Algebraic Topology (Master)

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Contents

These are the lecture notes for the Algebraic Topology Masters course at University of Hamburg in summer semester 2023. Many thanks to Birgit Richter on whose notes for the same course this is based. Her lecture notes are available at `https://www.math.uni-hamburg.de/home/richter/topo19.html`.

I'm always grateful for comments and corrections, please email `julian.holstein@uni-hamburg.de`. Many thanks to Patrick Antweiler for helpful comments!

The current version of this script is available as a pdf from `http://www.math.uni-hamburg.de/home/holstein/lehre/AT23notes.pdf`.

Note that internal references do not include the name of the chapter. The hyperlinks link to the correct place in the script; if you are using a paper copy of the notes there is at most two places to check (and it is usually clear from context which one it is).

Literature: Some useful books to complement the lectures notes are as follows.

- G. Bredon, *Topology and Geometry*, Springer 1993;
- A. Hatcher, *Algebraic Topology*, Cambridge University Press 2001;
- J. Munkres, *Topology*, Prentice-Hall 1975;
- E. Spanier, *Algebraic Topology*, Springer 1966;
- R. Stöcker, H. Zieschang, *Algebraische Topologie*, Teubner 1994;
- T. tom Dieck, *Algebraic Topology*, EMS 2008.

This list may be extended. The main reference for things treated in this course is Hatcher's Algebraic Topology.

CHAPTER 1

Introduction

This is a second course on topology with a focus on homology and cohomology theory.

I will assume you have taken a first course in topology (or done some equivalent reading) and know about

- Topological spaces, continuous maps, homeomorphisms,
- examples like Euclidean space $\mathbb{R}^n$, closed balls D^n, spheres $\mathbb{S}^n$, surfaces Σ_g, real projective space $\mathbb{R}P^n$,
- compactness and (path) connectedness,
- building topological spaces out of other spaces and maps by products, gluing along a map and taking the suspension,
- homotopies between continuous maps and homotopy equivalences between spaces
- the fundamental group of a space X with base point x as the set of homotopy classes of pointed maps from $\mathbb{S}^1$ to X, made into a group with the operation of concatenation,
- ideally you know the category of topological spaces and homotopy classes of maps between them and know that the fundamental group is a functor from the pointed homotopy category to the category of groups

If you know about these things you know how to show that the circle $\mathbb{S}^1$ is not homotopy equivalent to the point and the torus $T^2 \cong \mathbb{S}^1 \times \mathbb{S}^1 \cong \Sigma_1$ is not homotopy equivalent to the genus 2 surface Σ_2.

You probably don't know how to show that the sphere $\mathbb{S}^2$ is not homotopy equivalent to the point.

One way to prove that is to generalize the definition of the fundamental group to the higher homotopy groups, but they are very hard to compute.

To illustrate this, here are some homotopy groups of the 2-sphere $\mathbb{S}^2$ (writing $\mathbb{Z}_n$ for the cyclic groups $\mathbb{Z}/n\mathbb{Z}$.):

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\pi_n(\mathbb{S}^2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Instead of studying homotopy groups we will try to count holes in a computable way by linearizing the problem. (This sentence may not make sense right now.)

The basic idea is that when a loop γ in X is contractible we can extend $\gamma : S^1 \to X$ to a map $\theta : D^2 \to X$. Then the loop γ may be considered as the boundary of the disk θ. The boundary of γ itself is trivial as it is a loop. If instead we consider a general path $\beta : I \to X$ the boundary would be given by the restriction $\{0, 1\} \to X$. If these two points are the same the path is a loop and the boundary should be considered empty. So we say the boundary of β is the formal sum $\beta(1) - \beta(0)$ and it is 0 exactly if β is a loop.

A disk does not have a nice discrete boundary, but we can replace it (homeomorphically!) by a triangle with three sides. Then we define the boundary to be the alternating sum of the three sides, and if their concatenation is a given loop γ then γ is the boundary of the disk.

We will now develop this theory systematically.

In particular, if this motivational detour was mysterious to you, do not worry.

1. Chain complexes

DEFINITION 1.1. A *chain complex* is a sequence of abelian groups, $(C_n)_{n\in\mathbb{Z}}$, together with homomorphisms $d_n\colon C_n \to C_{n-1}$ for $n \in \mathbb{Z}$, such that $d_{n-1}\circ d_n = 0$.

Let R be an associative ring with unit 1_R. A *chain complex of R-modules* can analogously be defined as a sequence of R-modules $(C_n)_{n\in\mathbb{Z}}$ with R-linear maps $d_n\colon C_n \to C_{n-1}$ with $d_{n-1}\circ d_n = 0$.

DEFINITION 1.2.

- The d_n are *differentials* or *boundary operators*.
- $x \in C_n$ is called an *n-chain* and n is the *degree* of x.
- An $x \in C_n$ with $d_n x = 0$ is called an *n-cycle*.
$$Z_n(C) := \{x \in C_n \mid d_n x = 0\}.$$
- If $x \in C_n$ is of the form $x = d_{n+1}y$ for some $y \in C_{n+1}$, then x is an *n-boundary*.
$$B_n(C) := Im(d_{n+1}) = \{d_{n+1}y \mid y \in C_{n+1}\}.$$

Note that the cycles and boundaries form subgroups of the chains. As $d_n \circ d_{n+1} = 0$, we know that the image of d_{n+1} is a subgroup of the kernel of d_n and thus
$$B_n(C) \subset Z_n(C).$$

We will often often drop the subscript n from the boundary maps and write d. Other times we write d^C to emphasize that our differential belongs to a complex $(C_n)_{n\in\mathbb{Z}}$, which we often just write C_*.

DEFINITION 1.3. The abelian group $H_n(C) := Z_n(C)/B_n(C)$ is the *nth homology group of the complex C_**.

If $H_n(C) = 0$ we say C_* is *exact* at C_n. So the homology groups measure the extent to which C_* is not exact. The idea is that an exact chain complex may be large but it is boring, much like a contractible space in topology. If some element in C_* is a cycle it could be because it is a boundary, but that is not a very interesting reason, any boundary is a cycle by definition. But if there is an element x that is a cycle, i.e. it has no boundary, such that x is not itself a boundary, there may be something interesting going on. Much like the a loop in the fundamental group that cannot by contracted as there is a hole in our space.

If $c, c' \in C_n$ are such that $c - c'$ is a boundary, then we say c is *homologous* to c'. We denote by $[c]$ the equivalence class of a $c \in Z_n(C)$, or equivalently the image of c in $H_n(C)$.

EXAMPLE 1.4. (a) Consider
$$C_n = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and let d_1 be the multiplication with $N \in \mathbb{N}$, then

$$H_n(C) = \begin{cases} \mathbb{Z}/N\mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Take $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ and

$$d_n = \begin{cases} \mathrm{id}_{\mathbb{Z}} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

What is the homology of this chain complex?

(c) Consider $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ again, but let all boundary maps be trivial. What is the homology of this chain complex?

Definition 1.5. Let C_* and D_* be two chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence of homomorphisms $f_n\colon C_n \to D_n$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$ for all n, *i.e.*, the diagram

$$\begin{tikzcd} C_n \arrow[r, "d_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "d_n^D"] & D_{n-1} \end{tikzcd}$$

commutes for all n.

Such an f sends cycles to cycles and boundaries to boundaries. We therefore obtain an induced map

$$H_n(f)\colon H_n(C) \to H_n(D)$$

via $H_n(f)[c] = [f_n c]$.

There is a chain map from the chain complex mentioned in Example a) to the chain complex D_* that is concentrated in degree zero and has $D_0 = \mathbb{Z}/N\mathbb{Z}$. Note, that $H_0(f)$ is an isomorphism on 0th homology groups.

Are there chain maps between the complexes from Examples b) and c)?

Recall that a category is a collection (not necessarily a set) of *objects* and for every pair of objects A, B a collection of *morphisms* $\mathrm{Hom}(A, B)$, written $f : A \to B$, such that there is an associative composition and for each object there is a unit $\mathrm{id}_A \in \mathrm{Hom}(A, A)$.

You know many categories already, even if you don't know the word. for example you know the categories of topological spaces and continuous maps, vector spaces and linear maps or groups and homomorphisms.

Proposition 1.6. *There is a category* Ch *whose objects are chain complexes and whose morphisms are chain maps.*

Proof. To show the proposition we have to check that the composition of two chain maps is a chain map, and that the degree-wise identity map is a chain map. These are both immediate. □

From now on any map $f : C_* \to D_*$ between chain complexes will be assumed to be a chain map.

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ between two categories assigns to every object C of $\mathcal{C}$ a (unique) object $F(C)$ of $\mathcal{D}$ and to every morphism f in $\mathrm{Hom}(C, C')$ a morphism $F(f)$ in $\mathrm{Hom}(F(C), F(C'))$, such that composition and unit are respected: $F(g) \circ F(f) = F(g \circ f)$ whenever that is defined, and $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$.

Lemma 1.7. *For all n the rule $C_* \mapsto H_n(C)$ defines a functor from the category of chain complexes Ch to the category of abelian groups Ab.*

Proof. If $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ are two chain maps, we have to check that $H_n(g) \circ H_n(f) = H_n(g \circ f)$, but this is immediate from the definition: Both sides send $[c]$ to $[g(f(c))]$. We also have to check $H_n(\mathrm{id}_C) = \mathrm{id}_{H_n(C)}$, which is immediate. $\square$

When do two chain maps induce the same map on homology?

Definition 1.8. A *chain homotopy* H between two chain maps $f, g\colon C_* \to D_*$ is a sequence of homomorphisms $(H_n)_{n\in\mathbb{Z}}$ with $H_n\colon C_n \to D_{n+1}$ such that for all n

$$d^D_{n+1} \circ H_n + H_{n-1} \circ d^C_n = f_n - g_n.$$

$$\begin{tikzcd}
\ldots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[dl, "H_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[dl, "H_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[dl, "H_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \ldots \\
\ldots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \ldots
\end{tikzcd}$$

If such an H exists, then f and g are *(chain) homotopic*: $f \simeq g$.

The name is consciously chosen to remind you of homotopies between continuous maps and we will later see geometrically defined examples of chain homotopies.

Proposition 1.9. *(a) Being chain homotopic is an equivalence relation.*
(b) If f and g are homotopic, then $H_n(f) = H_n(g)$ for all n.

Proof. (a) If H is a homotopy from f to g, then $-H$ is a homotopy from g to f. Each f is homotopic to itself with $H = 0$. If f is homotopic to g via H and g is homotopic to h via K, then f is homotopic to h via $H + K$.

(b) We have for every cycle $c \in Z_n(C_*)$:

$$H_n(f)[c] - H_n(g)[c] = [f_n c - g_n c] = [d^D_{n+1} \circ H_n(c)] + [H_{n-1} \circ d^C_n(c)] = 0.$$

$\square$

Definition 1.10. Let $f\colon C_* \to D_*$ be a chain map. We call f a *chain homotopy equivalence*, if there is a chain map $g\colon D_* \to C_*$ such that $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$. The chain complexes C_* and D_* are then *chain homotopy equivalent*.

By Proposition 1.9 and functoriality of homology we see that if f is a chain homotopy equivalence with inverse g then $H_n(f)$ has inverse $H_n(g)$, thus we have:

Corollary 1.11. *If $f : C_* \to D_*$ is a chain homotopy equivalenve then $H_n(f)$ is an isomorphism for each n.*

However, chain complexes with isomorphic homology do not have to be chain homotopically equivalent. (Can you find a counterexample?)

Definition 1.12. If C_* and C'_* are chain complexes, then their *direct sum*, $C_* \oplus C'_*$, is the chain complex with

$$(C_* \oplus C'_*)_n = C_n \oplus C'_n = C_n \times C'_n$$

with differential $d = d_\oplus$ given by

$$d_\oplus(c, c') = (dc, dc').$$

Similarly, if $(C_*^{(j)}, d^{(j)})_{j \in J}$ is a family of chain complexes, then we can define their direct sum as follows:

$$(\bigoplus_{j \in J} C_*^{(j)})_n := \bigoplus_{j \in J} C_n^{(j)}$$

as abelian groups and the differential $d_\oplus$ is defined via the property that its restriction to the jth summand is $d^{(j)}$.

2. Singular homology

Definition 2.1. For every n we define the *(topological) n-simplex* Δ^n as

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum t_i = 1, t_i \geqslant 0\}.$$

Example 2.2. Δ^0 is a point, Δ^1 a line segment, Δ^2 a triangle, Δ^3 a tetrahedron.

By definition $\Delta^n \subset \mathbb{R}^{n+1}$, but we may always consider $\Delta^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \subset \ldots$.

The boundary of Δ^1 consists of two copies of Δ^0, the boundary of Δ^2 consists of three copies of Δ^1. In general, the boundary of Δ^n consists of $n+1$ copies of Δ^{n-1}. (Note this is not the boundary in the topological sense as subspaces of $\mathbb{R}^{n+1}$, but this is just intuition for the following formalization.)

We need the following *face maps* for $0 \leqslant i \leqslant n$

$$d_i = d_i^{n-1} \colon \Delta^{n-1} \hookrightarrow \Delta^n; (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

We will write e_i for the standard unit vector that is 1 in the i-th component and 0 otherwise. (We start counting at $i = 0$.) The image of d_i^{n-1} in Δ^n is the face that is opposite to e_i. It is the convex hull of $e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n$.

Lemma 2.3. *Concerning the composition of face maps, the following rule holds:*

$$d_i^{n-1} \circ d_j^{n-2} = d_j^{n-1} \circ d_{i-1}^{n-2}, \quad 0 \leqslant j < i \leqslant n.$$

For example we may consider the two maps $d_2 \circ d_0$ and $d_0 \circ d_1$from Δ^0 to Δ^2. We have $\Delta^0 = \{e_0\} = \{(1)\}$ and $d_2(d_0((1))) = d_2((0,1)) = (0,1,0)$ and $d_0(d_1(e_0)) = d_0((1,0)) = (0,1,0)$.

Proof. Both expressions yield

$$d_i^{n-1} \circ d_j^{n-2}(t_0, \ldots, t_{n-2}) = (t_0, \ldots, t_{j-1}, 0, \ldots, t_{i-2}, 0, \ldots, t_{n-2}) = d_j^{n-1} d_{i-1}^{n-2}(t_0, \ldots, t_{n-2}).$$

□

Remark 2.4. More generally any injection $f : \{0, \ldots, k\} \to \{0, \ldots, n\}$ induces a map $\Delta^k \to \Delta^n$ by sending e_i to $e_{f(i)}$ and extending linearly.

Let X be an arbitrary topological space, $X \neq \varnothing$.

Definition 2.5. A *singular n-simplex* in X is a continuous map $\alpha\colon \Delta^n \to X$.

Definition 2.6. Let $S_n(X)$ be the free abelian group generated by all singular n-simplices in X. We call $S_n(X)$ the *nth singular chain module of X*.

Elements of $S_n(X)$ are finite formal sums $\sum_{i\in I} \lambda_i\alpha_i$ with $\lambda_i = 0$ for almost all $i \in I$ and $\alpha_i\colon \Delta^n \to X$.

For all $n \geqslant 0$ there are non-trivial elements in $S_n(X)$, because we assumed that $X \neq \varnothing$: we can always take an $x_0 \in X$ and the constant map $\kappa_{x_0}\colon \Delta^n \to X$ as α. By convention, we define $S_n(\varnothing) = 0$ for all $n \geqslant 0$. (It's the free abelian group on no generators.)

If we want to define maps from $S_n(X)$ to some abelian group then it suffices to define such a map on generators.

Example 2.7. What is $S_0(X)$? A continuous $\alpha\colon \Delta^0 \to X$ is determined by its value $\alpha(e_0) =: x_\alpha \in X$, which is a point in X. A singular 0-simplex $\sum_{i\in I} \lambda_i\alpha_i$ can thus be identified with the formal sum of points $\sum_{i\in I} \lambda_i x_{\alpha_i}$.

For instance if you count the zeroes and poles of a meromorphic function with multiplicities then this gives an element in $S_0(X)$. In algebraic geometry a divisor on a curve X is an element in $S_0(X)$.

Definition 2.8. We define $\partial_i\colon S_n(X) \to S_{n-1}(X)$ on generators

$$\partial_i(\alpha) = \alpha \circ d_i^{n-1}$$

and call it the *ith face of α*.

On $S_n(X)$ we therefore get $\partial_i(\sum_j \lambda_j\alpha_j) = \sum_j \lambda_j(\alpha_j \circ d_i^{n-1})$.

Lemma 2.9. *The face maps on $S_n(X)$ satisfy*

$$\partial_j \circ \partial_i = \partial_{i-1} \circ \partial_j, \quad 0 \leqslant j < i \leqslant n.$$

Proof. This follows directly from Lemma 2.3. □

Definition 2.10. We define the boundary operator on singular chains as $\partial\colon S_n(X) \to S_{n-1}(X)$, $\partial = \sum_{i=0}^n (-1)^i\partial_i$.

Lemma 2.11. *The map ∂* is *a boundary operator,* i.e., *$\partial \circ \partial = 0$.*

Proof. We calculate

$$\begin{aligned}
\partial \circ \partial = (\sum_{j=0}^{n-1}(-1)^j\partial_j) \circ (\sum_{i=0}^{n}(-1)^i\partial_i) &= \sum\sum(-1)^{i+j}\partial_j \circ \partial_i \\
&= \sum_{0\leqslant j<i\leqslant n}(-1)^{i+j}\partial_j \circ \partial_i + \sum_{0\leqslant i\leqslant j\leqslant n-1}(-1)^{i+j}\partial_j \circ \partial_i \\
&= \sum_{0\leqslant j<i\leqslant n}(-1)^{i+j}\partial_{i-1} \circ \partial_j + \sum_{0\leqslant i\leqslant j\leqslant n-1}(-1)^{i+j}\partial_j \circ \partial_i = 0.
\end{aligned}$$

Where in the last line we relabelled $i-1$ as j and j as i in the first summand to identify it with the negative of the second summand. □

We therefore obtain the *singular chain complex*, $S_*(X)$,

$$\ldots \longrightarrow S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \longrightarrow 0.$$

The singular chain complex is very large and unwieldy! But its homology contains important information about X and we will find many ways of computing this homology without ever having to worry about classifying all maps from Δ^k to X.

We abbreviate $Z_n(S(X))$ by $Z_n(X)$, $B_n(S(X))$ by $B_n(X)$ and $H_n(S(X))$ by $H_n(X)$.

DEFINITION 2.12. For a space X, $H_n(X)$ is the *nth singular homology group* of X.

Note that $Z_0(X) = S_0(X)$ as $S_{-1}(X) = 0$.

As an example of a 1-cycle consider a 1-chain $c = \alpha + \beta + \gamma$ where $\alpha, \beta, \gamma \colon \Delta^1 \to X$ such that $\alpha(e_1) = \beta(e_0)$, $\beta(e_1) = \gamma(e_0)$ and $\gamma(e_1) = \alpha(e_0)$ and calculate that $\partial c = 0$. (One way to obtain such a 1-cycle is to take a loop and divide it into three parts.)

We need to understand how continuous maps of topological spaces interact with singular chains and singular homology. So let $f \colon X \to Y$ be a continuous map.

DEFINITION 2.13. The map $f_n = S_n(f) \colon S_n(X) \to S_n(Y)$ is defined on generators $\alpha \colon \Delta^n \to X$ as

$$f_n(\alpha) = f \circ \alpha : \Delta^n \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

LEMMA 2.14. *The singular chain complex defines a functor* $S_* : \mathsf{Top} \to \mathsf{Ch}$. *For every* n *the singular homology* H_n *defines a functor* $\mathsf{Top} \to \mathsf{Ab}$.

PROOF. We have to show that for any continuous map $f : X \to Y$ the induced map $f_n : S_n(X) \to S_n(Y)$ assemble into a chain map f_*, i.e. we need

$$\begin{tikzcd}
S_n(X) \arrow[r, "f_n"] \arrow[d, "\partial^X"'] & S_n(Y) \arrow[d, "\partial^Y"] \\
S_{n-1}(X) \arrow[r, "f_{n-1}"] & S_{n-1}(Y).
\end{tikzcd}$$

But by definition

$$\partial^Y(f_n(\alpha)) = \sum_{i=0}^{n} (-1)^i (f \circ \alpha) \circ d_i = \sum_{i=0}^{n} (-1)^i f \circ (\alpha \circ d_i) = f_{n-1}(\partial^X \alpha).$$

The identity map on X induces the identity map on $S_n(X)$ for all $n \geqslant 0$ and if we have a composition of continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then $S_n(g \circ f) = S_n(g) \circ S_n(f)$.

As f_* is a chain map it induces a map on homology which is functorial by Lemma 1.7. □

In any category a morphism f with an inverse morphism g such that $f \circ g$ and $g \circ f$ is called an *isomorphism*. It follows directly from the definition that any functor preserves isomorphisms. Thus by Lemma 2.14 it follows that homeomorphic spaces have isomorphic homology groups:

$$X \cong Y \Rightarrow H_n(X) \cong H_n(Y) \text{ for all } n \geqslant 0.$$

Our first (not too exciting) calculation is the following. We will denote the 1 point space by $*$.

