Universität Hamburg
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Algebraic Topology (Master)
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Exercise sheet 5

Question 5.1

Consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5
\end{array}$$

Check under which assumptions on f_1, f_2, f_4, f_5 we can deduce that the map f_3 is injective respectively surjective.

Question 5.2

We consider the following commutative diagram with exact columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1 & \xrightarrow{\gamma_1} & C_2 & \xrightarrow{\gamma_2} & C_3 & \xrightarrow{\gamma_1} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

a) Prove that the top row is exact if the two bottom rows are exact and that the bottom row is exact if the two top rows are exact.

b) What happens if the top and bottom rows are exact? Can you deduce that the middle row is exact or do you need an extra condition?

Question 5.3

a) Let $A \in O(n+1)$. Then multiplication by A induces a continuous self-map on $\mathbb{S}^n$. What is its degree?

b) Construct a map $\mathbb{S}^n \to \mathbb{S}^n$ of degree k for every k.

Question 5.4

Let $f : K \to X$ be a map from a compact space to a CW complex. Show that $f(K)$ is contained in a finite subcomplex.

Question 5.5

* Let $f, g\colon X \to Y$ be two continuous maps. The *mapping torus of f and g* is the space $T(f, g)$ defined as the quotient of $X \times [0, 1] \sqcup Y$ by $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$. (Important special cases are if f is the identity and g is a homeomorphism.)

Prove that there is a long exact sequence

$$\ldots \longrightarrow H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{i_*} H_n(T(f,g)) \xrightarrow{\delta} H_{n-1}(X) \xrightarrow{f_* - g_*} \ldots$$

and verify your computation of the homology groups of the Klein bottle.

These questions will be discussed in the class on 10/5/23. You may hand in your solutions (in pairs) the day before.

Questions with an asterisk are more challenging.