

## Exercise sheet 10

### Question 10.1

Check that the swap map  $\tau : a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$  is a chain map  $V \otimes W \rightarrow W \otimes V$ , but the 'naive' swap  $a \otimes b \mapsto b \otimes a$  is not.

### Question 10.2

Show that the short exact sequence in the universal coefficient theorem for cohomology is split.

### Question 10.3

The Mayer Vietoris formula is missing from the axioms of a (co)homology theory. In this exercise you will deduce it from excision.

- a) Let  $X$  be a topological space with open subspaces  $U, V$  satisfying  $X = U \cup V$ . Show that  $H_n(U, U \cap V) \cong H_n(X, V)$  for all  $n$ .
- b) Consider the following diagram with exact rows

$$\begin{array}{ccccccccccc}
 \dots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n & \xrightarrow{d_n} & A_{n-1} & \longrightarrow & \dots \\
 & & \downarrow h_{n+1} & & \downarrow f_n & & \downarrow g_n & & \downarrow h_n & & \downarrow f_{n-1} & & \\
 \dots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & A'_n & \xrightarrow{i'_n} & B'_n & \xrightarrow{p'_n} & C'_n & \xrightarrow{d'_n} & A'_{n-1} & \longrightarrow & \dots
 \end{array}$$

where all  $h_n$  are isomorphisms. Then defining  $\delta = d_n \circ (h_n)^{-1} \circ p'_n$  there is an exact sequence

$$\dots \rightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{\delta} A_{n-1} \dots$$

Check that this sequence is exact at  $A_n$ .

- c) Deduce the Mayer-Vietoris sequence for homology.

### Question 10.4

Compute the cohomology ring of the  $n$ -torus  $T^n \cong (S^1)^n$ .

**These questions will be discussed in the class on 20/6/2023. You may hand in your solutions the day before.**

Questions with an asterisk are more challenging.