Proposition 2.15. *The homology groups of a one-point space $*$ are trivial but in degree zero,*

$$H_n(*) \cong \begin{cases} 0, & \text{if } n > 0, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. For every $n \geqslant 0$ there is precisely one continuous map $\alpha \colon \Delta^n \to *$, namely the constant map. We denote this map by κ_n. Then the boundary of κ_n is

$$\partial \kappa_n = \sum_{i=0}^{n} (-1)^i \kappa_n \circ d_i = \sum_{i=0}^{n} (-1)^i \kappa_{n-1} = \begin{cases} \kappa_{n-1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

For all n we have $S_n(\mathrm{pt}) \cong \mathbb{Z}$ generated by κ_n and therefore the singular chain complex looks as follows:

$$\ldots \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=\mathrm{id}_{\mathbb{Z}}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z}. \qquad \square$$

3. H_0 and H_1

Next we will compute the lowest homology groups. We begin by defining a map:

Proposition 3.1. *For any topological space X there is a homomorphism $\varepsilon \colon H_0(X) \to \mathbb{Z}$ with $\varepsilon \neq 0$ for $X \neq \varnothing$.*

Proof. For any topological space there is a unique projection map to the 1 point space. By Lemma 2.14 this induces a map on homology, so $H_0(X)$ maps to $H_0(*) = \mathbb{Z}$.

We can also construct ϵ more explicitly: By definition $S_0(\varnothing)$ is zero, so $H_0(\varnothing) = 0$ and in this case we define ε to be the zero map.

If $X \neq \varnothing$, then we define $\varepsilon(\alpha) = 1$ for any $\alpha \colon \Delta^0 \to X$, thus $\varepsilon(\sum_{i \in I} \lambda_i \alpha_i) = \sum_{i \in I} \lambda_i$ on $S_0(X)$. As only finitely many λ_i are non-trivial, this is in fact a finite sum.

We have to show that this map is well-defined on homology, i.e. that it vanishes on boundaries. Let $S_0(X) \ni c = \partial b$ be a boundary and write $b = \sum_{i \in I} \nu_i \beta_i$ with $\beta_i \colon \Delta^1 \to X$. Then we get

$$\partial b = \partial \sum_{i \in I} \nu_i \beta_i = \sum_{i \in I} \nu_i (\beta_i \circ d_0 - \beta_i \circ d_1) = \sum_{i \in I} \nu_i \beta_i \circ d_0 - \sum_{i \in I} \nu_i \beta_i \circ d_1$$

and hence

$$\varepsilon(c) = \varepsilon(\partial b) = \sum_{i \in I} \nu_i - \sum_{i \in I} \nu_i = 0.$$

$\square$

If $X \neq \varnothing$, then any $\alpha \colon \Delta^0 \to X$ can be identified with its image point, so the map ε on $S_0(X)$ counts points in X with multiplicities.

Proposition 3.2. *If X is a path-connected, non-empty space, then $\varepsilon \colon H_0(X) \cong \mathbb{Z}$.*

Proof. As X is non-empty, there is a point $x \in X$ and the constant map κ_x with value x is an element in $S_0(X)$ with $\varepsilon(\kappa_x) = 1$. Therefore ε is surjective. Any other generator of $S_0(X)$ is of the form κ_y for some point $y \in X$ and there is a continuous path $\omega\colon [0,1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. We define $\alpha_\omega\colon \Delta^1 \to X$ as

$$\alpha_\omega(t_0, t_1) = \omega(1 - t_0).$$

Then

$$\partial(\alpha_\omega) = \partial_0(\alpha_\omega) - \partial_1(\alpha_\omega) = \alpha_\omega(e_1) - \alpha_\omega(e_0) = \alpha_\omega(0,1) - \alpha_\omega(1,0) = \kappa_y - \kappa_x,$$

and the two generators κ_x, κ_y are homologous. This shows that ε is injective. $\square$

From now on we will identify paths w and their associated 1-simplices α_w.

Corollary 3.3. *If X is of the form $X = \bigsqcup_{i\in I} X_i$ such that the X_i are non-empty and path-connected, then*

$$H_0(X) \cong \bigoplus_{i\in I} \mathbb{Z}.$$

In this case, the zeroth homology group of X is the free abelian group generated by the path-components.

Proof. As the Δ^n are connected the singular chain complex of X splits as the direct sum of chain complexes of the X_i:

$$S_n(X) \cong \bigoplus_{i\in I} S_n(X_i)$$

for all n. Boundary summands ∂_i stay in a component, in particular,

$$\partial\colon S_1(X) \cong \bigoplus_{i\in I} S_1(X_i) \to \bigoplus_{i\in I} S_0(X_i) \cong S_0(X)$$

is the direct sum of the boundary operators $\partial\colon S_1(X_i) \to S_0(X_i)$ and the claim follows. $\square$

In fact the same proof shows that $H_n(X) = \oplus_{i\in I} H_n(X_i)$ for all n in the situation of the corollary.

Next, we want to study H_1. I have already been hinting it relates to the fundamental group. But the fundamental group is not abelian, while H_1 is, we have to fix that.

Definition 3.4. Let G be an arbitrary group, then its *abelianization*, G_{ab}, is $G/[G,G]$.

Recall that $[G,G]$ is the commutator subgroup of G. That is the smallest subgroup of G containing all commutators $ghg^{-1}h^{-1}, g, h \in G$. It is a normal subgroup of G: If $c \in [G,G]$, then for any $g \in G$ the element $gcg^{-1}c^{-1}$ is a commutator and also by the closure property of subgroups the element $gcg^{-1}c^{-1}c = gcg^{-1}$ is in the commutator subgroup. Thus G_{ab} is a group and since every commutator is contained in $[G,G]$ it is in fact abelian.

Let now X be path-connected and $x \in X$.

Definition 3.5. Let $h\colon \pi_1(X,x) \to H_1(X)$ be the map, that sends the homotopy class of a closed path ω, $[\omega]_{\pi_1}$, to its homology class $[\omega] = [\omega]_{H_1}$. This map is called the *Hurewicz-homomorphism.*

We will need a lemma to ensure that this is in fact well-defined!

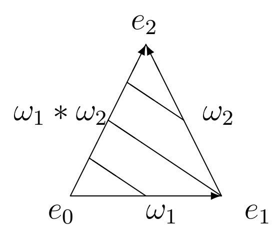
LEMMA 3.6. *Let $\omega_1, \omega_2, \omega$ be paths in X.*

(a) Constant paths are null-homologous.
(b) If $\omega_1(1) = \omega_2(0)$, then $\omega_1 * \omega_2 - \omega_1 - \omega_2$ is a boundary. Here $\omega_1 * \omega_2$ is the concatenation of ω_1 followed by ω_2.
(c) If $\omega_1(0) = \omega_2(0), \omega_1(1) = \omega_2(1)$ and if ω_1 is homotopic to ω_2 relative to $\{0, 1\}$, then ω_1 and ω_2 are homologous as singular 1-chains.
(d) Any 1-chain of the form $\bar{\omega} * \omega$ is a boundary. Here, $\bar{\omega}(t) := \omega(1 - t)$.

Note that I used the opposite convention for $\omega_1 * \omega_2$ in the lecture.

PROOF. For a), consider the constant singular 2-simplex $\alpha(t_0, t_1, t_2) = x$ and c_x, the constant path on x. Then $\partial\alpha = c_x - c_x + c_x = c_x$.

For b), we define a singular 2-simplex $\beta\colon \Delta^2 \to X$ as follows.



We define β on the boundary components of Δ^2 as indicated and prolong it constantly along the sloped inner lines. Then

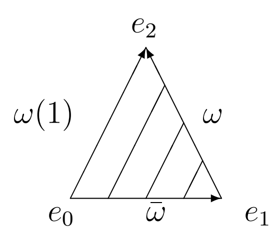
$$\partial\beta = \beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2 = \omega_2 - \omega_1 * \omega_2 + \omega_1.$$

For c): Let $H\colon [0, 1] \times [0, 1] \to X$ a homotopy from ω_1 to ω_2. As we have that $H(0, t) = \omega_1(0) = \omega_2(0)$, we can factor H over the quotient $[0, 1] \times [0, 1]/\{0\} \times [0, 1] \cong \Delta^2$ with induced map $h\colon \Delta^2 \to X$. Then

$$\partial h = h \circ d_0 - h \circ d_1 + h \circ d_2.$$

The first summand is null-homologous, because it's constant (with value $\omega_1(1) = \omega_2(1)$), the second one is ω_2 and the last is ω_1, thus $\omega_1 - \omega_2$ is null-homologous.

For d): Consider $\gamma\colon \Delta^2 \to X$ as indicated below.



Alternatively, remember from your topology course that $\tilde{\omega} \star \omega$ is homotopic to the constant map and apply (b). $\square$

COROLLARY 3.7. *The Hurewicz map is a well-defined homomorphism.*

Proof. By Lemma 3.6 (b)

$$h([\omega_1][\omega_2]) = h([\omega_1 * \omega_2]) = [\omega_1] + [\omega_2] = h([\omega_1]) + h([\omega_2])$$

Well-definedness is Lemma 3.6 (c). $\square$

Proposition 3.8. *Let X be path connected and $x \in X$. The Hurewicz homomorphism induces an isomorphism*

$$\pi_1(X,x)_{\mathrm{ab}} \cong H_1(X).$$

Proof. As $H_1(X)$ is abelian the commutator subgroup $[\pi_1(X,x), \pi_1(X,x)]$ must be sent to 0 and we have the following factorization:

$$\begin{tikzcd} \pi_1(X,x) \arrow[rr, "h"] \arrow[d, "p"'] & & H_1(X) \\ \pi_1(X,x)_{\mathrm{ab}} = \pi_1(X,x)/[\pi_1(X,x), \pi_1(X,x)] \arrow[urr, "\cong", "h_{\mathrm{ab}}"'] & & \end{tikzcd}$$

We will construct an inverse to h_{ab}. For any $y \in X$ we choose a path u_y from x to y. For $y = x$ we take u_x to be the constant path on x. Let α be an arbitrary singular 1-simplex and $y_i = \alpha(e_i)$. Define $\phi\colon S_1(X) \to \pi_1(X,x)_{\mathrm{ab}}$ on generators as $\phi(\alpha) = [u_{y_0} * \alpha * \bar{u}_{y_1}]$ and extend ϕ linearly to all of $S_1(X)$, keeping in mind that the composition in π_1 is written multiplicatively.

We have to show that ϕ is trivial on boundaries, so let $\beta\colon \Delta^2 \to X$. Then

$$\phi(\partial\beta) = \phi(\beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2) = \phi(\beta \circ d_0)\phi(\beta \circ d_1)^{-1}\phi(\beta \circ d_2).$$

Abbreviating $\beta \circ d_i$ with α_i and writing y_i for the vertices of β we get as a result

$$[u_{y_1} * \alpha_0 * \bar{u}_{y_2}][u_{y_0} * \alpha_1 * \bar{u}_{y_2}]^{-1}[u_{y_0} * \alpha_2 * \bar{u}_{y_1}] = [u_{y_0} * \alpha_2 * \bar{u}_{y_1} * u_{y_1} * \alpha_0 * \bar{u}_{y_2} * u_{y_2} * \bar{\alpha_1} * \bar{u}_{y_0}].$$

Here, we've used that the image of ϕ is abelian. We can reduce $\bar{u}_{y_1} * u_{y_1}$ and $\bar{u}_{y_2} * u_{y_2}$ and are left with $[u_{y_0} * \alpha_2 * \alpha_0 * \bar{\alpha_1} * \bar{u}_{y_0}]$ but $\alpha_2 * \alpha_0 * \bar{\alpha_1}$ is the closed path tracing the boundary of β and therefore it is null-homotopic in X. Thus $\phi(\partial\beta) = 0$ and ϕ passes to a map

$$\phi\colon H_1(X) \to \pi_1(X,x)_{\mathrm{ab}}.$$

The composition $\phi \circ h_{\mathrm{ab}}$ evaluated on the class of a closed path ω gives

$$\phi \circ h_{\mathrm{ab}}[\omega]_{\pi_1} = \phi[\omega]_{H_1} = [u_x * \omega * \bar{u}_x]_{\pi_1}.$$

But we chose u_x to be constant, thus $\phi \circ h_{\mathrm{ab}} = \mathrm{id}_{\pi_1(X,x)}$.

If $c = \sum \lambda_i \alpha_i$ is a cycle, then $h_{\mathrm{ab}} \circ \phi(c)$ is of the form $[c + D_c]$ where the D_c-part comes from the contributions of the u_{y_i}. The fact that $\partial(c) = 0$ implies that the summands in D_c cancel and thus $h_{\mathrm{ab}} \circ \phi = \mathrm{id}_{H_1(X)}$. $\square$

Note, that abelianization doesn't change anything for abelian groups, *i.e.*, whenever we have an abelian fundamental group, we know that $H_1(X) \cong \pi_1(X,x)$. In general we lose some information, which is the result of our linearization procedure.

Example 3.9. Knowledge of π_1 immediately gives the following:

(a) $H_1(\mathbb{S}^n) = 0$, for $n > 1$, $\quad H_1(\mathbb{S}^1) \cong \mathbb{Z}$.
(b) $H_1(\underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n}) \cong \mathbb{Z}^n$.

(c) $H_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong (\mathbb{Z} * \mathbb{Z})_{\text{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}$. It is an exercise in group theory to see that the natural map from $\mathbb{Z} * \mathbb{Z}$ to $\mathbb{Z} \oplus \mathbb{Z}$ induces an isomorphism on abelianizations.

(d) For real projective space we have

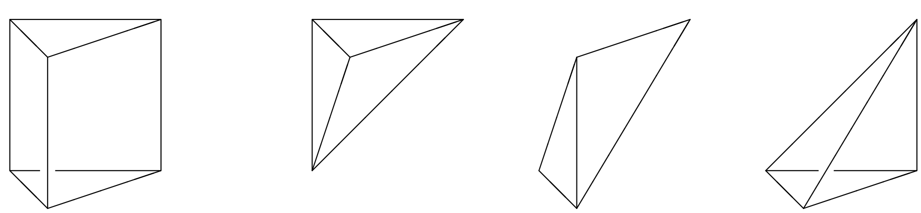
$$H_1(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & n > 1. \end{cases}$$

4. Homotopy invariance

Before exploring higher homology groups we will show that two continuous maps that are homotopic induce chain homotopic maps on singular chains and thus identical maps on the level of homology groups. Thus homology is homotopy invariant and a good tool to study spaces up to homotopy equivalence (rather than up to homeomorphism).

Heuristics: If $\alpha\colon \Delta^n \to X$ is a singular n-simplex and if f, g are homotopic maps from X to Y, then the homotopy from $f \circ \alpha$ to $g \circ \alpha$ is a map from $\Delta^n \times [0,1]$. We want to translate this geometric homotopy into a chain homotopy on the singular chain complex. To that end we have to cut the prism $\Delta^n \times [0,1]$ into $(n+1)$-simplices.

In low dimensions this is easy: $\Delta^0 \times [0,1]$ is homeomorphic to Δ^1, $\Delta^1 \times [0,1] \cong [0,1]^2$ and this can be cut into two copies of Δ^2 and $\Delta^2 \times [0,1]$ is a 3-dimensional prism and that can be glued together from three tetrahedra, e.g.



As you might guess now, we use $n+1$ copies of Δ^{n+1} to build $\Delta^n \times [0,1]$. We introduce some notation first. Embedding $\Delta^n \times [0,1] \subset \mathbb{R}^{n+1} \times \mathbb{R}$ we denote the vertices $(e_i, 0)$ of the bottom simplex by v_i and the vertices $(e_j, 1)$ of the top simplex by w_j.

Then any ordered subset $(q_0, \ldots, q_{n+1})$ of $n+2$ of the points $\{v_0, \ldots, v_n, w_0, \ldots, w_n\}$ determines a map $\Delta^{n+1} \to \Delta^n \times [0,1]$ by sending e_i to the point q_i and extending linearly. (Equivalently we send $(t_0, \ldots, t_{n+1})$ to $\sum t_i q_i$.

We denote this map by $[q_0, \ldots, q_{n+1}]$.

For $i = 0, \ldots, n$ define $p_i\colon \Delta^{n+1} \to \Delta^n \times [0,1]$ as the map $[v_0, \ldots, v_i, w_i, \ldots, w_n]$.

We then define maps $P_i\colon S_n(X) \to S_{n+1}(X \times [0,1])$ via $P_i(\alpha) = (\alpha \times \mathrm{id}) \circ p_i$:

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times [0,1] \xrightarrow{\alpha \times \mathrm{id}} X \times [0,1].$$

For $k = 0, 1$ let $j_k\colon X \to X \times [0,1]$ be the inclusion $x \mapsto (x, k)$. We will show that $P = \sum (-1)^i P_i$ gives a chain homotopy between $S_*(j_0)$ and $S_*(j_1)$.

Lemma 4.1. *The maps P_i satisfy the following relations*

(a) $\partial_0 \circ P_0 = S_n(j_1)$,
(b) $\partial_{n+1} \circ P_n = S_n(j_0)$,
(c) $\partial_i \circ P_i = \partial_i \circ P_{i-1}$ *for* $1 \leqslant i \leqslant n$.
(d)

$$\partial_j \circ P_i = \begin{cases} P_i \circ \partial_{j-1}, & \textit{for } i \leqslant j-2 \\ P_{i-1} \circ \partial_j, & \textit{for } i \geqslant j+1. \end{cases}$$

Proof. Note that it suffices to check the corresponding claims for the p_i's and d_j's, i.e. $\partial_0 \circ P_0 = S_n(j_1)$ if $p_0 \circ d_0 = (\mathrm{id}_{\Delta^n}, 1)$ etc.

It also suffices to check the claims on the vertices e_i as all maps are linear extensions of maps on the vertices.

For the first two points, we note that on Δ^n we have

$$p_0 \circ d_0(e_i) = p_0(e_{i+1}) = (e_i, 1)$$

and

$$p_n \circ d_{n+1}(e_i) = p_n(e_i) = (e_i, 0)$$

for all $0 \leqslant i \leqslant n$.

For c), one checks that $p_i \circ d_i = p_{i-1} \circ d_i$ on Δ^n: both send e_j to w_j if $i \leqslant j$ and to v_j otherwise.

For d) we first consider the case $i \geqslant j+1$. We need to compare $p_i \circ d_j$ and $(d_j \times \mathrm{id}) \circ p_{i-1}$. In other words, the following diagram commutes:

$$\begin{tikzcd} & \Delta^{n+1} \arrow[r, "p_i"] & \Delta^n \times [0,1] \arrow[d, equal] \\ \Delta^n \arrow[ur, "d_j"] \arrow[dr, "p_{i-1}"'] & & \\ & \Delta^{n-1} \times [0,1] \arrow[r, "d_j \times \mathrm{id}"] & \Delta^n \times [0,1] \end{tikzcd}$$

Indeed one checks that by both routes

$$e_k \mapsto \begin{cases} (e_k, 0) & \text{for } k < j \\ (e_{k-1}, 0) & \text{for } j \leqslant k < i \\ (e_{k-1}, 1) & \text{for } i \leqslant k \end{cases}$$

The remaining case follows similarly. $\square$

Lemma 4.2. *The map* $P = \sum_{i=0}^n (-1)^i P_i \colon S_n(X) \to S_{n+1}(X \times [0,1]$ *is a chain homotopy between* $(S_n(j_0))_n$ *and* $(S_n(j_1))_n$*,* i.e., $\partial \circ P + P \circ \partial = S_n(j_1) - S_n(j_0)$.

Proof. We take an $\alpha \colon \Delta^n \to X$ and calculate

$$\partial P\alpha + P\partial\alpha = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \partial_j P_i \alpha + \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} P_i \partial_j \alpha.$$

If we single out the terms involving the pairs of indices $(0,0)$ and $(n,n+1)$ in the first sum and use Lemma 4.1 (a) and (b), we are left with

$$S_n(j_1)(\alpha) - S_n(j_0)(\alpha) + \sum_{(i,j)\neq(0,0),(n,n+1)} (-1)^{i+j}\partial_j P_i \alpha + \sum_{i=0}^{n-1}\sum_{j=0}^{n} (-1)^{i+j} P_i \partial_j \alpha.$$

We now split the third sum according to the cases $i \leqslant j-2$, $i = j-1, j$ and $i \geqslant j+1$. By Lemma 4.1 (c) the cases $i = j-1, j$ cancel and we can use 4.1 (d) to cancel the other two cases with the last summand of the equation. Thus we see that only the first two summands survive. □

So, finally we can prove the main result of this section:

Theorem 4.3 (Homotopy invariance). *If $f, g\colon X \to Y$ are homotopic maps, then they induce the same map on homology.*

Proof. By Lemma 4.2 we know that $S(j_0)$ and $S(j_1)$ are chain homotopic. But composing a chain homotopy with a chain map gives another chain homotopy (check this!). Thus $S(f) = S(H \circ j_0) = S(H) \circ S(j_0) \simeq S(H) \circ S(j_1) = S(g)$. □

Corollary 4.4. *If two spaces X, Y are homotopy equivalent, then $H_*(X) \cong H_*(Y)$. In particular, if X is contractible, then*

$$H_*(X) \cong \begin{cases} \mathbb{Z}, & \textit{for } * = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Example 4.5.
(a) As $\mathbb{R}^n$, the closed disk $\mathbb{D}^n$ and the open disk $\mathring{\mathbb{D}}^n$ are contractible for all n, the above corollary gives that their homology groups are trivial except in degree zero where it consists of the integers.
(b) As the Möbius strip is homotopy equivalent to $\mathbb{S}^1$, we know that their homology groups are isomorphic (and we already know H_0 and H_1).
(c) If you know about vector bundles: the zero section of a vector bundle induces a homotopy equivalence between the base and the total space, hence these two have isomorphic homology groups.

