Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Julian Holstein
Algebraic Topology (Master)
Summer Semester 2023

Exercise sheet 6

Question 6.1

Let X be a finite CW complex. The *Euler characteristic* of X, $\chi(X)$, is then defined as

$$\chi(X) := \sum_{n\geq 0}(-1)^n \mathrm{rk}(H_n(X;\mathbb{Z})).$$

Here, rk denotes the rank of a finitely generated abelian group, i.e. the number of its free summands.

Let now C be a chain complex such that all but finintely many C_n are zero and all C_n have finite rank. Then we define $\chi(C_*) = \sum(-1)^n \mathrm{rk}(C_n)$.

a) Show that $\chi(C_*) = \sum(-1)^n \mathrm{rk}(H_n(C))$.

b) Let $c_n(X)$ denote the number of n-cells of X. Prove that

$$\chi(X) = \sum_{n\geq 0}(-1)^n c_n(X).$$

c) What is $\chi(X)$ for a sphere, torus, and for a general oriented compact closed surface of genus g, F_g?

d) What can you say about $\chi(X \sqcup Y)$ for two finite CW complexes X and Y? What about $\chi(X \cup Y)$ if X and Y are not necessarily disjoint? Assume that $X \cap Y$ is a subcomplex of X and Y.

e) For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

f) Assume that X is a finite CW complex and that $p\colon \tilde{X} \to X$ is an n-sheeted covering. Prove that $\chi(\tilde{X}) = n\chi(X)$.

Question 6.2

a) Give a CW model of $\mathbb{S}^n \times \mathbb{S}^m$ and use it to compute the homology of the space if $n, m \geq 2$.

b) Now compute the homology of $\mathbb{S}^1 \times \mathbb{S}^n$ using your CW model.

c) Compare the homology groups $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$. Are the spaces homotopy equivalent? Is there a cellular map inducing an isomorphism in homology?

Question 6.3

Let G be an arbitrary finitely generated abelian group.

a) Construct a CW complex $M(G, n)$ whose reduced homology is concentrated in degree n with $\tilde{H}_n(M(G, n)) \cong G$. Such a space is called a *Moore space of type* (G, n). You may begin with the case that G is a cyclic group.

b) What is $M(\mathbb{Z}/2\mathbb{Z}, 1)$?

Question 6.4

For $g \geq 2$ consider a regular $2g$-gon $P_{2g} \subset \mathbb{R}^2$ with vertices $z_1, \ldots, z_{2g}$. We identify edges according to

$$(1-t)z_{2j-1} + tz_{2j} \sim tz_{2j+1} + (1-t)z_{2j}$$

(here the indices are to be read mod $2g$) and call the quotient $N_g = P_{2g}/\sim$ the *closed non-orientable surface of genus* g. What is N_2? Calculate the homology of N_{2g} using the cellular chain complex.

Question 6.5

* The group of rotational symmetries of a Dodecahedron is isomorphic to A_5. It is naturally a subgroup of $SO(3)$. Compute the homology of the quotient space $X = SO(3)/A_5$.

These questions will be discussed in the class on 24/5/23. You may hand in your solutions (in pairs) the day before.

Questions with an asterisk are more challenging.