F \circ G : I \to \mathscr{D}$.*

*Dually, if $G$ is a right adjoint then $G$ preserves limits.*

**Remark A.39.** Under some assumption on the categories $\mathscr{C}$ and $\mathscr{D}$ there is even a converse to the lemma: Any functor preserving all colimits has a left adjoint. There are different theorems, depending on the precise assumptions made, but they are all called *adjoint functor theorems*.

We can even characterize limits using adjoints.

**Lemma A.40.** *Consider the category* $\mathrm{Fun}(I, \mathscr{C})$ *of $I$-shaped diagrams in $\mathscr{C}$. There is a diagonal functor $\Delta : \mathscr{C} \to$* $\mathrm{Fun}(I, \mathscr{C})$ *sending any object $C$ to the constant functor $c_C$. Then taking the limit of a diagram is right adjoint to $\Delta$, and taking the colimit is left adjoint.*

# B. Multiplicative matters

In this appendix we collect the necessary results to prove Theorem 6.16. The proof is very elegant but it needs some background and is quite abstract, so feel free to just take the result on trust!

## B.1. Simplicial sets as colimits

A remarkable consequence of the homotopical view-point is that simplicial sets (and thus spaces) are generated under (homotopy!) colimits by a single point. In other words every simplicial set is a homotopy colimit of a constant diagram taking value the point!

We begin by observing that any simplicial set can be written as a colimit over its simplicies. To make this precise, we introduce the following.

**Definition B.1.** Let $K$ be a simplicial set. The *simplex category* $\Delta K$ of $K$ is the category whose objects are the simplices of $\sigma : \Delta[n] \to K$ of $K$, with maps $\theta : \sigma \to \tau$ given by maps $\Delta[n] \to \Delta[m]$ making the triangle with vertices $\Delta[m]$, $\Delta[n]$ and $K$ commute.

**Remark B.2.** There is an elegant way of expressing this definition as the *slice category* $\Delta[\bullet] \downarrow K$ where $\Delta[\bullet]$ is the functor from $\mathbf{\Delta}$ to $\mathsf{sSet}$ sending $[n]$ to $\Delta[n]$.

There is a natural functor $K_\Delta$ from $\Delta K$ to $\mathsf{sSet}$ that sends any $n$-simplex of $K$ to $\Delta[n]$.

**Proposition B.3.** *For any simplicial set $K$ we have $K \cong \operatorname{colim}_{\Delta K} K_\Delta$.*

We also write the colimit as

$$\operatorname*{colim}_{\Delta K} K_\Delta = \operatorname*{colim}_{\Delta[n]\in\Delta K} \Delta[n]$$

to make it more intuitive.

*Proof.* This is a special case of a much more general result: Any functor $F : I^{\mathrm{op}} \to \mathsf{Set}$ from a small category to the category of sets is a colimit of representable functors.

We show it in this example, but the proof is much more general! We need to show that any family of maps $K_\Delta \to L$ uniquely factors through $K$. By Lemma 5.10 a map $\Delta[n] \to L$ is just an $n$-simplex of $L$. So for any element in $K_n$ we get an element in $L_n$, in other words we have maps $K_n \to L_n$ for all $n$. Moreover, all of these maps have to be compatible with the morphisms $[m] \to [n]$ in $\Delta K$, and that translates to compatibility with face and degeneracy maps in $K$ and $L$. This defines a unique map $K \to L$ and we have exhibited $K$ as the colimit. □

**Theorem B.4.** *There is an equivalence $K \simeq \operatorname{hocolim}_{\Delta K} *$ in $Ho(\mathsf{sSet})$.*

*About the proof.* We can rewrite the colimit over $\Delta K$ to look like the realization functor , just by unravelling definitions:

$$\operatorname*{colim}_{\Delta K} \Delta[n] = \operatorname{coeq}\left( \coprod_{\alpha: m \to k} \Delta[m] \times A_k \rightrightarrows \coprod_{n} \Delta[n] \times A_n \right)$$

And moreover if we replace all $\Delta[n]$ by a point we get a formula for $\operatorname{colim}_{\Delta K} *$.