5. The long exact sequence in homology

Our next goal is to compute singular homology groups by breaking up spaces into subspaces.

But before we can move on to topological applications we need some more algebra of chain complexes.

Definition 5.1. A sequence

$$\ldots \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i-1}} \ldots$$

of homomorphisms of abelian groups (indexed over the integers) is called *exact* at A_i if the image of f_{i+1} is the kernel of f_i.

The sequence is called *(long) exact*, if it is exact at every A_i.

An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence.*

EXAMPLE 5.2. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is a short exact sequence.

A short exact sequence $0 \to A \to B \to C \to 0$ is called *split* if $B \cong A \oplus C$.
The following lemma will be useful later.

LEMMA 5.3. *A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split if and only if there exists a right inverse r of g if and only if there exists a left inverse s of f.*

PROOF. Given r we note that $rg(b) - b$ is in the image of A, so we define $s : B \to A$ by $b \mapsto rg(b) - b$. It is a homomorphism and $sf(a) = a$.

Given s we define $r(c)$ as follows. Pick any b in $g^{-1}(c)$ and let $r(c) = b - fs(b)$. This is independent of b as $g(b') = g(b)$ implies $b' - b$ is in the image of A, und thus equal to $fs(b) - fs(b')$. It follows that $r(b) = r(b')$.

We define homomorphisms $f + r : A \oplus C \to B$ and $(s, g) : B \to A \oplus C$ and compute that $(s, g)(f + r)(a, c) = (sf(a), gr(c)) = (a, c)$ and $(f + r)(s, g)(b) = fs(b) + rg(b) = b$, providing the desired isomorphism.

If conversely $B \cong A \oplus C$ we let r be the inclusion from C and s the projection to A. $\square$

By definition a chain complex C_* (considered as the sequence of homomorphisms d_j) is exact at C_i if $H_i(C) = 0$. Thus homology measures failure of exactness.

If $\iota\colon U \to A$ is an injection/monomorphism, then $0 \to U \to A$ is exact at U and $0 \to U \to A \to A/U \to 0$ is a short exact sequence.

Similarly, a surjection/epimorphism $\varrho\colon B \to Q$ gives rise to a sequence $B \to Q \to 0$ exact at Q.

An isomorphism $\phi\colon A \cong A'$ gives rise to an exact sequence $0 \to A \xrightarrow{\phi} A' \to 0$.

DEFINITION 5.4. If A_*, B_*, C_* are chain complexes and $f_*\colon A_* \to B_*$, $g\colon B_* \to C_*$ are chain maps, then we call the sequence

$$A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$$

exact a B_*, if the image of f_n is the kernel of g_n for all $n \in \mathbb{Z}$.

Thus an exact sequence of chain complexes is a commuting double ladder

$$\begin{tikzcd}
\vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] \\
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
A_n \arrow[r, "f_n"] \arrow[d, "d"] & B_n \arrow[r, "g_n"] \arrow[d, "d"] & C_n \arrow[d, "d"] \\
A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d"] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

in which every row is exact.

EXAMPLE 5.5. Let p be a prime, then

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, "p"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "p^2"] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "p"] \arrow[d, "\pi"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\pi"] & \mathbb{Z}/p\mathbb{Z} \arrow[d, "\mathrm{id}"] \\
\mathbb{Z}/p\mathbb{Z} \arrow[r, "p"] \arrow[d] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r, "\pi"] \arrow[d] & \mathbb{Z}/p\mathbb{Z} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

has exact rows and columns, in particular it is an exact sequence of chain complexes. Here, π denotes varying canonical projection maps.

PROPOSITION 5.6. *If* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *is a short exact sequence of chain complexes, then there exists a homomorphism* $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ *for all* $n \in \mathbb{Z}$ *which is natural, i.e. if*

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "f"] \arrow[d, "\alpha"] & B_* \arrow[r, "g"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "f'"] & B'_* \arrow[r, "g'"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of chain maps in which the rows are exact then $H_{n-1}(\alpha) \circ \delta = \delta \circ H_n(\gamma)$*,*

$$\begin{tikzcd} H_n(C_*) \arrow[r, "\delta"] \arrow[d, "H_n(\gamma)"'] & H_{n-1}(A_*) \arrow[d, "H_{n-1}(\alpha)"] \\ H_n(C'_*) \arrow[r, "\delta"] & H_{n-1}(A'_*) \end{tikzcd}$$

The method of proof is an instance of a *diagram chase*. The homomorphism δ is called *connecting homomorphism*.

Proof. We show the existence of a δ first and then prove that the constructed map satisfies the naturality condition.

a) Definition of δ:

Is $c \in C_n$ with $d(c) = 0$, then we choose a $b \in B_n$ with $g_n b = c$. This is possible because g_n is surjective. We know that $dg_n b = dc = 0 = g_{n-1}db$ thus db is in the kernel of g_{n-1}, hence it is in the image of f_{n-1}. Thus there is an $a \in A_{n-1}$ with $f_{n-1}a = db$. We have that $f_{n-2}da = df_{n-1}a = ddb = 0$ and as f_{n-2} is injective, this shows that a is a cycle.

We define $\delta[c] := [a]$.

$$\begin{tikzcd} & B_n \ni b \arrow[r, mapsto, "g_n"] & c \in C_n \\ A_{n-1} \ni a \arrow[r, mapsto, "f_{n-1}"] & db \in B_{n-1} & \end{tikzcd}$$

In order to check that δ is well-defined, we assume that there are b and b' with $g_n b = g_n b' = c$. Then $g_n(b - b') = 0$ and thus there is an $\tilde{a} \in A_n$ with $f_n \tilde{a} = b - b'$. Define a' as $a - d\tilde{a}$. Then

$$f_{n-1}a' = f_{n-1}a - f_{n-1}d\tilde{a} = db - db + db' = db'$$

because $f_{n-1}d\tilde{a} = db - db'$. As f_{n-1} is injective, we get that a' is uniquely determined with this property. As a is homologous to a' we get that $[a] = [a'] = \delta[c]$, thus the latter is independent of the choice of b.

In addition, we have to make sure that the value stays the same if we add a boundary term to c, i.e. take $c' = c + d\tilde{c}$ for some $\tilde{c} \in C_{n+1}$. Choose preimages of $c, \tilde{c}$ under g_n and g_{n+1}, i.e. b and $\tilde{b}$ with $g_n b = c$ and $g_{n+1}\tilde{b} = \tilde{c}$. Then the element $b' = b + d\tilde{b}$ has boundary $db' = db$ and thus both choices will result in the same a.

Therefore $\delta \colon H_n(C_*) \to H_{n-1}(A_*)$ is well-defined.

b) We have to show that δ is natural with respect to maps of short exact sequences.

Let $c \in Z_n(C_*)$, then $\delta[c] = [a]$ for a $b \in B_n$ with $g_n b = c$ and an $a \in A_{n-1}$ with $f_{n-1}a = db$. Therefore, $H_{n-1}(\alpha)(\delta[c]) = [\alpha_{n-1}(a)]$.

On the other hand, we have

$$f'_{n-1}(\alpha_{n-1}a) = \beta_{n-1}(f_{n-1}a) = \beta_{n-1}(db) = d\beta_n b$$

and

$$g'_n(\beta_n b) = \gamma_n g_n b = \gamma_n c$$

and we can conclude that by the construction of δ

$$\delta[\gamma_n(c)] = [\alpha_{n-1}(a)]$$

and this shows $\delta \circ H_n(\gamma) = H_{n-1}(\alpha) \circ \delta$. $\square$

With this auxiliary result at hand we can now prove the main result in this section:

Proposition 5.7. *For any short exact sequence*

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

of chain complexes we obtain a long exact sequence of homology groups

$$\ldots \xrightarrow{\delta} H_n(A_*) \xrightarrow{H_n(f)} H_n(B_*) \xrightarrow{H_n(g)} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{H_{n-1}(f)} \ldots$$

Proof. a) Exactness at $H_n(B_*)$:

We have $H_n(g) \circ H_n(f)[a] = [g_n(f_n(a))] = 0$ because the composition of g_n and f_n is zero. This proves that the image of $H_n(f)$ is contained in the kernel of $H_n(g)$.

For the converse, let $[b] \in H_n(B_*)$ with $[g_n b] = 0$. Then there is a $c \in C_{n+1}$ with $dc = g_n b$. As g_{n+1} is surjective, we find a $b' \in B_{n+1}$ with $g_{n+1}b' = c$. Hence

$$g_n(b - db') = g_n b - dg_{n+1}b' = dc - dc = 0.$$

Exactness gives an $a \in A_n$ with $f_n a = b - db'$ and $da = 0$ and therefore $f_n a$ is homologous to b and $H_n(f)[a] = [b]$ thus the kernel of $H_n(g)$ is contained in the image of $H_n(f)$.

b) Exactness at $H_n(C_*)$:

Let $b \in H_n(B_*)$, then $\delta[g_n b] = 0$ because b is a cycle, so 0 is the only preimage under f_{n-1} of $db = 0$. Therefore the image of $H_n(g)$ is contained in the kernel of δ.

Now assume that $\delta[c] = 0$, thus in the construction of δ, the a is a boundary, $a = d\tilde{a}$. Then for a preimage of c under g_n, b, we have by the definition of a

$$d(b - f_n\tilde{a}) = db - df_n\tilde{a} = db - f_{n-1}a = 0.$$

Thus $b - f_n\tilde{a}$ is a cycle and $g_n(b - f_n\tilde{a}) = g_n b - g_n f_n \tilde{a} = g_n b - 0 = g_n b = c$, so we found a preimage for $[c]$ and the kernel of δ is contained in the image of $H_n(g)$.

c) Exactness at $H_{n-1}(A_*)$:

Let c be a cycle in $Z_n(C_*)$. Again, we choose a preimage b of c under g_n and an a with $f_{n-1}(a) = db$. Then $H_{n-1}(f)\delta[c] = [f_{n-1}(a)] = [db] = 0$. Thus the image of δ is contained in the kernel of $H_{n-1}(f)$.

If $a \in Z_{n-1}(A_*)$ with $H_{n-1}(f)[a] = 0$. Then $f_{n-1}a = db$ for some $b \in B_n$. Take $c = g_n b$. Then by definition $\delta[c] = [a]$. $\square$

6. The long exact sequence of a pair of spaces

Let X be a topological space and $A \subset X$ a subspace of X. Consider the inclusion map $i\colon A \to X$, $i(a) = a$. We obtain an induced map $S_n(i)\colon S_n(A) \to S_n(X)$, but we know that the inclusion of spaces doesn't have to yield a monomorphism on homology groups. For instance, we can include $A = \mathbb{S}^1$ into $X = \mathbb{D}^2$.

We consider pairs of spaces (X, A).

Definition 6.1. The *relative chain complex of* (X, A) is

$$S_*(X, A) := S_*(X)/S_*(A)$$

with differential induced by the differential on $S_*(X)$.

Note the differential on $S_*(X)$ descends to the quotient as it preserves $S_*(A)$.

$S_n(X,A)$ is isomorphic to the free abelian group generated by all n-simplices $\beta\colon \Delta^n \to X$ whose image is not completely contained in A, *i.e.*, $\beta(\Delta^n) \cap (X \setminus A) \neq \varnothing$.

- Elements in $S_n(X,A)$ are called *relative chains in* (X,A)
- Cycles in $S_n(X,A)$ are represented by chains c whose boundary lies in A. These are *relative cycles*.
- Boundaries in $S_n(X,A)$ are chains c in X of the form $\partial^X b + a$ where a is a chain in A, these are *relative boundaries*.

The following facts are immediate from the definition:

(a) $S_n(X,\varnothing) \cong S_n(X)$.
(b) $S_n(X,X) = 0$.
(c) $S_n(X \sqcup X', X') \cong S_n(X)$.

Definition 6.2. The *relative homology groups of* (X,A) are

$$H_n(X,A) := H_n(S_*(X,A)).$$

Theorem 6.3. *For any pair of topological spaces $A \subset X$ we obtain a long exact sequence*

$$\ldots \xrightarrow{\delta} H_n(A) \xrightarrow{H_n(i)} H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} \ldots$$

For a map of spaces $f\colon X \to Y$ with $f(A) \subset B \subset Y$, we get an induced map of long exact sequences

$$\begin{tikzcd}
\ldots \arrow[r, "\delta"] & H_n(A) \arrow[r, "H_n(i)"] \arrow[d, "H_n(f|_A)"] & H_n(X) \arrow[r] \arrow[d, "H_n(f)"] & H_n(X,A) \arrow[r, "\delta"] \arrow[d, "H_n(f)"] & H_{n-1}(A) \arrow[r, "H_{n-1}(i)"] \arrow[d, "H_{n-1}(f|_A)"] & \ldots \\
\ldots \arrow[r, "\delta"'] & H_n(B) \arrow[r, "H_n(i)"'] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\delta"'] & H_{n-1}(B) \arrow[r, "H_{n-1}(i)"'] & \ldots
\end{tikzcd}$$

A map $f\colon X \to Y$ with $f(A) \subset B$ is denoted by $f\colon (X,A) \to (Y,B)$.

Proof. By definition of $S_*(X,A)$ the sequence

$$0 \longrightarrow S_*(A) \xrightarrow{S_*(i)} S_*(X) \xrightarrow{\pi} S_*(X,A) \longrightarrow 0$$

is an exact sequence of chain complexes and by Proposition 5.7 we obtain the first claim.

For a map f as above the following diagram

$$\begin{tikzcd}
0 \arrow[r] & S_n(A) \arrow[r, "S_n(i)"] \arrow[d, "S_n(f|_A)"] & S_n(X) \arrow[r, "\pi"] \arrow[d, "S_n(f)"] & S_n(X,A) \arrow[r] \arrow[d, "S_n(f)/S_n(f|_A)"] & 0 \\
0 \arrow[r] & S_n(B) \arrow[r, "S_n(i)"] & S_n(Y) \arrow[r, "\pi"] & S_n(Y,B) \arrow[r] & 0
\end{tikzcd}$$

commutes, thus the second claim follows from naturality of the boundary map in Proposition 5.6. □

Example 6.4. Let $A = \mathbb{S}^{n-1}$ and $X = \mathbb{D}^n$, then we know that $H_j(i)$ is trivial for $j > 0$. From the long exact sequence we get that $\delta\colon H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$ for $j > 1$ and $n \geqslant 1$.

Proposition 6.5. *If* $i\colon A \hookrightarrow X$ *is a weak retract, i.e. if there is an* $r\colon X \to A$ *with* $r \circ i \simeq \mathrm{id}_A$*, then*

$$H_n(X) \cong H_n(A) \oplus H_n(X,A), \quad 0 \leqslant n.$$

Proof. From the assumption we get that $H_n(r) \circ H_n(i) = H_n(\mathrm{id}_A) = \mathrm{id}_{H_n(A)}$ for all n and hence $H_n(i)$ is injective for all n. Thus all boundary maps are trivial and $0 \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \to H_n(X,A) \to 0$ is exact for all n.

As $H_n(r)$ is a left-inverse for $H_n(i)$ we obtain a splitting

$$H_n(X) \cong H_n(A) \oplus H_n(X,A)$$

by Lemma 5.3. □

Let us now consider the case where A is just a point. In that case the projection $X \to *$ makes $x : * \to X$ into a weak retract and we have $H_n(X) \cong H_n(X,x) \oplus H_n(*)$. For a path connected space this just splits off $H_0(X) \cong \mathbb{Z}$ and allows us to concentrate on the more interesting parts.

In fact $H(X,x)$ is isomorphic to another construction:

Definition 6.6. We define $\widetilde{H}_n(X) := \ker(H_n(\varepsilon)\colon H_n(X) \to H_n(*))$ and call it the *reduced nth homology group of the space X*.

We have the following straightforward observations:

- Note that $\widetilde{H}_n(X) \cong H_n(X)$ for all positive n.
- If X is path-connected, then $\widetilde{H}_0(X) = 0$.
- For any choice of a base point $x \in X$ we get

$$\widetilde{H}_n(X) \cong H_n(X,x)$$

- We can also augment the singular chain complex $S_*(X)$ and consider $\widetilde{S}_*(X)$:

$$\ldots \longrightarrow S_1(X) \longrightarrow S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

  where $\varepsilon(\alpha) = 1$ for every singular 0-simplex α. Then for all $n \geqslant 0$,

$$\widetilde{H}_n(X) \cong H_*(\widetilde{S}_n(X)).$$

Lemma 6.7. *The assignment* $X \mapsto \widetilde{H}_n(X))$ *is a functor* $\mathsf{Top} \to \mathsf{Ab}$.

Proof. This just means that for a continuous $f\colon X \to Y$ we get an induced map $\widetilde{H}_n(f)\colon \widetilde{H}_*(X) \to \widetilde{H}_n(Y)$ such that the identity on X induces the identity and composition of maps is respected.

All maps $f : X \to Y$ are compatible with the projections $p_X : X \to *$, thus f induces a map $\widetilde{H}(X) \to \widetilde{H}(Y)$ on the kernels of $H_*(p_X)$. Functoriality follows from functoriality of H_n. □

We can also define relative reduced homology:

Definition 6.8. For $\varnothing \neq A \subset X$ we define

$$\widetilde{H}_n(X,A) := H_n(X,A).$$

Proposition 6.9. *For each pair of spaces, there is a long exact sequence*

$$\ldots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X,A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \ldots .$$

Proof. If $A = \emptyset$ the result is trivial. If $A \neq \emptyset$ we consider the short exact sequence $\widetilde{S}_*(A) \to \widetilde{S}_*(X) \to S_*(X,A)$ (note there is no tilde on the rightmost term) and use Proposition 5.7. $\square$

We have one more long exact sequene for relative homology:

Definition 6.10. If X has two subspaces $A, B \subset X$, then (X,A,B) is called a *triple*, if $B \subset A \subset X$.

Any triple gives rise to three pairs of spaces (X,A), (X,B) and (A,B) and accordingly we have three long exact sequences in homology. But there is another one.

Proposition 6.11. *For any triple (X,A,B) there is a natural long exact sequence*

$$\ldots \longrightarrow H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \xrightarrow{\delta} H_{n-1}(A,B) \longrightarrow \ldots$$

Proof. Consider the sequence

$$0 \longrightarrow S_n(A)/S_n(B) \longrightarrow S_n(X)/S_n(B) \longrightarrow S_n(X)/S_n(A) \longrightarrow 0.$$

This sequence is exact by basic algebra, because $S_n(B) \subset S_n(A) \subset S_n(X)$. $\square$

Corollary 6.12. *Let (X,A,B) be a triple with $i\colon B \subset A$ a homotopy equivalence. Then $H_n(X,A) \cong H_n(X,B)$ for all n.*

Proof. By Theorem 4.3 $H_n(i)$ is an isomorphism for all n, thus by Theorem 6.3 $H_n(A,B) = 0$ for all n and by Proposition 6.11 we have $H_n(X,B) \cong H_n(X,A)$ for all n. $\square$

In fact, the sequence in Proposition 6.11 is part of the following commutative diagram displaying four long exact sequences braided together.

$$\begin{tikzcd}
\ldots \arrow[dr, red] & & & & & & \ldots \\
& H_{n+1}(X,A) \arrow[rr, bend left, green] \arrow[dr, red] & & H_n(A,B) \arrow[rr, bend left, blue] \arrow[dr, green] & & H_{n-1}(B) \arrow[ur] \arrow[dr, blue] & \\
\ldots \arrow[ur, green] \arrow[dr] & & H_n(A) \arrow[ur, blue] \arrow[dr, red] & & H_n(X,B) \arrow[ur] \arrow[dr, green] & & \ldots \\
& H_n(B) \arrow[ur, blue] \arrow[rr, bend right] & & H_n(X) \arrow[ur] \arrow[rr, bend right, red] & & H_n(X,A) \arrow[ur, red] \arrow[dr, green] & \\
\ldots \arrow[ur, blue] & & & & & & \ldots
\end{tikzcd}$$

In particular, the connecting homomorphism $\delta\colon H_n(X,A) \to H_{n-1}(A,B)$ is the composite $\delta = \pi_*^{(A,B)} \circ \delta^{(X,A)}$ (unravelling definitions).

7. Barycentric subdivision

We will now simplify relative homology groups in order to compute them. The key will be to replace spaces by smaller spaces by gluing pieces together or removing (excising) pieces. The problem is that we might have some “large” singular simplex that does not land neatly within one of the pieces. The solution is to replace singular simplices by smaller ones by a process called barycentric subdivision.

We will restrict ourselves to a special kind of simplex first.

DEFINITION 7.1. A singular n-simplex $\alpha\colon \Delta^n \to \Delta^p$ is called *affine*, if

$$\alpha(\sum_{i=0}^n t_i e_i) = \sum_{i=0}^n t_i \alpha(e_i).$$

We denote by $S_*^{aff}(\Delta^p)$ the subcomplex of affine simplices of Δ^p.

