Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Julian Holstein
Algebraic Topology (Master)
Summer Semester 2023

Exercise sheet 7

Question 7.1

Compute the homology of the Klein bottle with coefficients in $\mathbb{Z}/2$, $\mathbb{Z}/3$,$\mathbb{Z}/4$ and $\mathbb{C}$.

Question 7.2

a) Let n, m be natural numbers larger than one. Compute $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$.

b) Let A be a finitely generated abelian torsion group. What is $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$?

c) Is the abelian group $\mathbb{Q}$ free?

Question 7.3

a) Prove that for a split-exact sequence $0 \to A \to B \to C \to 0$, the sequence

$$0 \longrightarrow A \otimes D \xrightarrow{f \otimes \mathrm{id}} B \otimes D \xrightarrow{g \otimes \mathrm{id}} C \otimes D \longrightarrow 0$$

is exact and splits.

b) Let C_* be a chain complex and A a free abelian group. Prove directly that $H_n(C_* \otimes A) = H_n(C) \otimes A$.

Question 7.4

Give an example of a chain complex (C_*, d) with trivial homology, such that the chain complex $C_* \otimes \mathbb{Z}/2\mathbb{Z}$ has non-vanishing homology in every degree.

Question 7.5

a) Assume that R is a commutative ring with unit. If you don't know what R-modules are, then look up the definition. Note that a $\mathbb{Z}$-module is just an abelian group. Let M and N be two R-modules. Define $M \otimes_R N$.

b) Can you define Tor for R-modules in the same way as we did for $R = \mathbb{Z}$? What is different?

c) What can you say if R is a field? What about $R = \mathbb{Z}/4\mathbb{Z}$?

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These questions will be discussed in the class on 31.5.23. You may hand in your solutions (in pairs) the day before.

Questions with an asterisk are more challenging.