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Algebraic Topology (Master)
Summer Semester 2023

Exercise sheet 12

Question 12.1

You know the spaces N_g from Exercise 6.4. We called N_g the non-orientable surface of genus g. Justify that name.

Question 12.2

Let M be an m-dimensional connected topological manifold.

a) Prove that there is an oriented manifold $\hat{M}$ and a 2-fold covering $p\colon \hat{M} \to M$ called the orientation covering.

b) Are the following statements equivalent?

- M is orientable.
- The orientation covering is a trivial covering, i.e. $\hat{M} \cong M \times \mathbb{Z}/2\mathbb{Z}$ as spaces over M.

c) Assume that M is finite dimensional, path connected with $\pi_1(M) = 1$. Is M orientable?

d) What is the orientation covering of $\mathbb{R}P^n$ for even n? What about the Klein bottle and the open Möbius strip?

Question 12.3

Let M and N be two oriented compact connected manifolds of the same dimension $m \geq 1$ and let $f\colon M \to N$ be continuous. Define the *degree of* f.

a) Consider the projection map $\pi\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1 / \mathbb{S}^1 \vee \mathbb{S}^1 \cong \mathbb{S}^2$. What is the degree?

b) Let $f\colon M \to N$, $g\colon N \to L$ be continuous maps between oriented compact connected manifolds and let Show that degree is multiplicative, i.e.

$$\deg(g \circ f) = \deg(g)\deg(f).$$

c) If $\bar{M}$ is the same manifold as M but with opposite orientation, then

$$\deg(f) = \deg(f\colon \bar{M} \to \bar{N}_1) = -\deg(f\colon \bar{M} \to N_1) = -\deg(f\colon M \to \bar{N}_1).$$

d) If the degree of f is not trivial, then f is surjective.

Question 12.4

Let M be a compact connected 3-manifold. Its first homology group is a finitely generated abelian group and is hence of the form

$$H_1(M) \cong \mathbb{Z}^n \oplus T$$

where T denotes the finite torsion part of $H_1(M)$.

a) Determine $H_2(M)$ if M is orientable.

b) Does $\pi_1(M)$ determine $H_*(M)$ in this case?

c) What happens if we drop the assumption that M is orientable? Can you still say something about $H_2(M)$?

Question 12.5

Compute $H_c^*(X, \mathbb{Z})$ for

a) X equal to the open cylinder,

b) X equal to the open Möbius band.

These questions will be discussed in the class on 5 July 23. You may hand in your solutions the day before.

Questions with an asterisk are more challenging.