If we write $\alpha(e_i)$ as v_i then $\alpha(\sum_{i=0}^n t_i e_i) = \sum_{i=0}^n t_i v_i$. The map α is determind by the v_i which we call the *vertices of* α.

Similar to Section 4 we also write $\alpha = [v_0, \ldots, v_n]$. We note that $\partial_i \alpha = [v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ where $\widehat{v_i}$ indicates the entry with index i is skipped. Note that with this notation $[v]$ is the constant function with value v.

First, we construct the cone of a simplex. Let $v \in \Delta^p$ and let $\alpha\colon \Delta^n \to \Delta^p$ be a singular n-simplex in Δ^p.

DEFINITION 7.2. The *cone of* $\alpha = [v_0, \ldots, v_n]$ *with respect to* v is $K_v = [v_0, \ldots, v_n, v]$.

We could also defines this for a general singular simplex as

$$K_v(\alpha)\colon (t_0, \ldots, t_{n+1}) \mapsto \begin{cases} (1-t_{n+1})\alpha(\frac{t_0}{1-t_{n+1}}, \ldots, \frac{t_n}{1-t_{n+1}}) + t_{n+1}v, & t_{n+1} < 1, \\ v, & t_{n+1} = 1. \end{cases}$$

$K_v(\alpha)$ is again affine if α is, so extending K_v linearly gives a map

$$K_v\colon S_n^{aff}(\Delta^p) \to S_{n+1}^{aff}(\Delta^p).$$

LEMMA 7.3. *The map K_v satisfies*

(a) $\partial K_v(c) = \varepsilon(c)[v] - c$ where $c \in S_0(\Delta^p)$ and ε is the augmentation.

(b) For $n > 0$ we have that $\partial \circ K_v - K_v \circ \partial = (-1)^{n+1}\mathrm{id}$.

PROOF. For a singular 0-simplex $[v_0]\colon \Delta^0 \to \Delta^p$ we have $\varepsilon([v_1]) = 1$ and we calculate $\partial[v_0, v] = v - v_0$. The result follows by extending linearly.

For $n > 0$ we have to calculate $\partial_i K_v(\alpha)$ and it is straightforward to see that $\partial_{n+1} K_v([v_0, \ldots, v_n]) = \partial_{n+1}[v_0, \ldots, v_n, v] = [v_0, \ldots, v_n]$ and $\partial_i(K_v([v_0, \ldots, v_n])) = [v_0, \ldots, \widehat{v_i}, \ldots v] = K_v(\partial_i \alpha)$ for all $i < n+1$. □

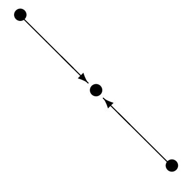
DEFINITION 7.4. For $\alpha\colon \Delta^n \to \Delta^p$ let $b(\alpha) = b := \frac{1}{n+1}\sum_{i=0}^n \alpha(e_i)$ be the barycenter of α. The *barycentric subdivision* $B\colon S_n^{aff}(\Delta_p) \to S_n^{aff}(\Delta_p)$ is defined inductively as $B(\alpha) = \alpha$ for $\alpha \in S_0(\Delta_p)$ and $B(\alpha) = (-1)^n K_b(B(\partial\alpha))$ for $n > 0$.

For $n \geqslant 1$ this yields $B(\alpha) = \sum_{i=0}^n (-1)^{n+i} K_b(B(\partial_i \alpha))$.

If we take $n = p$ and $\alpha = \mathrm{id}_{\Delta^n}$, then for small n this looks as follows:

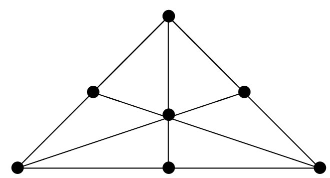
For $n = 0$ we have $B(c) = c$, you cannot subdivide a point any further.

For $n = 1$ we get



Note here the arrows are the direction of the simplices making up the barycentric subdivision, in the barycentric subdivision of the 1-simplex considered as a 1-chain the two simplices are oriented in parallel.

And for $n = 2$ we get (up to tilting)



LEMMA 7.5. *The barycentric subdivision is a chain map.*

PROOF. We have to show that $\partial B = B\partial$. If α is a singular zero chain, then $\partial B\alpha = \partial\alpha = 0$ and $B\partial\alpha = B(0) = 0$.

Let $n = 1$. Then $\alpha = [v_0, v_1]$ and

$$\partial B[v_0, v_1] = -\partial K_v B([v_1]) + \partial K_b B([v_0]).$$

But the boundary terms are zero chains and there B is the identity so we get

$$-\partial K_b([v_1]) + \partial K_b([v_0]) = -[b] + [v_1] + [b] - [v_0] = \partial\alpha = B\partial\alpha$$

where we used Lemma 7.3 (a). (Note that b is always $b(\alpha)$, not a $b(\partial_i\alpha)$.)

We prove the claim inductively on n, so let $\alpha \in S_n^{aff}(\Delta^p)$. Then

$$\begin{aligned}
\partial B\alpha =& (-1)^n \partial K_b(B\partial\alpha) \\
=& (-1)^n((-1)^n B\partial\alpha + K_b\partial B\partial\alpha) \\
=& B\partial\alpha + (-1)^n K_b B\partial\partial\alpha = B\partial\alpha.
\end{aligned}$$

Here, the first equality is by definition, the second one follows by Lemma 7.3 and then we use the induction hypothesis and the fact that $\partial\partial = 0$. $\square$

Subdividing chains should not change anything on the level of homology groups and to prove that we show that B is chain homotopic to the identity.

We construct $\psi_n \colon S_n(\Delta^p) \to S_{n+1}(\Delta^p)$ again inductively by

$$\psi_0([v]) = -K_{b(v)}([v]) = -[v, v]$$

and

$$\psi_n(\alpha) = (-1)^{n+1} K_b(\alpha - \psi_{n-1}\partial\alpha)$$

LEMMA 7.6. *The sequence $(\psi_n)_n$ is a chain homotopy from B to the identity.*

PROOF. So we claim that $\partial\psi_n + \psi_{n-1}\partial = \mathrm{id} - B_n$.
For $n = 0$ we have $\partial\psi_0([v]) = -\partial([v, v]) = 0$ and this agrees with $B_0 - \mathrm{id}$.
For $n = 1$ we compute

$$\begin{aligned}\partial\psi_n(\alpha) &= (-1)^{n+1}K_b(\alpha - \psi_{n-1}\partial\alpha)\\ &= \alpha - \psi_{n-1}\partial\alpha + (-1)^{n+1}K_b\partial(\alpha - \psi_{n-1}\partial\alpha)\\ &= \alpha - \psi_{n-1}\partial\alpha + (-1)^{n+1}K_b\partial(\alpha - (\alpha - B\alpha - \partial\psi_{n-2}\alpha)\end{aligned}$$

by first Lemma 7.3 and then the induction assumption. We cancel $\alpha-\alpha$ and note $K_b\partial\partial\psi_{n-2} = 0$. Then using that B is a chain may by Lemma 7.5 we may rearrange and are left with the identity

$$\partial\psi_n + \psi_{n-1}\partial = \mathrm{id} - (-1)^{n+1}K_bB_{n-1}\partial.$$

But the rightmost term is $-B$ by definition and we are done. □

DEFINITION 7.7. Let A be a subset of a metric space (X, d). The *diameter of A* is

$$\sup\{d(x, y) \mid x, y \in A\}$$

and we denote it by $\mathrm{diam}(A)$.

Accordingly, the *diameter of an affine n-simplex α in Δ^p* is the diameter of its image (with the metric induced from $\mathbb{R}^{p+1}$), and we abbreviate that with $\mathrm{diam}(\alpha)$.

LEMMA 7.8. *For any affine α every simplex in the chain $B\alpha$ has diameter $\leqslant \frac{n}{n+1}\mathrm{diam}(\alpha)$.*

PROOF. Do it yourself or see [Bredon], proof of Lemma IV.17.3. □

We may iterate the application of B and find that the k-fold iteration, $B^k(\alpha)$, has diameter at most $\left(\frac{n}{n+1}\right)^k \mathrm{diam}(\alpha)$.

In the following we use the deceptively easy trick to write α as

$$\alpha = \alpha \circ \mathrm{id}_{\Delta^n} = S_n(\alpha)(\mathrm{id}_{\Delta^n}).$$

This allows us to use the barycentric subdivision for general simplices in general spaces.

DEFINITION 7.9.
(a) We define $B_n^X : S_n(X) \to S_n(X)$ as

$$B_n^X(\alpha) := S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}).$$

(b) Similarly, $\psi_n^X : S_n(X) \to S_{n+1}(X)$ is

$$\psi_n^X(\alpha) := S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}).$$

LEMMA 7.10. *The maps B^X are natural in X and are chain maps homotopic to the identity on $S_n(X)$ via ψ_n^X.*

PROOF. Naturality follows directly from the definition, let $f : X \to Y$ be a continuous map. We have

$$\begin{aligned}S_n(f)B_n^X(\alpha) =&S_n(f) \circ S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n})\\ =&S_n(f \circ \alpha) \circ B(\mathrm{id}_{\Delta^n})\\ =&B_n^Y(f \circ \alpha).\end{aligned}$$

As α induces a chain map we have

$$\partial\psi_n^X(\alpha) = \partial \circ S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}) = S_n(\alpha) \circ \partial \circ \psi_n(\mathrm{id}_{\Delta^n})$$

and thus we can check the chain homotopy

$$\partial\psi_n^X + \psi_{n-1}^X\partial = S_n(\alpha)\circ(\partial\circ\psi_n(\mathrm{id}_{\Delta^n})+\psi_{n-1}\circ\partial(\mathrm{id}_{\Delta^n})) = S_n(\alpha)\circ(B-\mathrm{id})(\mathrm{id}_{\Delta^n}) = B_n^X(\alpha)-\alpha.$$

□

We now drop the superscript X from B^X. Now we consider singular n-chains that are spanned by 'small' singular n-simplices.

Definition 7.11. Let $\mathfrak{U} = \{U_i, i \in I\}$ be an open covering of X. Then $S_n^{\mathfrak{U}}(X)$ is the free abelian group generated by all $\alpha\colon \Delta^n \to X$ such that the image of Δ^n under α is contained in one of the $U_i \in \mathfrak{U}$.

Note that $S_n^{\mathfrak{U}}(X)$ is an abelian subgroup of $S_n(X)$. As we will see now, these chains see the whole singular homology of X.

Lemma 7.12. *Every chain in $S_n(X)$ is homologous to a chain in $S_n^{\mathfrak{U}}(X)$ and $H_n(X) \cong H_n(S^{\mathfrak{U}}(X))$.*

Proof. Let $\alpha = \sum_{j=1}^m \lambda_j\alpha_j \in S_n(X)$ and for each j let L_j for $1 \leqslant j \leqslant m$ be the Lebesgue number for the covering $\{\alpha_j^{-1}(U_i), i \in I\}$ of Δ^n. I.e. L_j is such that any ball with diameter less than L_j is entirely contained in one of the $\alpha_j^{-1}(U)$. It exists as Δ^n is compact.

Choose a k, such that $\left(\frac{n}{n+1}\right)^k \leqslant \min(L_1, \dots, L_m)$. Then $B^k\alpha_1, \dots, B^k\alpha_m$ are all in $S_n^{\mathfrak{U}}(X)$. Therefore

$$B^k(\alpha) = \sum_{j=1}^m \lambda_j B^k(\alpha_j) \in S_n^{\mathfrak{U}}(X).$$

As B is chain homotopic to the identity we see that

$$\alpha \simeq B\alpha \simeq \ldots \simeq B^k\alpha$$

are all homologous and we are done.

This shows surjectivity of the natural map $i : H_n(S^{\mathfrak{U}}(X)) \to H_n(X)$. To show injectivity let $i(\alpha) = \partial\beta$ in $H_n(X)$. Using the previous argument $\beta = \beta' + \partial\gamma$ with $\beta' \in S_{n+1}^{\mathfrak{U}}(X)$. But then $[\alpha] = [\partial\beta'] = 0 \in H_n(S^{\mathfrak{U}}(X))$. □

8. Excision

With the technical work form the last section we can prove one of the main results of this part of the course:

Theorem 8.1 (Excision). *Let $W \subset A \subset X$ such that $\bar{W} \subset \mathring{A}$. Then the inclusion $i\colon (X \setminus W, A \setminus W) \hookrightarrow (X, A)$ induces an isomorphism*

$$H_n(i)\colon H_n(X \setminus W, A \setminus W) \cong H_n(X, A)$$

for all $n \geqslant 0$.

Proof. Consider the open covering $\mathfrak{U} = \{\mathring{A}, X \setminus \bar{W}\} =: \{U, V\}$.

We first prove that $H_n(i)$ is surjective, so consider a relative cycle in $S_n(X, A)$ represented by $c \in S_n(X)$ which satisfies $\partial c \in S_{n-1}(A)$.

By Lemma 7.12 there is a k such that $c' := B^k c$ is a chain in $S_n^{\mathfrak{U}}(X)$. We decompose c' as $c' = c^U + c^V$ with c^U and c^V being elements in the corresponding chain complex. (This decomposition is not unique.)

We know that the boundary of c' is $\partial c' = \partial B^k c = B^k \partial c$ and by assumption this is a chain in $S_{n-1}(A)$. But $\partial c' = \partial c^U + \partial c^V$ with $\partial c^U \in S_{n-1}(U) \subset S_{n-1}(A)$. Thus, $\partial c^V \in S_{n-1}(A)$ also, in fact, $\partial c^V \in S_{n-1}(A \setminus W)$ and therefore c^V is a relative cycle in $S_n(X \setminus W, A \setminus W)$. We compute $H_n(i)[c^V] = [c - c^U] = [c] \in H_n(X, A)$ because $[c^U]$ lies in $S_n(U) \subset S_n(A)$.

We consider injectivity of $H_n(i)$. Assume that there is a $c \in S_n(X \setminus W)$ with $\partial c \in S_{n-1}(A \setminus W)$ and assume $H_n(i)[c] = 0$, i.e. c is of the form $c = \partial b + a'$ with $b \in S_{n+1}(X)$ and $a' \in S_n(A)$. Write b as $b^U + b^V$ with $b^U \in S_{n+1}(U) \subset S_{n+1}(A)$ and $b^V \in S_{n+1}(V) \subset S_{n+1}(X \setminus W)$. Then

$$c - \partial b^V = \partial b^U + a'.$$

Here ∂b^U and a' are elements in $S_n(A)$ and as the left hand side lies in $S_n(X \setminus W)$ so does the right hand side. Thus $[c] = [\partial b^V] = 0 \in H_n(X \setminus W, A \setminus W)$. $\square$

Example 8.2. $X = \Sigma_g$ and A a subspace homeomorphic to a surface of genus $h < g$ with one boundary component. We can equip shrink A a little bit to define a subset W. Then $H_n(X, A) \cong H_n(X \setminus W, A \setminus W)$. We can also consider $Y = \Sigma_{g-h}$ with a subset B homeomorphic to a disk. Picking $V \subset B$ a smaller disk we see $H_n(Y, B) \cong H_n(Y \setminus V, B \setminus V)$. But the pairs $(Y \setminus V, B \setminus V)$ and $(X \setminus W, A \setminus W)$ are homeomorphic, thus $H^n(X, A) \cong H^n(Y, B)$ or, in more suggestive notation:

$$H_n(\Sigma_g, \Sigma_h^{\partial}) \cong H_n(\Sigma_{g-h}, \mathbb{D}^2).$$

Now we can finally compute some relative homology.

Definition 8.3. We call (X, A) a *good pair* if $A \subset X$ is a closed subspace A is a deformation retract of an open neighbourhood $A \subset U \subset X$.

Here we say A is a deformation retract of U if there is $r : U \to A$ such that $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_U$ via a homotopy h with $h_t(a) = a$ for all t. It then in in particular follows that U/A deformation retracts to $A/A = *$. This is the key point why good pairs are good, the proof is a little subtle and many places gloss over it, see Lemma A.1.2 for details.

Proposition 8.4. *Let (X, A) be a good pair. Then*

$$H_n(X, A) \cong \widetilde{H}_n(X/A), \quad 0 \leqslant n.$$

Proof. Let $\pi \colon X \to X/A$ be the canonical projection. Let U be a neighbourhood of A such that A is a deformation retract of U.

Consider the following diagram:

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\cong"] \arrow[d, "H_n(\pi)"'] & H_n(X, U) & H_n(X \setminus A, U \setminus A) \arrow[l, "\cong"'] \arrow[d, "\cong"', "H_n(\pi)"] \\
H_n(X/A, A/A) \arrow[r, "\cong"] & H_n(X/A, U/A) & H_n(X/A \setminus A/A, U/A \setminus A/A) \arrow[l, "\cong"']
\end{tikzcd}$$

The upper left arrow is an isomorphism by Corollary 6.12 because A is a deformation retract of U. The isomorphism in the upper right is a consequence of excision, because $A = \bar{A} \subset U$. The right vertical map is an isomorphism as π induces a homeomorphis of pairs $(X \setminus A, U \setminus A) \cong (X/A \setminus A/A, U/A \setminus A/A)$. The lower right map is an isomorphism by excision again.

Finally, for the lower left map we need to use that A is a deformation retract of U. Thus A/A is homotopy equivalent to U/A and the last map is an isomorphism. $\square$

We now return to Example 6.4. We had shown $H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$ for all $n \geqslant 1$ and $j > 1$. But $(\mathbb{D}^n, \mathbb{S}^{n-1})$ is a good pair, so the right hand side is $\widetilde{H}_j(\mathbb{D}^n/\mathbb{S}^{n-1}) \cong \widetilde{H}_j(\mathbb{S}^n)$.

So we may compute homology groups of $\mathbb{S}^n$ inductively. As $\mathbb{S}^0$ is just a disjoint union of two points we know $\widetilde{H}_i(\mathbb{S}^0) \cong \mathbb{Z}$ if $i = 0$ and 0 if $i > 0$.

Thus $\widetilde{H}_i(\mathbb{S}^n)$ is 0 in degrees higher than n and $\mathbb{Z}$ in degree n. If $0 < i < n$ we may reduce $\widetilde{H}_i(\mathbb{S}^n)$ to $\widetilde{H}_1(\mathbb{S}^{n-i+1})$ which is 0 by the computation in Example 3.9.

In fact, revisiting Example 6.4 and considering the long exact sequence of reduced homology we can directly compute $H_1(\mathbb{D}^n/\mathbb{S}^{n-1}) \cong \widetilde{H}_0(\mathbb{S}^{n-1}) \cong 0$.

If we pick a generator $\mu_0 := (1, -1)$ of $\widetilde{H}^0(S^0)$ we may thus define generators μ_n of $H^n(S^2)$ for all $n > 0$ by $D\mu_n = \mu_{n-1}$ where $D : \widetilde{H}^n(S^n) \cong \widetilde{H}^{n-1}(S^{n-1})$ is the isomorphism we just constructed.

As this is arguably the most important computation in the course we state the result as a theorem:

Theorem 8.5. *For all $n \geqslant 0$ we have*

$$\widetilde{H}_i(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

We can thus prove topological invariance of dimension:

Corollary 8.6. *If $\mathbb{R}^m \cong \mathbb{R}^n$ then $m = n$.*

Proof. The case $m = 0$ is straightforward so assume $m \geqslant 1$ and $n \geqslant 1$. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a homeomorphism, this induces a homeomorphism $\mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{f(0)\}$ and a homotopy equivalence $S^{m-1} \simeq S^{n-1}$. But reduced homology groups are homotopy invariant, so Theorem 8.5 implies $m = n$. $\square$

We can also compute the homology groups of bouquets of spaces. Let $(X_i)_{i \in I}$ be a family of topological spaces with chosen basepoints $x_i \in X_i$. Consider

$$X = \bigvee_{i \in I} X_i.$$

Proposition 8.7. *If there are open neighbourhoods U_i of $x_i \in X_i$ together with a deformation of U_i to $\{x_i\}$, then we have*

$$\widetilde{H}_n(\bigvee_{i \in I} X_i) \cong \bigoplus_{i \in I} \widetilde{H}_n(X_i).$$

Proof. We may define a deformation retract of $\amalg U_i$ to $\amalg\{x_i\}$.

We then have $\widetilde{H}_n(\bigvee_i X_i) = H_n(\amalg_i X_i, \amalg_i\{x_i\})$ as $(\amalg X_i, \amalg\{x_i\})$ is a good pair. But the right hand side is isomorphic to $\oplus_i \widetilde{H}_n(X_i)$ by splitting $S_*(\amalg_i X_i, \amalg_i\{x_i\})$ into $\oplus S_*(X_i, x_i)$ as in the proof of Corollary 3.3 and observing that taking homology commutes with taking a direct sum. (To convince yourself define a comparison map from $\oplus_i H_n(C_i)$ to $H_n(\oplus_i C_i)$ and check it is an isomorphism.) $\square$

9. Mayer-Vietoris sequence

We consider the following situation: there are subspaces $U, V \subset X$ such that U and V are open in X and such that $X = U \cup V$. We consider the open covering $\mathfrak{U} = \{U, V\}$. We need the following maps:

$$\begin{tikzcd} & U \arrow[dr, "j^U"] & \\ U \cap V \arrow[ur, "i^U"] \arrow[dr, "i^V"'] & & X = U \cup V \\ & V \arrow[ur, "j^V"'] & \end{tikzcd}$$

Note that by definition, the sequence

$$0 \to S_*(U \cap V) \xrightarrow{(i_*^U, i_*^V)} S_*(U) \oplus S_*(V) \xrightarrow{j_*^U - j_*^V} S_*^{\mathfrak{U}}(X) \to 0 \tag{9.1}$$

is exact. Here we write j_*^U for $S_*(j^U)$ etc. for better legibility.