At this stage I would like to say that the introduction of the $\Delta[n]$, which are of course all weakly equivalent to $*$, makes the big diagram cofibrant in the projective model strucure and thus its colimit computes the homotopy colimit.

Unfortunately the conclusion is true, but the diagram is not cofibrant in the projective model structure!

What actually happens is that there is a different model structure (called the Reedy model structure) on the diagram category $\mathsf{sSet}^{\Delta^{op} \times \Delta}$ such that forming the big colimit (called a coend) is a left Quillen functor. In the Reedy model structure weak equivalences are still defined objectwise. In this model structure the diagram $([m], [n]) \mapsto \Delta[m] \times A_n$ is cofibrant, so it computes the homotopy colimit for the Reedy model structure.

But since the homotopy category of a model category (and its diagram categories) only depends on weak equivalences one can show that homotopy colimits only depend on the choice of weak equivalences in the diagram category. Thus the mysterious Reedy model structure is good enough to construct the homotopy colimit (up to isomorphism in the homotopy category).

A precise reference for this is [AØ18]. □

## B.2. Some related model structures

We noted already in Example 3.33 that the fibrations and weak equivalences in $\mathsf{dgMod}_{\mathbb{Q}}^{\geq 0}$ and in $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ are the same and the forgetful functor $U$ is right Quillen.

Moreover, the proof that $\mathsf{dgMod}_{\mathbb{Q}}^{\geq 0}$ is a model category can be carried out quite similarly to the case of $\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0}$ in Theorem 4.1.

The same ideas are used to construct the model structure on dg algebras that are not necessarily commutative, writen $\mathsf{dgA}_{\mathbb{Q}}^{\geq 0}$. Recall from Example 3.9 that we define weak equivalences as quasi-isomorphisms and fibrations as degreewise surjections.

**Remark B.5.** If you want to verify that these are model categories the first step is to identify the generating cofibrations and acyclic cofibrations. This is a good exercise!

We will later use the following proposition:

**Proposition B.6.** *There are right Quillen functors*

$$\mathsf{cdgA}_{\mathbb{Q}}^{\geq 0} \xrightarrow{\iota} \mathsf{dgA}_{\mathbb{Q}}^{\geq 0} \xrightarrow{U'} \mathsf{dgMod}_{\mathbb{Q}}^{\geq 0}$$

*given by the inclusion functor and the forgetful functor.*

*Sketch of proof.* The "Quillen" part is clear, it only remains to present the left adjoints. For $\iota$ we define the left adjoint by sending a dg algebra to the quotient by its dg commutator ideal, generated by all terms $[a, b] := ab - (-1)^{|a||b|}ba$.

For $U'$ the left adjoint is given by a free algebra functor $T$, analogous to Definition 2.8 except that we never divide out by the action of the symmetric group. □

**Remark B.7.** In fact, the most systematic way is to first prove the model structure on $\mathsf{dgMod}_{\mathbb{Q}}^{\geq 0}$ and then show this structure can be transferred across an adjunction. This allows one to construct model structures on many different kinds of algebraic categories quite easily. Note that some assumptions will be needed. For example $\mathsf{dgMod}_{\mathbb{F}_p}$ has a model structure in the usual way, but it cannot be transferred to $\mathsf{cdgA}_{\mathbb{F}_p}$ by Remark 4.4.

We also have the following result:

**Lemma B.8.** *The map $C^* : \mathsf{sSet}_*^{op} \to \mathsf{dgA}_{/\mathbb{Q}}^{\geq 0}$ is left Quillen.*

*Sketch of proof.* The same proof as in Lemma 6.17 works once we define the right adjoint. But the right adjoint is defined exactly in the same way as for $A$, by considering morphisms into the simplicial dg algebra $C^*(\Delta^n)$. The only difference is the target category: general cochains are not commutative, so we have to map to the category of all dg algebras. □

## B.3. Cochain multiplication

We still need to compare the the multiplicative structure of $AX$ and $C^*X$ and so far all we know ist that the natural map $\rho : AX \to C^*X$ is not multiplicative.

