Universität Hamburg
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Exercise sheet 3

Question 3.1

Compute the homology groups H_0 and H_1 of $GL_2(\mathbb{R})$.

*Compare the homology groups of $GL_n(\mathbb{R})$ and $O(n)$ for all n.

Question 3.2

Prove the famous Snake Lemma:

If

$$\begin{array}{ccccccccc} & & A' & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\alpha'} & B & \xrightarrow{\beta'} & B'' & & \end{array}$$

is a commutative diagram of abelian groups with exact rows, then there is an exact sequence

$$\ker(f') \longrightarrow \ker(f) \longrightarrow \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(f'').$$

Define δ explicitly in this case.

Question 3.3

Let M be an abelian group and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of abelian groups.

What can you say about the exactness of the sequence

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}(M, C) \longrightarrow 0?$$

Question 3.4

Consider the following short exact sequence of complexes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

where p and q denote multiplication by p respectively q and π is the canonical projection. Compute the long exact sequence on homology groups in the following cases:

a) p and q are coprime,

b) $q = p$,

c) $q = p^2$.

Can you modify the middle term in example (b), leaving the outer terms the same but changing the boundary map?