Theorem 9.1 (The Mayer-Vietoris sequence). *There is a long exact sequence*

$$\dots \xrightarrow{\delta} H_n(U \cap V) \xrightarrow{(i_*^U, i_*^V)} H_n(U) \oplus H_n(V) \xrightarrow{j_*^U - j_*^V} H_n(X) \xrightarrow{\delta} H_{n-1}(U \cap V) \to \dots$$

Proof. By Lemma 7.12 $H_n^{\mathfrak{U}}(X) \cong H_n(X)$, thus the theorem follows from Theorem 6.3 and Equation 9.1. $\square$

There is also a short exact sequence

$$0 \longrightarrow \widetilde{S}_*(U \cap V) \xrightarrow{(i_1, i_2)} \widetilde{S}_*(U) \oplus \widetilde{S}_*(V) \longrightarrow \widetilde{S}_*^{\mathfrak{U}}(X) \longrightarrow 0 \tag{9.2}$$

which is just $\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{-} \mathbb{Z}$ in degree -1. Thus we similarly obtain a Mayer-Vietoris sequence in reduced homology (just put $\widetilde{H}$ instead of H everywhere in Theorem 9.1).

Example 9.2. We calculate the homology groups of spheres again. Let $X = \mathbb{S}^m$ and for $m \geqslant 1$ let $X^{\pm} := \mathbb{S}^m \setminus \{\mp e_{m+1}\}$ with inclusion $i^{\pm} : X^{\pm} \to \mathbb{S}^m$. The subspaces X^+ and X^- are homeomorphic to open balls and contractible, therefore $H_n(X^{\pm}) = 0$ for all positive n. Moreover. $X^+ \cap X^- \simeq S^{m-1}$.

The Mayer-Vietoris sequence is as follows

$$\dots \xrightarrow{\delta} H_n(X^+ \cap X^-) \longrightarrow H_n(X^+) \oplus H_n(X^-) \longrightarrow H_n(\mathbb{S}^m) \xrightarrow{\delta} H_{n-1}(X^+ \cap X^-) \longrightarrow \dots$$

We consider $m = 1$ first where $H_n(\mathbb{S}^1) = 0$ if $n > 1$ as it lies between zeros in an exact sequence. In low degrees we have

$$0 \to H_1(\mathbb{S}^1) \xrightarrow{\delta} H_0(\mathbb{S}^0) \xrightarrow{\iota} H_0(X^+) \oplus H_0(X^-) \cong \mathbb{Z}^2 \to H_0(\mathbb{S}^1) \cong \mathbb{Z}^1 \to 0$$

which is entirely determined by $\iota = (i_*^+, i_*^-)$.

$H_0(X^+ \cap X^-)$ is $\mathbb{Z}^2$ generated by $[e_1]$ and $[-e_1]$. The map ι sends both $[e_1]$ and $[-e_1]$ to $(1,1) \in H_0(X^+) \oplus H_0(X^-) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Thus $H_1(S^1) \cong \ker((i_U, i_V)) \cong \mathbb{Z}$ is generated by $\delta^{-1}([e_1] - [-e_1])$.

For $n > 1$ we can deduce

$$H_n(\mathbb{S}^m) \cong H_{n-1}(X^+ \cap X^-) \cong H_{n-1}(\mathbb{S}^{m-1}).$$

The first map is the connecting homomorphism and the second map is the inverse of $H_{n-1}(i)\colon H_{n-1}(\mathbb{S}^{m-1}) \to H_{n-1}(X^+ \cap X^-)$ where i is the inclusion of $\mathbb{S}^{m-1}$ into $X^+ \cap X^-$ and this inclusion is a homotopy equivalence. Thus define $D' := H_{n-1}(i)^{-1} \circ \delta$. This D' is an isomorphism for all $n \geqslant 2$.

Thus $H_n(\mathbb{S}^m) = H_{n-m+1}(\mathbb{S}^1) = 0$ for $n > m$ and $H_m(\mathbb{S}^m) \cong H_1(\mathbb{S}^1) \cong \mathbb{Z}$. Finally $H_n(\mathbb{S}^m) \cong H_1(\mathbb{S}^{m-n+1})$ if $0 < n < m$ and it remains to compute $H_1(\mathbb{S}^m)$ for $m > 1$.

Again we have $H_1(\mathbb{S}^m) \cong \ker(\iota : H_0(X^+ \cap X^-) \to H_0(X^+) \oplus H_0(X^-))$ and $\iota : 1 \mapsto (1,1)$ is injective.

Thus $H_1(\mathbb{S}^m) = 0$ for $m > 1$, confirming the earlier computation via Hurewicz' theorem.

We can summarize the result as follows.

$$H_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = m = 0, \\ \mathbb{Z}, & n = 0, m > 0, \\ \mathbb{Z}, & n = m > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 9.3. Let $\mu_0' := -[e_1] + [-e_1] \in H_0(X^+ \cap X^-) \cong H_0(\mathbb{S}^0)$. Then a diagram chase shows that $\mu_1' \in H_1(\mathbb{S}^1)$ given by the loop $t \mapsto e^{2\pi i t}$, aka the identity, satisfies $D'\mu_1' = \mu_0'$.

We define the higher μ_n' via $D'\mu_n' = \mu_{n-1}'$. Then μ_n' is called *the fundamental class in* $H_n(\mathbb{S}^n)$.

We could have simplified our live by using the reduced Mayer-Vietoris sequence. We shall do this for our next example.

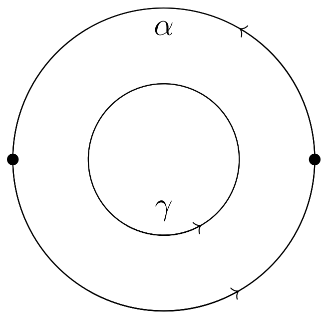
Example 9.4. Recall that we can express $\mathbb{R}P^2$ as the quotient space of $\mathbb{S}^2$ modulo antipodal points or as a quotient of $\mathbb{D}^2$:

$$\mathbb{R}P^2 \cong \mathbb{S}^2/\pm\mathrm{id} \cong \mathbb{D}^2/z \sim -z \text{ for } z \in \mathbb{S}^1.$$

We use the latter definition and set $X = \mathbb{R}P^2$, $U = X \setminus \{[0,0]\}$ (which is an open Möbius strip and hence homotopy equivalent to $\mathbb{S}^1$) and $V = \mathring{\mathbb{D}}^2$. Then

$$U \cap V = \mathring{\mathbb{D}}^2 \setminus \{[0,0]\} \simeq \mathbb{S}^1.$$

Thus we know that $H_1(U) \cong \mathbb{Z}$, $H_1(V) \cong 0$ and $H_2U = H_2V = 0$. We choose generators α for $H_1(U)$ and γ for $H_1(U \cap V)$ as follows:



Let α be the path that runs along the outer circle in mathematical positive direction half around starting from the point $(1,0)$. Let γ be the loop that runs along the inner circle in mathematical positive direction. It thus runs around the boundary of the Möbius map, which corresponds to running around the equator of the Möbius band twice. Thus the inclusion $i\colon U \cap V \to U$ induces

$$i_*[\gamma] = 2[\alpha].$$

This suffices to compute $H_*(\mathbb{R}P^2)$ up to degree two because the long exact sequence is

$$\widetilde{H}_2(U) \oplus \widetilde{H}_2(V) \to \widetilde{H}_2(X) \to \widetilde{H}_1(U \cap V) \xrightarrow{i_*} \widetilde{H}_1(U) \to \widetilde{H}_1(X) \to \widetilde{H}_0(U \cap V)$$

which becomes

$$0 \to \widetilde{H}_2(X) \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \widetilde{H}_1(X) \to 0.$$

We obtain:

$$\begin{aligned} H_2(\mathbb{R}P^2) &\cong \ker(2\cdot\colon \mathbb{Z} \to \mathbb{Z}) = 0, \\ H_1(\mathbb{R}P^2) &\cong \operatorname{coker}(2\cdot\colon \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \\ H_0(\mathbb{R}P^2) &\cong \mathbb{Z}. \end{aligned}$$

The higher homology groups are trivial, because there $H_n(\mathbb{R}P^2)$ is located in a long exact sequence between trivial groups.

We next consider a relative version of the Mayer-Vietoris sequence. For this we need some tools from homological algebra.

LEMMA 9.5 (The 5-lemma). *Let*

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\ B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \end{tikzcd}$$

be a commutative diagram of exact sequences. If f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

PROOF. Again, we are chasing diagrams.

In order to prove that f_3 is injective, assume that there is an $a \in A_3$ with $f_3 a = 0$. Then $\beta_3 f_3 a = f_4 \alpha_3 a = 0$, as well. But f_4 is injective, thus $\alpha_3 a = 0$. Exactness of the top row gives, that there is an $a_2 \in A_2$ with $\alpha_2 a_2 = a$. This implies

$$f_3 \alpha_2 a_2 = f_3 a = 0 = \beta_2 f_2 a_2.$$

Exactness of the bottom row gives us a $b \in B_1$ with $\beta_1 b = f_2 a_2$, but f_1 is an isomorphism so we can lift b to $a_1 \in A_1$ with $f_1 a_1 = b$.

Thus $f_2 \alpha_1 a_1 = \beta_1 b = f_2 a_2$ and as f_2 is injective, this implies that $\alpha_1 a_1 = a_2$. So finally we get that $a = \alpha_2 a_2 = \alpha_2 \alpha_1 a_1$, but the latter is zero, thus $a = 0$.

For the surjectivity of f_3 assume $b \in B_3$ is given. Move b over to B_4 via β_3 and set $a := f_4^{-1} \beta_3 b$. (Note here, that if $\beta_3 b = 0$ we actually get a shortcut: Then there is a $b_2 \in B_2$ with $\beta_2 b_2 = b$ and thus an $a_2 \in A_2$ with $f_2 a_2 = b_2$. Then $f_3 \alpha_2 a_2 = \beta_2 b_2 = b$.)

Consider $f_5 \alpha_4 a$. This is equal to $\beta_4 \beta_3 b$ and hence trivial. Therefore $\alpha_4 a = 0$ and thus there is an $a_3 \in A_3$ with $\alpha_3 a_3 = a$. Then $b - f_3 a_3$ is in the kernel of β_3 because

$$\beta_3(b - f_3 a_3) = \beta_3 b - f_4 \alpha_3 a_3 = \beta_3 b - f_4 a = 0.$$

Hence we get a $b_2 \in B_2$ with $\beta_2 b_2 = b - f_3 a_3$. Define a_2 as $f_2^{-1}(b_2)$, so $a_3 + \alpha_2 a_2$ is in A_3 and

$$f_3(a_3 + \alpha_2 a_2) = f_3 a_3 + \beta_2 f_2 a_2 = f_3 a_3 + \beta_2 b_2 = f_3 a_3 + b - f_3 a_3 = b.$$

$\square$

The next lemma has an easier proof, left as an exercise (on one of the example sheets).

Lemma 9.6 (The 9-lemma). *Consider the following commutative diagram such that all columns and the first two rows are exact.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
 & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then the bottom row is also exact.

Theorem 9.7 (Relative Mayer-Vietoris sequence). *If $A, B \subset X$ are open in $A \cup B$, then the following sequence is exact:*

$$\ldots \xrightarrow{\delta} H_n(X, A \cap B) \longrightarrow H_n(X, A) \oplus H_n(X, B) \longrightarrow H_n(X, A \cup B) \xrightarrow{\delta} \quad \ldots$$

Proof. Set $\mathfrak{U} := \{A, B\}$. This is an open covering of $A \cup B$.

The following diagram of exact sequences combines absolute chains with relative ones:

$$
\begin{tikzcd}
 & & & & & 0 \arrow[d] \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & S_n(A \cup B) \arrow[d] \\
0 \arrow[r] & S_n(A \cap B) \arrow[r] \arrow[d] & S_n(A) \oplus S_n(B) \arrow[r] \arrow[d] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] \arrow[urr, "\varphi"] & 0 \arrow[d] & S_n(X) \arrow[d] \\
0 \arrow[r] & S_n(X) \arrow[r, "\Delta"] \arrow[d] & S_n(X) \oplus S_n(X) \arrow[r, "-"] \arrow[d] & S_n(X) \arrow[r] \arrow[d] \arrow[urr, equal] & 0 \arrow[d] & S_n(X, A \cup B) \arrow[d] \\
0 \arrow[r] & S_n(X, A \cap B) \arrow[r] \arrow[d] & S_n(X, A) \oplus S_n(X, B) \arrow[r] \arrow[d] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] \arrow[urr, "\psi"] & 0 & 0 \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

Here, ψ is induced by the inclusion $\varphi\colon S_n^{\mathfrak{U}}(A \cup B) \to S_n(A \cup B)$, Δ denotes the diagonal map and $-$ the difference map. It is clear that the first two rows are exact, thus the third row is exact by Lemma 9.6.

Consider the two right-most non-trivial columns in this diagram. Each gives a long exact sequence in homology and we focus on five terms.

$$
\begin{tikzcd}
H_n(S_*^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "\varphi_*"'] & H_n(X) \arrow[r] \arrow[d, equal] & H_n(S_*(X)/S_*^{\mathfrak{U}}(A \cup B)) \arrow[r, "\delta"] \arrow[d, "\psi_*"'] & H_{n-1}(S_*^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "\varphi_*"'] & H_{n-1}(X) \arrow[d, equal] \\
H_n(A \cup B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A \cup B) \arrow[r, "\delta"] & H_{n-1}(A \cup B) \arrow[r] & H_{n-1}(X)
\end{tikzcd}
$$

Then by the five-lemma 9.5, as $H_n(\varphi)$ and $H_{n-1}(\varphi)$ are isomorphisms, so is $H_n(\psi)$. Thus the bottom row gives a short exact sequence

$$0 \to S_*(X, A \cap B) \to S_*(X, A) \oplus S_*(X, B) \to S_*(X, A \cup B) \to 0$$

which gives the theorem by Proposition 5.7. $\square$

Example 9.8. We compute the homology $H_n(\mathbb{S}^3, \mathbb{S}^1)$. Let A, B be two arcs homeomorphc to $[0, 1]$ connecting the two points of a copy of $\mathbb{S}^0$ in $\mathbb{S}^3$. We really want to take open neighbourhoods, but this won't affect the homotopy type and thus won't affect the homology groups.

$$\cdots \to H_n(\mathbb{S}^3, \mathbb{S}^0) \to H_n(\mathbb{S}^3, B) \oplus H_n(\mathbb{S}^3, A]) \to H_n(\mathbb{S}^3, \mathbb{S}^1) \to H_{n-1}(\mathbb{S}^3, \mathbb{S}^0) \to \dots$$

From the relative homology sequence $H^n(\mathbb{S}^3, \mathbb{S}^0)$ is $\mathbb{Z}$ in degrees 3 and 1. Thus we have

$$
\begin{aligned}
0 \to 0 \to H_4(\mathbb{S}^3, \mathbb{S}^1) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \oplus \mathbb{Z} \to H_3(\mathbb{S}^3, \mathbb{S}^1) \\
\xrightarrow{\delta} 0 \to 0 \to H_2(\mathbb{S}^3, \mathbb{S}^1) \xrightarrow{\delta} \mathbb{Z} \to 0 \to H_1(\mathbb{S}^3, \mathbb{S}^1) \xrightarrow{\delta} 0 \to 0 \to H_0(\mathbb{S}^3, \mathbb{S}^1)
\end{aligned}
$$

Thus we immediately see that $H_n(\mathbb{S}^3, \mathbb{S}^1) = \mathbb{Z}$ if $n = 2$ and 0 if $n \neq 2, 3, 4$.

To further analyze this we need to work out ι. It is induced by the inclusions of $(\mathbb{S}^3, \mathbb{S}^0)$ into $(\mathbb{S}^3, A)$ and $(\mathbb{S}^3, A)$. But $H_3 \cong \mathbb{Z}$ is generated by the image of the generator μ_3 of $\mathbb{S}^3$ for all of these spaces (the subspace does not have any influence on H^3). Thus $\iota(\bar{\mu}_3) = (\bar{\mu}_3, \bar{\mu}_3)$ and $H_4(\mathbb{S}^3, \mathbb{S}^1) \cong 0$, $H_3(\mathbb{S}^3, \mathbb{S}^1) = \mathbb{Z}$.

We check that this makes sense topologically: $\mathbb{S}^3/\mathbb{S}^1$ squeezes a loop in $\mathbb{S}^3$ down to a point, thus the interior of the loop bubbles out to give a copy of $\mathbb{S}^2$, wedged together with $\mathbb{S}^3/\mathbb{D}^2 \cong \mathbb{S}^3$. By our computation of the homology group of wedge sums $\widetilde{H}_n(\mathbb{S}^3/\mathbb{S}^1) = \widetilde{H}_n(\mathbb{S}^3) \oplus \widetilde{H}_2(\mathbb{S}^2)$, which agrees with our Mayer-Vietoris computation.

10. Mapping degree

Recall that we defined fundamental classes $\mu_n \in \tilde{H}_n(\mathbb{S}^n)$ for all $n \geqslant 0$. In fact there are two reasonable solutions: By the boundary map of the Mayer-Vietoris sequence and by the boundary map of the relative homology exact sequence.

Definition 10.1. Let $\mu_0 := [e_1] - [-e_1] \in H_0(\mathbb{S}^0)$. We define the *the fundamental class* $\mu_n \in H_n(\mathbb{S}^n)$ via $D\mu_n = \mu_{n-1}$.

Here we used

$$D : \widetilde{H}_n(\mathbb{S}^n) \cong \widetilde{H}_n(\mathbb{D}^n/\mathbb{S}^{n-1}) \cong \widetilde{H}_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \xrightarrow{\delta} H_{n-1}(\mathbb{S}^{n-1})$$

and then let the fundamental class be $\mu_n = D^{-1}\mu_{n-1}$.

Here we have to fix the first isomorphism, and we choose it to be induced by the map from $\mathbb{D}^n \subset \mathbb{R}^n$ to $\mathbb{S}^n + e_{n+1}\mathbb{R}^{n+1}$ that wraps the disk around the ball in an upwards direction. As a formula $(x_1, \ldots, x_n, 0) \mapsto (ux_1, \ldots, ux_n, 2t)$ where $t = \sum x_i^2$ and $u = \sqrt{1 - (2t-1)^2}$. Call this map $u_n : \mathbb{D}^n/\mathbb{S}^{n-1} \to \mathbb{S}^n$ for future reference.

Remark 10.2. With our conventions the closed interval $[-1, 1]$ in $\mathbb{D}^1$ generates the loop $e^{2\pi i}$ in mathematically positive direction and by a diagram chase this is sent to μ_0 by D. Contrast this with the situation in Definition 9.3!

Let $f \colon \mathbb{S}^n \to \mathbb{S}^n$ be any continuous map.

Definition 10.3. The map f induces a homomorphism

$$\tilde{H}_n(f) \colon \tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_n(\mathbb{S}^n)$$

and therefore we get

$$\tilde{H}_n(f)\mu_n = \deg(f)\mu_n$$

with $\deg(f) \in \mathbb{Z}$. We call this integer *the degree of* f.

In the case $n = 1$ we can relate this notion of a mapping degree to the one defined via the fundamental group of the 1-sphere: if we represent the generator of $\pi_1(\mathbb{S}^1, 1)$ as the class given by the loop

$$\omega \colon [0, 1] \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i t},$$

then the abelianized Hurewicz, $h_{\mathrm{ab}}\colon \pi_1(\mathbb{S}^1,1) \to H_1(\mathbb{S}^1)$, sends the class of ω precisely to μ_1 and therefore the naturality of h_{ab}

$$\begin{tikzcd}
\pi_1(\mathbb{S}^1,1) \arrow[r, "\pi_1(f)"] \arrow[d, "h_{\mathrm{ab}}"'] & \pi_1(\mathbb{S}^1,1) \arrow[d, "h_{\mathrm{ab}}"] \\
H_1(\mathbb{S}^1) \arrow[r, "H_1(f)"] & H_1(\mathbb{S}^1)
\end{tikzcd}$$

shows that

$$\deg(f)\mu_1 = H_1(f)\mu_1 = h_{\mathrm{ab}}(\pi_1(f)[w]) = h_{\mathrm{ab}}(\deg_{\pi(f)}[w]) = \deg_{\pi(f)}\mu_1.$$

where $\deg_{\pi(f)}$ is the degree of f defined via the fundamental group. Thus both notions coincide for $n = 1$.

The degree of self-maps of $\mathbb{S}^n$ satisfies the following properties:

Proposition 10.4.
(a) If f is homotopic to g, then $\deg(f) = \deg(g)$.
(b) The degree of the identity on $\mathbb{S}^n$ is one.
(c) The degree is multiplicative, i.e. $\deg(g \circ f) = \deg(g)\deg(f)$.
(d) If f is not surjective, then $\deg(f) = 0$.