There are two approaches here. We could show that while $\rho$ is not mulitplicative it is multiplicative up to homotopy, to be precise it is a *strong homotopy multplicative* map, also know as an $A_\infty$*-map*. This will ensure that $H(\rho)$ is in fact a homomorphism of graded algebras.

This is a very interesting construction, but it takes some work to set up, so we will instead use a clever trick I learned from lecture notes of Thomas Nikolaus.

We rely on three basic facts:

1. any simplicial set $K$ is of the form $\operatorname{hocolim}_{\Delta K} *$ by Theorem B.4,
2. the polynomial de Rham functor and the cochain functor both send homotopy colimits to homotpy limits by Lemmas 6.17 and B.8,
3. the inclusion $\iota : \mathsf{cdgA}_{\mathbb{Q}}^{\geq 0} \to \mathsf{dgA}_{\mathbb{Q}}^{\geq 0}$ preserves homotopy limits by Proposition B.6,
4. for a point we have $A(*) \cong \mathbb{Q} \cong C^*(*)$ by the definitions.

**Theorem B.9** (= Theorem 6.16)**.** *For any simplicial set $K$ there is a natural zig-zag of quasi-isomorphisms of dg algebras between $A(K)$ and $C^*(K)$. In particular there is a natural isomorphism of cohomology rings.*

*Proof.* Natural means that the quasi-isomorphisms are compatible with maps of simplicial sets, so let us fix $f : K \to L$. Then $f$ induces a natural map of simplex categories $\Delta f : \Delta K \to \Delta L$ and this in turn induces a map of $\mathrm{hocolim}_{\Delta K} * \to \mathrm{hocolim}_{\Delta L} *$ that is equivalent in the homotopy category to $f : K \to L$.

We recall the inclusion $\iota : \mathsf{cdgA}_{\mathbb{Q}}^{\geq 0} \to \mathsf{dgA}_{\mathbb{Q}}^{\geq 0}$ and recall that this $\iota$ is right Quillen.

We get natural maps

$$\begin{tikzcd}
\iota A(K) \arrow[r, "\sim"] \arrow[d, "\iota A(f)"] & \iota\, \mathrm{holim}_{\Delta K} A(*) \arrow[r, "\sim"] \arrow[d] & \mathrm{holim}_{\Delta K} \iota A(*) \arrow[r, "\sim"] \arrow[d] & \mathrm{holim}_{\Delta K} C^*(*) \arrow[d] & C^*(K) \arrow[l, "\sim"'] \arrow[d, "C^*(f)"] \\
\iota A(L) \arrow[r, "\sim"] & \iota\, \mathrm{holim}_{\Delta L} A(*) \arrow[r, "\sim"] & \mathrm{holim}_{\Delta L} \iota A(*) \arrow[r, "\sim"] & \mathrm{holim}_{\Delta L} C^*(*) & C^*(L) \arrow[l, "\sim"']
\end{tikzcd}$$

where all horizontal maps are zig-zags of weak equivalences by repeatedly using the fact that $\iota$, $A$ and $C^*$ preserves homotopy limits. This is the promised zig-zag of quasi-isomorphisms. Taking cohomology all these maps become isomorphisms, proving the second statement of the proof. □

**Remark B.10.** We can compose $\iota$ with the forgetful functor $U' : \mathsf{dgA}_{\mathbb{Q}}^{\geq 0} \to \mathsf{dgMod}_{\mathbb{Q}}^{\geq 0}$ and this again preserves homotopy limits by Proposition B.6. But on $\mathsf{dgMod}_{\mathbb{Q}}$ we know that there is a map $A(K) \to C^*(K)$ induced by $\rho$ that is also compatible with colimits. This shows that in fact $H(\rho)$ is a ring isomorphism (rather than some other abstract map).

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