Proof. The first three properties follow directly from the definition of the degree. If f is not surjective, then it is homotopic to a constant map and this has degree zero. Alternatively we have a factorization of f through $\mathbb{S}^n \setminus \{x\}$), which has no n-th homology, thus $f_* is 0$ on $\widetilde{H}_n$. □

It is true that the group of (pointed) homotopy classes of self-maps of $\mathbb{S}^n$ is isomorphic to $\mathbb{Z}$ and thus the first property can be upgraded to an 'if and only if', but we won't prove that here.

We use the mapping degree to show some geometric properties of self-maps of spheres.

Proposition 10.5. *Let $f^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map*

$$(x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n).$$

Then $f^{(n)}$ has degree -1.

Proof. We prove the claim by induction. μ_0 was the difference class $[+1] - [-1]$, and

$$f^{(0)}([+1] - [-1]) = [-1] - [+1] = -\mu_0.$$

We defined μ_n in such a way that $D\mu_n = \mu_{n-1}$. Therefore, as D is natural and $f^{(n)}|_{\mathbb{S}^{n-1}} = f^{(n-1)}$ we have

$$H_n(f^{(n)})\mu_n = H_n(f^{(n)})D^{-1}\mu_{n-1} = D^{-1}H_{n-1}(f^{(n-1)})\mu_{n-1} = D^{-1}(-\mu_{n-1}) = -\mu_n.$$

□

Corollary 10.6. *The antipodal map $A\colon \mathbb{S}^n \to \mathbb{S}^n$, $A(x) = -x$, has degree $(-1)^{n+1}$.*

Proof. Let $f_i^{(n)} \colon \mathbb{S}^n \to \mathbb{S}^n$ be the map $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$. As all $f_i^{(n)}$ are homotopic to each other (by continuously varying the plane of reflection) we see that by Proposition 10.5 the degree of $f_i^{(n)}$ is -1. As $A = f_n^{(n)} \circ \ldots \circ f_0^{(n)}$, the claim follows. $\square$

In particular, the antipodal map cannot be homotopic to the identity as long as n is even!

Proposition 10.7. *Let $f, g \colon \mathbb{S}^n \to \mathbb{S}^n$ with $f(x) \neq g(x)$ for all $x \in \mathbb{S}^n$, then f is homotopic to $A \circ g$. In particular,*

$$\deg(f) = (-1)^{n+1} \deg(g).$$

Proof. By assumption the segment $t \mapsto (1-t)f(x) - tg(x)$ doesn't pass through the origin for $0 \leqslant t \leqslant 1$. Thus the homotopy

$$H(x,t) = \frac{(1-t)f(x) - tg(x)}{||(1-t)f(x) - tg(x)||}$$

connects f to $-g = A \circ g$. $\square$

Corollary 10.8. *For any $f \colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = 0$ there is an $x_+ \in \mathbb{S}^n$ with $f(x_+) = x_+$ and an x_- with $f(x_-) = -x_-$.*

Proof. If $f(x) \neq x$ for all x, then $\deg(f) = \deg(A) \neq 0$. If $f(x) \neq -x$ for all x, then $\deg(f) = (-1)^{n+1}\deg(A) \neq 0$. $\square$

Corollary 10.9. *Assume that n is even and let $f \colon \mathbb{S}^n \to \mathbb{S}^n$ be any continuous map. Then there is an $x \in \mathbb{S}^n$ with $f(x) = x$ or $f(x) = -x$.*

Proof. Assume $f(x) \neq x$ for all n. Then by Proposition 10.7 f is homotopic to $A \circ \mathrm{id}_{\mathbb{S}^n}$. If $f(x) \neq -x$ for all n then f is also homotopy to $A \circ A = \mathrm{id}$. As n is even this is a contradiction. $\square$

Finally, we can say the following about hairstyles of hedgehogs of arbitrary even dimension. For this we need to define a hairstyle, aka a tangential vector field.

The *tangent bundle* of a manifold $M \subset \mathbb{R}^N$ is the subspace of $M \times \mathbb{R}^N$ consisting of pairs (m, T) with $m \in M$ and T a vector in $\mathbb{R}^N$ tangent to M at m. See your differential geometry course for what that means in general, in the case of the sphere it gives

$$T\mathbb{S}^n = \{x, v \mid x \in \mathbb{S}^n \ \&\ (x,v) = 0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$

The tangent space ot x is $T_x\mathbb{S}^n = \{v \mid (x,v) = 0\}$. The tangent bundle has a natural projection $T\mathbb{S}^n \to \mathbb{S}^n$ and a *tangential vector field is a section $x \mapsto (x, V(x)) : \mathbb{S}^n \to T\mathbb{S}^n$.*

Proposition 10.10. *Any tangential vector field on $\mathbb{S}^{2k}$ is trivial in at least one point.*

Proof. Assume that V is a tangential vector field which does not vanish, *i.e.*, $V(x) \neq 0$ for all $x \in \mathbb{S}^{2k}$ and $V(x) \in T_x(\mathbb{S}^{2k}) \subset \mathbb{R}^{2k+1}$ for all x.

Define $f : \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ by $k \mapsto \frac{V(x)}{||V(x)||}$. Assume $f(x) = x$, hence $V(x) = ||V(x)||x$. But this means that $V(x)$ points into the direction of x and thus it cannot be tangential. Similarly, $f(x) = -x$ yields the same contradiction. Thus such a V cannot exist. $\square$

We now consider a way of determining the degree, which depends globally on the map f by a local computation, just considering what happens in the neighbourhoods of some points.

DEFINITION 10.11. For any topological space X and $x \in X$ we call $H_n(X, X \setminus \{x\})$ the *local homology groups of X at x*.

By excision this really only depends on an open neighbourhood of x in X.

If $X = \mathbb{S}^n$ then by excision $H_i(\mathbb{S}^n, \mathbb{S}^n \setminus \{x\}) \cong H_i(\mathbb{D}^n, \mathbb{S}^{n-1})$ is $\mathbb{Z}$ if $i = n$ and 0 otherwise. Assume $f : \mathbb{S}^n \to \mathbb{S}^n$ and $y \in \mathbb{S}^n$ are such that there is $y \in V \subset \mathbb{S}^n$ and $U \subset \mathbb{S}^n$ with $f(U) \subset V$ and $f^{-1}(y) \cap U = \{x\}$. Then there is an induced map

$$f_x : \widetilde{H}_n(\mathbb{S}^n) \cong H_n(\mathbb{S}^n, \mathbb{S}^n \backslash \{x\}) \cong H_n(U, U \backslash \{x\}) \xrightarrow{f_*} H_n(V, V \backslash \{y\}) \cong H_n(\mathbb{S}^n, \mathbb{S}^n \backslash \{y\}) \cong \widetilde{H}_n(\mathbb{S}^n)$$

which is given by multiplication of some integer d.

DEFINITION 10.12. In the situation as above we call the integer d the *local degree of f at x* and denote it by $\deg(f)|_{x_i}$.

PROPOSITION 10.13. *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a map and $y \in \mathbb{S}^n$ is such that $f^{-1}(y)$ is finite. Then* $\deg(f) = \sum_{x_i \in f^{-1}(y)} \deg(f)|_{x_i}$.

PROOF. By excision

$$H_n(\mathbb{S}^n, \mathbb{S}^n \setminus f^{-1}Y) \cong H_n(\amalg U_i, \amalg U_i \setminus \{x_i\}) \cong \oplus_i H_n(U_i, U_i \setminus \{x_i\})$$

for some collection of disjoint neighbourhoods of the $x_i \in f^{-1}(y)$.

In the following diagram the horizontal maps are induced by the long exact sequence of relative homology and by excision and all vertical maps are induced by f. Thus by naturality it commutes (the rightmost square commutes by definition).

$$\begin{tikzcd}
\widetilde{H}_n(\mathbb{S}^n) \arrow[r] \arrow[d, "f_*"'] & H_n(\mathbb{S}^n, \mathbb{S}^n \setminus f^{-1}(y)) \arrow[r, "\cong"] \arrow[d, "f_*"] & \oplus H_n(U_i, U_i \setminus \{x_i\} \arrow[r, "\cong"] \arrow[d, "f_*"'] & \oplus \widetilde{H}_n(\mathbb{S}^n) \arrow[d, "(\deg f|_{x_i})"'] \\
\widetilde{H}_n(\mathbb{S}^n) \arrow[r, "\cong"] & H_n(\mathbb{S}^n, \mathbb{S}^n \setminus \{y\} \arrow[r, "\cong"] & H_n(V, V \setminus \{y\}) \arrow[r, "\cong"] & \widetilde{H}_n(\mathbb{S}^n)
\end{tikzcd}$$

We denote the composition of isomorphisms at the bottom by v. (It can only by $+1$ or -1 and in fact it is the identity as the composition of inverse isomorphisms but that is not needed for the proof.) The composition of the top maps induces the diagonal map $1 \mapsto (v, \ldots, v)$ as it is equal to the map v on each summand as it is constructed in exactly the same way as the map on the bottom.

The rightmost map is just the local degree at x_i in the i-th coordinate. Thus commutativity of the diagram then gives $\deg(f) = \sum \deg(f)|_{x_i}$ (as the v's cancel). $\square$

EXAMPLE 10.14. Let $f, g : \mathbb{S}^n \to \mathbb{S}^n$ be maps that fix a point (which we will declare to be the base-point). Then we have an induced map $f \vee g : \mathbb{S}^n \vee \mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$. We consider the pinch map $P : \mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$ that contracts the equator down to a point and the fold map $\nabla : \mathbb{S}^n \vee \mathbb{S}^n$ induced by identity map on both summands.

We define $f+g := \nabla \circ (f \vee g) \circ P$. Then it follows from Proposition 10.13 that $\deg(f+g) = \deg f + \deg g$: Any non-base point has two pre-images x, y under the fold map and we compute the degree of f and g by considerinng their preimages $\{x_i\}$ respectively $\{y_i\}$ under f and g respectively. Then $\deg(f+g) = \sum_{x_i} \deg f|_{x_i} + \sum_{y_i} \deg g|_{y_i} = \deg f + \deg g$.

11. CW complexes

We now define an important class of topological spaces. They are flexible enough to cover most reasonable spaces, in particular all the spaces we are interested in in this course. At the same time they have very useful inductive description.

First we recall the notion of a colimit of topological spaces. (Replacing Top by another category we obtain the general definition of colimits.)

Definition 11.1. Let I be a small category (i.e. a collection of objects and morphisms). Then a *diagram* of topological spaces of shape I is a functor $I \to \mathsf{Top}$.

Example 11.2. A map of topological spaces is nothing but a diagram in the shape of the category $\bullet \to \bullet$ of two objects and non non-identity morphism.

Definition 11.3. Let $X : I \to \mathsf{Top}$ be a diagram. The *colimit* $\operatorname{colim}_I X$ of the diagram is a topological space C together with maps $\iota_i : X(i) \to C$ for all objects i of I such that

(a) $\iota_j \circ X(f) = \iota_i$ for any morphism $f : i \to j$ in I
(b) for any other topological space D with a maps $\phi_i : X(i) \to Y$ satisfying $\phi_j \circ X(f) = \phi_i$ there is a unique map $c : C \to D$ satisfying $\phi_i = \phi \circ \iota_i$ for all i.

We say C is the universal object under the diagram X.

The corresponding diagram looks like this:

$$\begin{tikzcd}
& X(i) \arrow[dl, "X(f)"'] \arrow[d, "\iota_i"] \arrow[ddr, bend left, "\phi_i"] & \\
X(j) \arrow[r, "\iota_j"'] \arrow[drr, bend right, "\phi_j"'] & \operatorname{colim}_I X \arrow[dr, dashed, "\exists!\varphi"] & \\
& & D
\end{tikzcd}$$

Let $I = \bullet \leftarrow \bullet \to \bullet$ be a category with three objects and two non-identity morphisms. A diagram of shape I is called a *pushout diagram* and its colimit a *pushout*.

Proposition 11.4. *Pushouts exist in* Top.

Proof. Let $I \to \mathsf{Top}$ be pushout diagram, we write it as $Y \xleftarrow{f} X \xrightarrow{g} Z$. Consider $C = Y \amalg Z/\sim$ where $y \sim z$ if there is x such that $f(x) = y$ and $g(x) = z$ and equip it with the quotient topology.

Let D be some other object under the pushout diagram, with maps ψ_X, ψ_Y, ψ_Z to B. It is easy to see from the definition that there is a unique map of sets $C \to B$ making everything commute, just define ψ by ψ_Y on $[y]$ and ψ_Z on $[z]$, it is well defined as $\psi_Y(f(x)) = \psi_X(x) = \psi_Z(g(z))$.

Moreover ψ is continuous by the universal property of the disjoint union and quotient topology on C (these are final topologies, the finest topologies making all the canonical incoming maps continuous). $\square$

Example 11.5. (a) The colimit of a discrete diagram (where I only has identity morphisms) is called a *coproduct*. In the category Top this is the disjoint union $\amalg_i X_i$.

(b) The colimit over the empty diagram is an object with a unique morphism to every other object. In Top this is the empty space.
(c) You have probably met some version of the gluing $X \cup_f Y$ where $f : A \to Y$. This is just the pushout of $X \xleftarrow{\iota} A \xrightarrow{f} Y$.
In particular $\mathbb{S}^n$ is the pushout of $\mathbb{D}^n \leftarrow \mathbb{S}^{n-1} \to \mathbb{D}^n$.
(d) Let I now be the category $\mathbb{N}$ with one object for every natural number and a unique morphism $i \to j$ if and only if $i \leqslant j$. A colimit of a diagram $I \to \mathsf{Top}$ is called a *direct limit* (my apologies, this is a terrible name).

PROPOSITION 11.6. *Direct limits exist in* Top.

PROOF. Let $X : \mathbb{N} \to \mathsf{Top}$ and define $\operatorname{colim}_{\mathbb{N}} X$ by $\amalg_n X(n)/\sim$ with $x_n \sim x_m$ if $x_n = X(m \leqslant n)x_n$ for $x_i \in X(i)$.
The proof now proceeds as for pushouts. □

REMARK 11.7. In fact, all colimits exist in Top, and they may be constructed in a similar fashion to pushouts and direct limits.

REMARK 11.8. One may dualize the notion of a colimit to define a limit. For example the limit over a discrete diagram of topological spaces is their product.

DEFINITION 11.9. An *CW complex* is a topological space X with a filtration by subspaces $\emptyset = X^{-1} \subset X^0 \subset X^1 \subset X^2 \dots$ such that
(a) every X^k is a pushout of a diagram

$$X^{k-1} \xleftarrow{q^k} \amalg_{i\in I_k} \partial\mathbb{D}^k \to \amalg_{i\in I_k} \mathbb{D}^k,$$

where I_k is some (possibly empty) indexing set and $q : \amalg_{i\in I_k}\partial\mathbb{D}^k_i \to X_{k-1}$ is a continuous map on the boundaries
(b) $X = \operatorname{colim}_k X^k$.

In particular by this definition $X^0 = (\amalg_{I_0}\mathring{\mathbb{D}}^0) \amalg_{\emptyset} \emptyset$ is a disjoint union of points, or a discrete topological space. (Noting $\mathbb{D}^0 = \mathring{\mathbb{D}}^0 = *$ and $\partial\mathbb{D}^0 = \emptyset$.)
Next we introduce some vocabulary:

DEFINITION 11.10. (a) We call X^n the *n-skeleton* of X. If $X = X^n$ for some n but $X \neq X^{n-1}$ we say X is *n-dimensional*. A CW complex is called *finite* if it has finitely many cells.
(b) We call the maps $q^k_i : \mathbb{S}^{n-1} \to X_{k-1}$ making up q^k the *attachment maps*.
(c) The induced maps $Q^k_i : \mathbb{D}^k \to X^k$ are called *characteristic maps*. We observe that the composition with the natural inclusion of the interior $\mathring{\mathbb{D}}^k{}_i$ gives a a homeomorphism onto a subset e^k_i of X that we call an *k-cell*. By construction X has a (set-theoretic!) *cell decomposition*

$$X = \bigsqcup_{k\geqslant 0} \bigsqcup_{i\in I^k} e^k_i, \quad e^k_i \cong \mathbb{R}^k.$$

(d) A closed subspace $A \subset X$ of a CW complex is called a *subcomplex* if it is a union of cells of X. In particular every n-skeleton X^n of X is a subcomplex of X (and of every m-skeleton with $m \geqslant n$).

Example 11.11. $\mathbb{S}^n$ has many cell decompositions. We can have $\mathbb{S}^n = e^0 \cup e^n$ with the unique attachment map $\mathbb{S}^{n-1} \to *$.

Alternatively we can inductively define $\mathbb{S}^n$ as $\mathbb{S}^{n-1} \cup e^n \cup e^n$ where both n-cells are attached via the identity map to $\mathbb{S}^{n-1}$.

Taking the colimit as $n \to \infty$ we obtain the infinite CW complex $\mathbb{S}^\infty$.

In the case $n = 2$ we can also obtain a CW structure by projecting our favourite dice out to $\mathbb{S}^2$, the vertices give X_0, adding the the edges gives X_1 and adding the faces gives $X_2 = \mathbb{S}^2$.

Example 11.12. $\mathbb{R}P^n$ is a CW complex with $X_k = \mathbb{R}P^k$ and the attachment map q^k is the canonical 2:1 map $\mathbb{S}^{k-1} \to \mathbb{R}P^{k-1}$. Then $\mathbb{R}P^n = e^0 \cup e^1 \cdots \cup e^n$.

One can relate this to other definitions of $\mathbb{R}P^n$ for example by considering the cell structure on $\mathbb{S}^n$ with $X^k = \mathbb{S}^k$ and taking the image in $\mathbb{R}P^n$ under the canonical map.

Remark 11.13. CW stands for *closure-finite weak-topology.* Closure-finite means that the closure of each cell is covered by finitely many open cells. This follows from a general result that any compact subspace of a CW complex (like the closure of a cell) is contained in a finite subcomplex.

Weak topology denotes the following equivalent definition of the topology on the colimit: A subset $A \subset X$ is closed if and only if it intersect each closure of a cell in a closed set.

Remark 11.14. The characteristic maps Q_i^k satisfy the following properties:

(a) $Q_i^k|_{\mathring{\mathbb{D}}^k}$ is a homeomorphism onto its image, the cell e_i^k, and the e_i^k are disjoint and exhaust X.
(b) $Q_i^k(\partial \mathbb{D}^k)$ is contained in the union of a finite number of cells of dimension less than k.
(c) A subset of X is closed iff it meets the closure of each cell in a closed set.

In fact a Hausdorff space X together with a collection of characteristic maps $Q_i^k : \mathbb{D}^k \to X$ is a CW complex if and only if these conditions hold. See Proposition A.2 in [Hatcher].

Example 11.15. The unit interval $[0,1]$ has a CW structure with two zero cells and one 1-cell. But for instance the decomposition $\sigma_0^0 = \{0\}$, $\sigma_k^0 = \{\frac{1}{k}\}, k > 0$ and $\sigma_k^1 = (\frac{1}{k+1}, \frac{1}{k})$ does not give a CW structure on $[0,1]$. The 0-skeleton is not discrete.

Another way to see this is to cconsider the $A \subset [0,1]$ given by

$$A := \left\{ \frac{1}{2} \left(\frac{1}{k} + \frac{1}{k+1} \right) | k \in \mathbb{N} \right\}.$$

Then $A \cap \bar{\sigma}_k^1$ is precisely the point $\frac{1}{2}(\frac{1}{k} + \frac{1}{k+1})$ and this is closed, but A isn't. Thus $[0,1]$ does not have the weak topology.

Let X and Y be CW complexes. A continuous map $f\colon X \to Y$ is called *cellular* if it is compatible with the filtration, i.e. $f(X^n) \subset Y^n$ for all $n \geqslant 0$.

The category of CW complexes together with cellular maps is rather flexible. Most of the classical constructions don't lead out of it (except mapping spaces), but one has to be careful with respect to products.

Example 11.16. Whenever X and Y are CW complexes and Y is locally compact then $X \times Y$ is a CW complex.

We can always define a cell decomposition of $X \times Y$ with n-cells given by the products of cells of X and Y, i.e. if e_X^k is a k-cell of X and , e_Y^{n-k} an $(n-k)$-cell of Y, then $e_X^k \times e_Y^{n-k}$ is an n-cell of the product.

We have to be careful though, the product $X \times Y$ is only guaranteed to carry the weak topology if X or Y is locally compact or has countably many cells! If X and Y don't satisfy these conditions it is best to re-topologize $X \times Y$ with the weak topology. So there is a product of CW spaces, it is just not the naive product in topological spaces.

Lemma 11.17. *For any CW complex X we get for the skeleta:*

(a)
$$X^n \setminus X^{n-1} \cong \bigsqcup_{I^n} \mathring{\mathbb{D}}^n.$$

(b)
$$X^n / X^{n-1} \cong \bigvee_{I^n} \mathbb{S}^n.$$

Proof. The first claim follows directly from the definition of a CW complex. For the second claim note that the characteristic maps send the boundary $\partial \mathbb{D}^n$ to the $n-1$-skeleton and hence for every n-cell we get a copy of $\mathbb{S}^n$ in the quotient. $\square$

Example 11.18. Consider the hollow cube W^2 as a cell complex. Then $W^2/W^1 \cong \bigvee_{i=1}^6 \mathbb{S}^2$ and $W^1/W^0 \cong \bigvee_{i=1}^{12} \mathbb{S}^1$.

The following is a key fact about the topology of CW complexes, that I won't prove:

Lemma 11.19. *Let X be a CW complex. Then (X, A) is a good pair for any subcomplex $A \subset X$. In particular, for each skeleton (X^n, X^{n-1}) is a good pair. Recall that this means A has a neighbourhood in X which deformation retracts onto A.*

Proof. Proposition A.5 in [Hatcher]. $\square$

Remark 11.20. CW complexes are nice topological spaces in the following sense: They are normal (and thus Hausdorff), locally contractible, locally path-connected and paracompact. This is all shown in Appendix A of [Hatcher].

12. Cellular homology

In the following, X will always be a CW complex.

Lemma 12.1. *For all $q \neq n \geqslant 1$, $H_q(X^n, X^{n-1}) = 0$. For $q = n$ $H_q(X^n, X^{n-1})$ is a free abelian group with one generator of each n-cell of X.*

Proof. By Lemma 11.19 we may use Proposition 8.4 to compute relative homology via the quotient, which is determined by Lemma 11.17 and Proposition 8.7:
$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{I^n} \tilde{H}_q(\mathbb{S}^n). \qquad \square$$

Lemma 12.2. *Consider the inclusion $i^n \colon X^n \to X$ and let $q \leqslant n$.*

(a) The induced map $i_*^q \colon H_q(X^n) \to H_q(X)$ is surjective.

(b) On the $(n+1)$-skeleton we get an isomorphism

$$i^q_*\colon H_q(X^{n+1}) \cong H_q(X).$$

Proof. (a) We can factor i^n as

$$\begin{tikzcd}
X^n \arrow[rrr, "i^n"] \arrow[d, "\alpha_1"] & & & X \\
X^{n+1} \arrow[urrr, "i^{n+1}"] \arrow[r, "\alpha_2"'] & X^{n+2} \arrow[urr, "i^{n+2}"] \arrow[r, "\alpha_3"'] & X^{n+3} \arrow[ur, "i^{n+3}"] \arrow[r, "\alpha_4"'] & \ldots
\end{tikzcd}$$

The map $H_q(\alpha_1)\colon H_q(X^n) \to H_q(X^{n+1})$ is surjective, because $H_q(X^{n+1}, X^n) = 0$. For $i > 1$ we have the following piece of the long exact sequence of the pair (X^{n+i}, X^{n+i-1})

$$0 \cong H_{q+1}(X^{n+i}, X^{n+i-1}) \longrightarrow H_q(X^{n+i-1}) \xrightarrow{H_q(\alpha_i)} H_q(X^{n+i}) \longrightarrow H_q(X^{n+i}, X^{n+i-1}) \cong 0.$$

Therefore $H_q(\alpha_i)$ is an isomorphism in this range. If X is finite-dimensional, this already proves the claim.

Every singular simplex in X has an image that is contained in one of the X^n because the standard simplices are compact. If $a \in S_q(X)$ is a chain, $a = \sum_{i=1}^m \lambda_i \beta_i$ then we can find an M such that the images of all the β_i's are contained in X^M, say for $M = n + k$. Therefore every $[a] \in H_q(X)$ can be written as $i^M[b]$, but $\alpha_k \circ \ldots \circ \alpha_1$ is surjective, hence $[b]$ is of the form $\alpha_k \circ \ldots \circ \alpha_1[c]$ but then

$$[a] = i^M \circ \alpha_k \circ \ldots \circ \alpha_1[c] = i^q[c]$$

thus i^q is surjective.

(b) If $[a] = i^{n+1}_*[u] = 0$, then we have $a = dc$ and as c can be defined on some M-skeleton of X as in (a) we have $c = i^M c'$ and $a = i^M \circ \alpha_q \circ \ldots \circ \alpha_2[u]$ where $\alpha_q \circ \ldots \circ \alpha_2[u] = dc' = [0]$. As the α_i are injective $[u] = 0$ also and i^n_* is injective.

□

Corollary 12.3. *For CW complexes X, Y we have*

(a) If the n-skeleta X^n and Y^n are homeomorphic, then $H_q(X) \cong H_q(Y)$, for all $q < n$.
(b) If X has no q-cells, then $H_q(X) \cong 0$.
(c) In particular, if q exceeds the dimension of X, then $H_q(X) \cong 0$.

Proof. The first claim is a direct consequence of the lemma above.

By assumption in (b) $X^{q-1} = X^q$, therefore we have $H_q(X^{q-1}) \cong H_q(X^q)$ and the latter surjects onto $H_q(X)$. We show that $H_n(X^r) \cong 0$ for $n > r$. To that end we use the chain of isomorphisms

$$H_n(X^r) \cong H_n(X^{r-1}) \cong \ldots \cong H_n(X^0)$$

which holds because the adjacent relative groups $H_n(X^i, X^{i-1})$ are trivial for $i < n$. □

Again, X is a CW complex.

Definition 12.4. The *cellular chain complex* $C_*(X)$ consists of $C_n(X) := H_n(X^n, X^{n-1})$ with boundary operator

$$d\colon H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is the map induced by the projection map $S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2})$.

We have observed that $C_n(X)$ is a free abelian group with

$$C_n(X) \cong \bigoplus_{I^n} \tilde{H}_n(\mathbb{S}^n) \cong \bigoplus_{I^n} \mathbb{Z}.$$

For $n < 0$, $C_n(X)$ is trivial. If X has only finitely many n-cells, then $C_n(X)$ is finitely generated. If X has finitely many n-cells and $(n-1)$-cells the boundary operator d_n can be calculated using matrices over the integers. We will soon analyze it.

Let us first check that our definition is right.

Lemma 12.5. *The map d is a boundary operator.*

Proof. The composition d^2 is $\varrho \circ \delta \circ \varrho \circ \delta$, but $\delta \circ \varrho$ is a composition in an exact sequence, the homology exact sequence of the pair (X^{n-1}, X^{n-2}). $\square$

Theorem 12.6 (Comparison of cellular and singular homology). *For every CW complex X, there is an isomorphism $\Upsilon\colon H_*(C_*(X), d) \cong H_*(X)$.*

Proof. Consider the diagram

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, equal] \arrow[dd, "d"] & H_{n+1}(X^{n+1}, X^n) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_{n+1}(X, X^n) \arrow[r, "\delta'"] & H_n(X^n) \arrow[r, "i_*^n"] \arrow[dl, "\varrho"] & H_n(X) \\
C_n(X) \arrow[r, equal] \arrow[dd, "d"] & H_n(X^n, X^{n-1}) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_n(X, X^{n-1}) \arrow[r, "\delta'"] & H_{n-1}(X^{n-1}) \arrow[r, "i_*^{n-1}"] \arrow[dl, "\varrho"] & H_{n-1}(X) \\
C_{n-1}(X) \arrow[r, equal] \arrow[dd, "d"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_{n-1}(X, X^{n-1}) \arrow[r, "\delta'"] & H_{n-2}(X^{n-2}) \arrow[r, "i_*^{n-2}"] \arrow[dl, "\varrho"] & H_{n-2}(X) \\
\ldots & \ldots & &
\end{tikzcd}$$

We now make the following series of observations:

- All occurring ϱ-maps are injective because $H_k(X^{k-1}) \cong 0$ for all k.
- For every $a \in H_n(X^n)$ $\varrho(a)$ is a cycle for d:

$$d\varrho(a) = \varrho\delta\varrho(a) = 0.$$

- Let $c \in C_n(X)$ be a d-cycle, thus $0 = dc = \varrho\delta c$ and as ϱ is injective we obtain $\delta c = 0$. Exactness for the homology of the pair (X^n, X^{n-1}) yields that $c = \varrho(a)$ for an $a \in H_n(X^n)$. Hence,

$$H_n(X^n) \cong \ker(d\colon C_n(X) \to C_{n-1}(X)).$$

- We define $\Upsilon\colon \ker(d) \to H_n(X)$ as $\Upsilon[c] = i_*^n(a)$ for $c = \varrho(a)$ and $i_*^n\colon H_n(X^n) \to H_n(X)$.
- The map Υ is surjective because i_*^n is surjective.
- In the diagram, the triangles commute, i.e. $\delta = \delta' \circ \lambda$ by naturality of the boundary map.
- The sequence

  $$H_{n+1}(X^{n+1}) \twoheadrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, X^{n+1}) \longrightarrow H_n(X^{n+1}) \xrightarrow{\cong} H_n(X)$$

  tells us that $H_{n+1}(X, X^{n+1}) = 0$ and this in turn implies that λ is surjective.
- Using this we obtain

  $$\operatorname{im}(\delta) = \operatorname{im}(\delta') = \ker(i_*^n).$$

  As $d = \varrho \circ \delta$ and ρ is injective, the map ϱ induces an isomorphism between the image of d and the image of δ.
- Thus we have determined both the kernel and the image of d in terms of expressions on the right of our diagram. Taking quotients ϱ induces an isomorphism

  $$\frac{\ker(d\colon C_n(X) \to C_{n-1}(X))}{\operatorname{im}(d\colon C_{n+1}(X) \to C_n(X))} \cong \frac{H_n(X^n)}{\ker(i_*^n)}$$

  But the exact sequence

  $$0 \longrightarrow \ker(i_*^n) \longrightarrow H_n(X^n) \longrightarrow \operatorname{im}(i_*^n) \longrightarrow 0$$

  gives us

  $$H_n(X^n)/\ker(i_*^n) \cong \operatorname{im}(i_*^n) \cong H_n(X).$$ □

It is clear from the definition that any cellular map $f : X \to Y$ induces a map f_* of cellular homology $H_*(C_*(X), d) \to H_*(C_*(Y), d)$.

Lemma 12.7. *The isomorphism in Theorem 12.6 is natural, i.e.* $\Upsilon \circ f_* = f_* \circ \Upsilon$.

Proof. Observe that every map in the large diagram in the proof of Theorem 12.6 is natural. □

To use cellular homology we next need to be able to compute d. We've already observed that the (closed) n-cells give a natural basis of $C_n(X) = H_n(X^n, X^{n-1})$ and the (closed) $(n-1)$-cells give a basis of $C_{n-1}(X)$. So the question is what happens to an n-cell under d.

We consider the following diagram:

$$\begin{tikzcd}
\widetilde{H}_n(\mathbb{S}^n) \arrow[r, "\cong"] & H_n(\mathbb{D}^n, \partial\mathbb{D}^n) \arrow[r, "\delta"] \arrow[d, "(Q_i)_*"] & \widetilde{H}_{n-1}(\mathbb{S}^{n-1}) \arrow[d, "(q_i)_*"] & & \\
& H_n(X^n, X^{n-1}) \arrow[r, "\delta"] & H_{n-1}(X^{n-1}) \arrow[r, "\rho"] \arrow[rr, bend right, "(p_j)_*"'] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\pi_j"] & \widetilde{H}_{n-1}(\mathbb{S}^{n-1})
\end{tikzcd}$$

Here Q_i is the canonical map of pairs from from the i-th n-cell (and its boundary) to (X^n, X^{n-1}). The map π_j is projection onto the j-th factor. Geometrically we may describe the map p_j as projection onto the j-th $(n-1)$-cell (i.e. we collapse the $n-2$-skeleton and all other $(n-1)$-cells to a point and are left witth one copy of S^{n-1}). The square in the middle commutes by naturality of the connecting homomorphism δ.

The generator μ_n for $\widetilde{H}_n(\mathbb{S}^n)$ is sent by Q_i to one of the generators of $C_n(X)$, and the image under $\rho\delta$ may be computed as $(q_i)_* \circ \delta(\mu_n) = (q_i)_*(\mu_{n-1})$. Projecting to the j-th $(n-1)$-cell gives $(p_j \circ j_i)_*(\mu_{n-1})$.

Thus the (i,j)-component of the boundary map $d : \oplus_{I^n}\mathbb{Z} \to \oplus_{I^{n-1}}\mathbb{Z}$ is the degree of $p_j \circ q_i$.

As we compute d on the boundary of the cell representing the n-th homology it is indeed a boundary operator in the topological sense and does provide a nice conceptual description of homology.

Example 12.8. We compute the homology of projective Spaces.

Let K be the reals $\mathbb{R}$, complex numbers $\mathbb{C}$ or quoternions $\mathbb{H}$ with $m := \dim_{\mathbb{R}}(K)$ and let $K^* = K \setminus \{0\}$. We let K^* act on K^{n+1} via

$$K^* \times K^{n+1} \setminus \{0\} \to K^{n+1} \setminus \{0\}, \quad (\lambda, v) \mapsto \lambda v.$$

We define $KP^n = (K^{n+1} \setminus \{0\})/K^*$ (with the quotient topology) and we denote the equivalence class of $(x_0, \ldots, x_n)$ in KP^n by $[x_0 : \ldots : x_n]$.

We define a filtration by

$$X^{mi} := \{[x_0 : \ldots : x_n] \mid x_{i+1} = \ldots = x_n = 0\}$$

and note that $X^{mi} \cong KP^i$. We see that $\{[x_0 : \ldots : x_n] \mid x_i \neq 0, x_{i+1} = \ldots = x_n = 0\}$ is an open mi-cell.

An explicit characteristic map is $Q_i : \mathbb{D}^{mi} \to KP^n$ given by $(y_0, \ldots, y_{i-1}) \mapsto [y_0 : \cdots : y_{i-1} : 1 - ||y|| : 0 : \cdots : 0]$.

Thus attachment map $\partial\mathbb{D}^{mi} \to X^{m(i-1)}$ is given by the composition $\mathbb{S}^{mi-1} \to K^i \setminus \{0\} \to KP^{i-1} \cong X^{m(i-1)}$.

Here the map from the sphhere to projective space is well known in some examples: It specializes to the 2:1 map $\mathbb{S}^{i-1} \to \mathbb{R}P^{i-1}$ if $K = \mathbb{R}$ and to the quotient map by the $U(1)$ action from $\mathbb{S}^{2i-1} \to \mathcal{C}P^{i-1}$ if $K = \mathcal{C}$.. (The case $i = 2$ is the Hopf fibration.)

(a) First we consider the case $K = \mathbb{C}$. Here, we have a cell in each even dimension $0, 2, 4, \ldots, 2n$ for $\mathbb{C}P^n$. Therefore the cellular chain complex is

$$C_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k = 2i, 0 \leqslant i \leqslant n, \\ 0 & k = 2i-1 \text{ or } k > 2n. \end{cases}$$

The boundary operator is zero in each degree (as it always has source or target equal to 0) and thus

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & k = 2i, 0 \leqslant k \leqslant 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) The case of the quaternions is similar. Here the cells are spread in degrees congruent to zero modulo four, thus

$$H_k(\mathbb{H}P^n) = \begin{cases} \mathbb{Z}, & k = 4i, 0 \leqslant k \leqslant 4n, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Non-trivial boundary operators occur in the case of the real numbers. Here, we have a cell in each dimension up to n and thus the homology of $\mathbb{R}P^n$ is the homology of the chain complex

$$0 \longrightarrow C_n \cong \mathbb{Z} \xrightarrow{d} C_{n-1} \cong \mathbb{Z} \xrightarrow{d} \ldots \xrightarrow{d} C_0 \cong \mathbb{Z}.$$

For the computation of d_n we have to compute the degree of $\phi := p \circ q$ in the diagram $\mathbb{S}^{n-1} \xrightarrow{q} \mathbb{R}P^{n-1} \xrightarrow{p} \mathbb{S}^{n-1}$ where q is the canonical quotient map and p is obtained by collapsing the subcomplex $\mathbb{R}P^{n-2}$ to a point.

In coordinates we send $(x_1, \ldots, x_n)$ to $[x_1 : x_2 \cdots : x_n]$ where we moreover identify all points with $x_n = 0$. The point $[e_n]$ has thus preimage e_n and $-e_n$ and we may use the local formula for degrees: In the neighbourhood $\{x_n > 0\}$ of e_n the map ϕ is a local homeomorphism so we must have $\deg(\phi)_{e_n} = \pm 1$. As the sign of d will be irrelevant for our computations we just assume the degree is $+1$. (Or we check it is indeed $+1$.) But $\phi|_{x_n>0} = \phi|_{x_n<0} \circ A$ thus $\deg(\phi)|_{-e_n} = \deg(\phi)|_{e_n} \deg(A) = (-1)^n$.

Together we have $\deg(\phi) = \deg(\mathrm{id}) + \deg(A) = 1 + (-1)^n$.

Thus $d[e_i] = 2[e_{i-1}]$ if i is even and 0 if i is odd.

Thus, depending on n we compute

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k \leqslant n, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

for n even.

For odd dimensions n we get

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathbb{R}P^1 \cong \mathbb{S}^1$ and $\mathbb{R}P^3 \cong SO(3)$.

13. Homology with coefficients

Let G be an arbitrary abelian group.

Definition 13.1. The *singular chain complex of a topological space X with coefficients in G*, $S_*(X;G)$, has as elements in $S_n(X;G)$ finite sums of the form $\sum_{i=1}^N g_i\alpha_i$ with g_i in G and $\alpha_i \colon \Delta^n \to X$. Addition in $S_n(X;G)$ is given by

$$\sum_{i=1}^N g_i\alpha_i + \sum_{i=1}^N h_i\alpha_i = \sum_{i=1}^N (g_i + h_i)\alpha_i.$$

The *nth (singular) homology group of X with coefficients in G* is

$$H_n(X;G) := H_n(S_*(X;G))$$

where the boundary operator $\partial \colon S_n(X;G) \to S_{n-1}(X;G)$ is given by

$$\partial(\sum_{i=1}^N g_i\alpha_i) = \sum_{j=0}^n (-1)^j (\sum_{i=1}^N g_i(\alpha_i \circ d_j)).$$

We use a similar definition for *cellular homology of a CW complex X with coefficients in G*. Recall, that $C_n(X) = H_n(X^n, X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \mathbb{Z}$.

Definition 13.2. We define $C_n(X; G) = \bigoplus_{\sigma \text{ an } n\text{-cell}} G$. On $c \in C_n(X; G)$ written as $c = \sum_{i=1}^N g_i \sigma_i$ we define the boundary operator $\tilde{d}$ by $\tilde{d}c = \sum_{i=1}^N g_i d(\sigma_i)$ where $d\colon C_n(X) \to C_{n-1}(X)$ is the boundary in the cellular chain complex of X.

We can transfer Theorem 12.6 (and every other general theorem we have proven, like Excision and Mayer-Vietoris) to the case of homology with coefficients:

$$H_n(X; G) \cong H_n(C_*(X; G), \tilde{d})$$

for every CW complex X and therefore we denote the latter by $H_n(X; G)$ as well.

Note, that $H_n(X; \mathbb{Z}) = H_n(X)$ for every space X.

Example 13.3. If we consider the case $X = \mathbb{R}P^2$, then we see that coefficients really make a difference. Thus while theorems translate, computations have to be re-checked.

Recall that for $G = \mathbb{Z}$ we had that $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$, $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(\mathbb{R}P^2) = 0$. However, for $G = \mathbb{Z}/2\mathbb{Z}$ the cellular chain complex looks as follows:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2=0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and therefore $H_i(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for $0 \leqslant i \leqslant 2$.

If we consider $H_*(\mathbb{R}P^2; \mathbb{Q})$ we obtain the cellular complex

$$0 \longrightarrow \mathbb{Q} \xrightarrow{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \longrightarrow 0$$

But here, multiplication by 2 is an isomorphism and we get $H_0(\mathbb{R}P^2; \mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{R}P^2; \mathbb{Q}) = \mathbb{Q}/2\mathbb{Q} = 0$ and $H_2(\mathbb{R}P^2; \mathbb{Q}) = 0$.

Thus we see that homology with coefficients can be very different from the homology with integer coefficients we first met.

However, somewhat surprisingly, $H_*(X, G)$ is computable from $H_*(X)$ and G. But we need some basics from algebra to see that.

Let A and B be abelian groups.

Definition 13.4. The *tensor product of A and B*, $A \otimes B$, is the quotient of the free abelian group generated by $A \times B$ by the subgroup generated by

(a) $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$,
(b) $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$

for $a_1, a_1, a \in A$ and $b_1, b_2, b \in B$.

We denote an equivalence class of (a, b) in $A \otimes B$ by $a \otimes b$.

Note, that relations (a) and (b) imply that $\lambda(a \otimes b) = (\lambda a) \otimes b = a \otimes (\lambda b)$ for any integer $\lambda \in \mathbb{Z}$ and $a \in A$, $b \in B$. Elements in $A \otimes B$ are finite sums of equivalence classes $\sum_{i=1}^n \lambda_i a_i \otimes b_i$.

- Of course, $A \otimes B$ is generated by $a \otimes b$ with $a \in A$, $b \in B$.

- The tensor product is symmetric up to isomorphism and the isomorphism $A \otimes B \cong B \otimes A$ is given by
$$\sum_{i=1}^{n} \lambda_i a_i \otimes b_i \mapsto \sum_{i=1}^{n} \lambda_i b_i \otimes a_i.$$
- It is associative up to isomorphism:
$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$
for all abelian groups A, B, C.
- For homomorphisms $f\colon A \to A'$ and $g\colon B \to B'$ we get an induced homomorphism
$$f \otimes g\colon A \otimes B \to A' \otimes B'$$
which is given by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ on generators.
- In particular we may tensor a chain complex C_* with an abelian group G by defining $(C \otimes G)_n = C_n \otimes G$ and setting the differential to be $d \otimes \mathrm{id}$. We've already seen this tensor product: $S_n(X) \otimes G$ is isomorphic to $S_n(X, G)$.

Remark 13.5. The tensor product has the following universal property. For abelian groups A, B, C, the bilinear maps from $A \times B$ to C are in bijection with the linear maps from $A \otimes B$ to C.

There is another closely related universal property. For two abelian groups A, B the set of homomorphisms has a natural structure of abelian group by pointwise addition. Denoting this abelian group by $\underline{\mathrm{Hom}}$ we have
$$\mathrm{Hom}_{\mathsf{Ab}}(A \otimes B, C) = \mathrm{Hom}_{\mathsf{Ab}}(A, \underline{\mathrm{Hom}}(B, C))$$

We collect the following properties of tensor products:

(a) For every abelian group A, we have
$$A \otimes \mathbb{Z} \cong A \cong \mathbb{Z} \otimes A.$$
(b) For every abelian group A, we have
$$A \otimes \mathbb{Z}/n\mathbb{Z} \cong A/nA.$$
Here, note that $nA = \{na \mid a \in A\}$ makes sense in any abelian group. The isomorphism above is given by
$$a \otimes \bar{i} \mapsto \bar{ia}$$
where $\bar{i}$ denotes an equivalence class of $i \in \mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$ and $\bar{ia}$ the class of $ia \in A$ in A/nA.
(c) If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is a short exact sequence, then in general,
$$0 \longrightarrow A \otimes D \xrightarrow{\alpha \otimes \mathrm{id}} B \otimes D \xrightarrow{\beta \otimes \mathrm{id}} C \otimes \mathrm{id} \longrightarrow 0$$
is not exact for D abelian. For example,
$$0 \to \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \to 0$$
is exact, but
$$0 \to \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to 0$$

isn't, because $\mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \cong 0$.

When tensoring complexes with G it is often interesting to ask when a complex stays exact.

Lemma 13.6. *For every abelian group D, $(-)\otimes D$ is right exact,* i.e., *if* $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is a short exact sequence, then*

$$A \otimes D \xrightarrow{\alpha\otimes\mathrm{id}} B \otimes D \xrightarrow{\beta\otimes\mathrm{id}} C \otimes D \longrightarrow 0$$

is exact. If the exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is a split short exact sequence, then*

$$0 \longrightarrow A \otimes D \xrightarrow{\alpha\otimes\mathrm{id}} B \otimes D \xrightarrow{\beta\otimes\mathrm{id}} C \otimes D \longrightarrow 0$$

is exact.

Proof. It is easy to check surjectivity of $\beta \otimes \mathrm{id}$: It is enough to show that $c \otimes d$ is in the image, so just $b \in \beta^{-1}(c)$ and by definition $\beta \otimes \mathrm{id}(b \otimes d) = c \otimes d$.

It is a non-trivial exercise to directly show that $-\otimes D$ is also exact in the middle. Instead we can use some abstract machinery (feel free to ignore this if you haven't seen the categorical tools before). We need to show that $\ker(\beta \otimes \mathrm{id}) = \mathrm{im}(\alpha \otimes \mathrm{id})$. That means we want to show $B \otimes D/\mathrm{im}(A \otimes D) \cong C \otimes D$.

The left hand side is a colimit in abelian groups of the diagram $(0, \alpha\otimes\mathrm{id}) : A\otimes D \rightrightarrows B\otimes D$.

By the universal property in Remark 13.5 we have that $-\otimes D$ commutes with all colimits, as it follows directly from unravelling definitions that

$$\begin{aligned} \mathrm{Hom}(\mathrm{colim}(A_i \otimes D), B) &\cong \lim \mathrm{Hom}(A_i \otimes D, B) \cong \lim \mathrm{Hom}(A_i, \underline{\mathrm{Hom}}(D, B)) \\ &\cong \mathrm{Hom}(\mathrm{colim}\, A_i, \underline{\mathrm{Hom}}(D, B)) \cong \mathrm{Hom}(\mathrm{colim}\, A_i \otimes D, B) \end{aligned}$$

holds for all B. Thus maps out of $\mathrm{colim}\, A_i \otimes D$ agree with maps out of $\mathrm{colim}(A_i \otimes D)$. As everything is natural under the colimit diagram this implies that $\mathrm{colim}\, A_i \otimes D$ is a colimit of the diagram $A_i \otimes D$. Alternatively it follows from the Yoneda lemma that the two expressions agree.

The second part is left as an exercise. □

The failure of the functor $(-)\otimes D$ to be exact on the left hand side means that $H_n(X, G) = H_n(S_*(X) \otimes G)$ is not always isomorphic to $H_n(X) \otimes G = H_n(S_*(X)) \otimes G$.

Definition 13.7. Let A be an abelian group. A short exact sequence $0 \to F_1 \longrightarrow F_0 \longrightarrow A \to 0$ with F_0 and F_1 free abelian groups is called a *free resolution of* A.

Note that whenever F_0 is free then F_1 is automatically free abelian because it can be identified with a subgroup of F_0, recalling from algebra that a subgroup of a free abelian group is free. (This is not true for modules over a general ring R!)

Here we may see $F_1 \to F_0$ as a chain complex with homology A concentrated in degree 0. We replace A by the complex with the same homology.

Example 13.8. For every $n \geqslant 1$, the sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$.

Proposition 13.9. *Every abelian group possesses a free resolution.*

The resolution that we will construct in the proof is called the *standard resolution of A*.

PROOF. Let F_0 be the free abelian group generated by the elements of the underlying set of A. We denote by y_a the basis element in F_0 corresponding to $a \in A$. Define a homomorphism

$$p\colon F_0 \to A,\ p\left(\sum_{a\in A} \lambda_a y_a\right) = \sum_{a\in A} \lambda_a a.$$

Here, $\lambda_a \in \mathbb{Z}$ and this integer is non-trivial for only finitely many $a \in A$. By construction, p is an epimorphism. We set F_1 to be the kernel of p and in that way obtain the desired free resolution of A. $\square$

DEFINITION 13.10. For two abelian groups A and B and for $0 \longrightarrow F_1 \xrightarrow{i} F_0 \longrightarrow A \longrightarrow 0$ the standard resolution of A we define

$$\mathrm{Tor}(A, B) := \ker(i \otimes \mathrm{id}\colon F_1 \otimes B \to F_0 \otimes B) = H_1(F_* \otimes B)).$$

Here we write $F_* \otimes B$ for the complex $F_1 \otimes B \xrightarrow{i\otimes \mathrm{id}} F_0 \otimes B$.

As $i \otimes \mathrm{id}$ doesn't have to be injective, thus $\mathrm{Tor}(A, B)$ need not be trivial.

We will show that we can calculate $\mathrm{Tor}(A, B)$ via an arbitrary free resolution of A. To that end we prove the following result.

PROPOSITION 13.11. *For every homomorphism $f\colon A \to B$ and for free resolutions $0 \longrightarrow F_1 \xrightarrow{i} F_0 \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow F_1' \xrightarrow{i'} F_0' \longrightarrow B \longrightarrow 0$ we have:*

(a) There is achain map $g : F_* \to F_*$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{i} & F_0 & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & F_1' & \xrightarrow{i'} & F_0' & \xrightarrow{p'} & B & \longrightarrow & 0 \end{array}$$

commutes.

Any two such chain maps are chain homotopic, i.e. if h_0, h_1 are also homomorphisms with this property, then there is an $\alpha\colon F_0 \to F_1'$ with $i' \circ \alpha = g_0 - h_0$ and $\alpha \circ i = g_1 - h_1$.

(b) For every abelian group D the map $g_1 \otimes \mathrm{id}$ induces a map $H_1(F_* \otimes D) \to H_1(F_*' \otimes D)$ that is independent of the choice of g. We denote this map by $\varphi(f, F, F')$.

(c) For a homomorphism $f'\colon B \to C$ the map $\varphi(f' \circ f, F, F'')$ is equal to the composition $\varphi(f', F', F'') \circ \varphi(f, F, F')$.

PROOF. For (a) let $\{x_i\}$ be a basis of F_0 and choose $y_i \in F_0'$ with $p'(y_i) = fp(x_i)$. We define $g_0\colon F_0 \to F_0'$ via $g_0(x_i) = y_i$. For every $r \in F_1$ we obtain $p' \circ g_0(i(r)) = f \circ p \circ i(r) = 0$ and therefore $g_0(i(r))$ is contained in the kernel of p' which is equal to the image of i'. As i' is injective we may define $g_1(r)$ as the unique preimage of $g(i(r))$ under i'.

For h and g as in (a) we get for $x \in F_0$ that $g_0(x) - h_0(x)$ is in the kernel of p' which is the image of the injection i'. Define α as $(i')^{-1}(h_0 - h_0)$. Then by construction $i'\alpha = g_0 - h_0$ and

$$i'(g_1 - h_1) = (g_0 - h_0)i = i'\alpha i.$$

As i' is injective, this yields $g_1 - h_1 = \alpha i$.

For (b) it is easy to see that $g \otimes \mathrm{id}$ defines a chain map and thus induces a map on H_1 and that $g \otimes \mathrm{id}$ is chain homotopic to $h \otimes \mathrm{id}$ via $\alpha \otimes \mathrm{id}$.

For (c) we note that the uniqueness in (b) implies (c). □

COROLLARY 13.12. *For every free resolution* $0 \longrightarrow F_1' \xrightarrow{i'} F_0' \longrightarrow A \longrightarrow 0$ *we get a unique isomorphism*

$$\varphi(\mathrm{id}_A, F', F)\colon \ker(i' \otimes \mathrm{id}) \to \mathrm{Tor}(A, D).$$

PROOF. By the proposition we obtain $\phi(\mathrm{id}_A, F, F')$ which is an inverse of $\phi(\mathrm{id}_A, F, F')$. □

Thus we can calculate $\mathrm{Tor}(A, D)$ with every free resolution of A.

EXAMPLE 13.13.
(a) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D) \cong \{d \in D \mid nd = 0\}$ for all $n \geqslant 1$. That's why Tor is sometimes called torsion product. For the calculation we use the resolution $0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$. By definition and by Corollary 13.12 we have
$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D) \cong \ker(n \otimes \mathrm{id}\colon \mathbb{Z} \otimes D \to \mathbb{Z} \otimes D).$$
As $\mathbb{Z} \otimes D \cong D$ and as $n \otimes \mathrm{id}$ induces the multiplication by n, we get the claim.
(b) From the first example we obtain $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\mathrm{gcd}(m, n)\mathbb{Z}$ because the n-torsion subgroup in $\mathbb{Z}/m\mathbb{Z}$ is $\mathbb{Z}/\mathrm{gcd}(m, n)\mathbb{Z}$.
(c) For A free abelian, $\mathrm{Tor}(A, D) \cong 0$ for arbitrary D. For this note that $0 \to 0 \to A = A \to 0$ is a free resolution of A and the kernel is a subgroup of $0 \otimes D = 0$ and hence trivial.
(d) For two abelian groups A_1, A_2, D there is an isomorphism
$$\mathrm{Tor}(A_1 \oplus A_2, D) \cong \mathrm{Tor}(A_1, D) \oplus \mathrm{Tor}(A_2, D).$$
If we have free resolutions
$$0 \to F_1^i \to F_0^i \to A_i \to 0$$
for $i = 1, 2$ then the direct sum is a free resolution of $A_1 \oplus A_2$ and
$$\ker((i_1 \oplus i_2) \otimes \mathrm{id}) = \ker(i_1 \otimes \mathrm{id}) \oplus \ker(i_2 \otimes \mathrm{id}).$$
It follows that tensoring with a free abelian group preserves exact sequences.

From Example (c) we get the following useful corallary:

LEMMA 13.14. *Let C_* be a chain complex and A a free abelian group. Then $H_n(C_* \otimes A) = H_n(C) \otimes A$.*

PROOF. The proof is left as an exercise. □

We can now state the following powerful theorem:

THEOREM 13.15 (Universal coefficient theorem). *For every space X there is a split short exact sequence*

$$0 \to H_n(X) \otimes G \to H_n(X; G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0,$$

and therefore we get an isomorphism

$$H_n(X; G) \cong H_n(X) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X), G).$$

The proof will need some further work in algebra.

EXAMPLE 13.16. For $X = \mathbb{R}P^2$ we obtain

$$H_n(\mathbb{R}P^2; G) \cong H_n(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_{n-1}(\mathbb{R}P^2), G)$$

thus

$$H_0(\mathbb{R}P^2; G) \cong H_0(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_{-1}(\mathbb{R}P^2), G) \cong G,$$
$$H_1(\mathbb{R}P^2; G) \cong H_1(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_0(\mathbb{R}P^2), G) \cong G/2G \oplus 0 \cong G/2G,$$

and

$$H_2(\mathbb{R}P^2; G) \cong H_2(\mathbb{R}P^2) \otimes G \oplus \mathrm{Tor}(H_1(\mathbb{R}P^2), G) \cong \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, G).$$

And this agrees with our earlier computations!

REMARK 13.17. Note that the splitting in the unvirsal coefficient theorem is not natural. This means for example that a map $f : X \to Y$ may induce the zero map on $H_n(X) \otimes G \to H_n(Y) \otimes G$ and on $\mathrm{Tor}(H_{n-1}(X), G) \to \mathrm{Tor}(H_{n-1}(Y), G)$ yet be nonzero on $H_n(-, G)$! (This situation is compatible with the short exact sequence being natural, but not the splitting being natural.)

For example consider the map $\mathbb{R}P^2 \to \mathbb{S}^2$ collapsing the 1-cell. It is non-trivial on homology with $\mathbb{Z}/2$ coefficients (as is apparent from cellular homology), yet on H_1 and H_2 with integer coefficients, and thus on the outer terms of the short exact sequence, it must induce the zero map.

14. Algebraic Künneth theorem

We extend the definition of tensor products to chain complexes.

DEFINITION 14.1. Are (C_*, d) and (C'_*, d') two chain complexes, then $(C_* \otimes C'_*, d_\otimes)$ is the chain complex with

$$(C_* \otimes C'_*)_n = \bigoplus_{p+q=n} C_p \otimes C'_q$$

and with $d_\otimes(c_p \otimes c'_q) = (dc_p) \otimes c'_q + (-1)^p c_p \otimes d'c'_q$.

Note the sign in the definition, which is needed to make $d_\otimes$ a differential:

LEMMA 14.2. *The map $d_\otimes$ is a differential.*

PROOF. The composition is

$$d_\otimes((dc_p) \otimes c'_q + (-1)^p c_p \otimes d'c'_q) = 0 + (-1)^{p-1}(dc_p) \otimes (d'c'_q) + (-1)^p (dc_p) \otimes (d'c'_q) + 0 = 0.$$

□

In particular the abelian group G may be viewed as a chain complex that is G in degree 0 and 0 in all other degrees. We will abuse notation and denote the chain complex and the abelian group by the same letter. Then for every chain complex (C_*, d) we recover our definition

$$(C_* \otimes G)_n = C_n \otimes G, \quad d_\otimes = d \otimes \mathrm{id}.$$

In particular, for every topological space X,

$$S_*(X) \otimes G \cong S_*(X, G).$$

Similarly, for a CW complex X we get $C_*(X;G) = C_*(X) \otimes G$.

For every pair of spaces (X, A) we have

$$S_*(X, A; G) := S_*(X, A) \otimes G.$$

As tensoring with G is right exact this is equivalent to defining it as the quotient of $S_*(X;G)$ by $S_*(A;G)$

A map $f\colon (C_*, d_C) \to (D_*, d_D)$ induces a map of chain complexes

$$f \otimes \mathrm{id}\colon C_* \otimes C'_* \to D_* \otimes C'_*.$$

In particular, for every continuous (cellular) map we get induced maps on singular (cellular) homology with coefficients.

We may similarly define $f \otimes g : C \otimes C' \to D \otimes D'$ for $f : C \to C'$, $g : D \to D'$ by sending $c \otimes c'$ to $f(c) \otimes g(c')$.

Definition 14.3. A chain complex C_* is called *free*, if C_n is a free abelian group for all $n \in \mathbb{Z}$.

The complexes $S_*(X, A)$ and $C_*(X)$ are free.

Theorem 14.4 (Universal coefficient theorem (algebraic version)). *Let C_* be a free chain complex and G an abelian group, then for all $n \in \mathbb{Z}$ we have a split short exact sequence*

$$0 \to H_n(C_*) \otimes G \to H_n(C_* \otimes G) \to \mathrm{Tor}(H_{n-1}(C_*), G) \to 0,$$

in particular

$$H_n(C_* \otimes G) \cong H_n(C_*) \otimes G \oplus \mathrm{Tor}(H_{n-1}(C_*), G).$$

Unravelling the definitions we can deduce the topological universal coefficient theorem form the algebraic version.

The algebraic universal coefficient theorems itself is a corollary of the following more general statement.

Theorem 14.5. *(Künneth formula) For a free chain complex C_* and a chain complex C'_* we have the following split exact sequence for every integer n*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \xrightarrow{\lambda} H_n(C_* \otimes C'_*) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)) \longrightarrow 0,$$

i.e.*,*

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)).$$

The map $\lambda\colon \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \to H_n(C_* \otimes C'_*)$ in the theorem is given on the (p, q)-summand by

$$\lambda([c_p] \otimes [c'_q]) := [c_p \otimes c'_q]$$

for $c_p \in C_p$ and $c'_q \in C'_q$. By the definition of the tensor product of complexes, this map is well-defined.

Proof of Theorems 13.15 and 14.4. To recover the algebraic universal coefficient theorem we just set $C'_* = G$. To recover the topological version we set $C_* = S_*(X)$, which is free by definition. □

LEMMA 14.6. *Let $0 \to A \to B \xrightarrow{g} C \to 0$ be a short exact sequence where C is free. Then the short exact sequence is split.*

PROOF. By Lemma 5.3 it suffices to provide a right inverse r of $g : B \to C$. But as C is free we may just pick a basis $\{c\}$ of C, let $r(c)$ to be an arbitrary element of $g^{-1}(c)$ for each c and extend to all of C. $\square$

LEMMA 14.7. *For any free chain complex C_* with trivial differential and an arbitrary chain complex, C'_*, λ is an isomorphism*

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \cong H_n(C_* \otimes C'_*).$$

PROOF. We note $C_* \cong \bigoplus C_p[-p]$ where $C_p[-p]$ denotes the chain complex which is C_p in degree p and 0 otherwise.

It is easy to show from the definition of the tensor product that it commutes with direct sums. As homology also commutes with direct sums we find $H_n(C \otimes C') = H_n((\oplus_p C_p[-p]) \otimes C'_*) \cong \oplus_p H_n(C_p[-p] \otimes C'_*)$.

As C_p is free we have $H_n(C_p[-p] \otimes C'_*) \cong C_p \otimes H_{n-p}(C'_*)$ by Lemma 13.14 and this completes the proof. $\square$

PROOF OF THEOREM 14.5. We abbreviate the subgroup of cycles in C'_q with Z'_q and the subgroup of boundaries in C'_q with B'_q and use analogous abbreviations for C_*. As C_p is free so are the subgroups Z_p and B_p.

We consider the short exact sequence $0 \to Z_p \longrightarrow C_p \longrightarrow B_{p-1} \to 0$ and tensor it with C'_q and sum over $p + q = n$. Since B_{p-1} is free, the original sequence is split by Lemma 14.6 and hence the resulting sequence is exact by Lemma 13.6.

We define two free chain complexes Z_* and D_* via

$$(Z_*)_p = Z_p, (D_*)_p = B_{p-1}$$

with trivial differential.

Collecting our short exact sequences for all values of n we obtain a short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{p+q=n} Z_p \otimes C'_q \arrow[r] \arrow[d, "(-1)^p \mathrm{id}\otimes d'"] & \bigoplus_{p+q=n} C_p \otimes C'_q \arrow[r, "d\otimes\mathrm{id}"] \arrow[d, "d\otimes\mathrm{id}+(-1)^p\mathrm{id}\otimes d'"] & \bigoplus_{p+q=n} B_{p-1} \otimes C'_q \arrow[r] \arrow[d, "(-1)^p\mathrm{id}\otimes d'"] & 0 \\
0 \arrow[r] & \bigoplus_{p+q=n-1} Z_p \otimes C'_q \arrow[r] & \bigoplus_{p+q=n-1} C_p \otimes C'_q \arrow[r, "d\otimes\mathrm{id}"'] & \bigoplus_{p+q=n-1} B_{p-1} \otimes C'_q \arrow[r] & 0
\end{tikzcd}$$

We have to verify that the two squares commute. This is clear for the left one and a quick computation for the right one. Note that as B_{p-1} is the degree p part of D we do indeed have the sign $(-1)^p$ in front of the rightmost differential.

This gives a long exact sequence

$$\ldots \longrightarrow H_{n+1}(D_* \otimes C'_*) \xrightarrow{\delta_{n+1}} H_n(Z_* \otimes C'_*) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow H_n(D_* \otimes C'_*) \xrightarrow{\delta_n} H_{n-1}(Z_* \otimes C'_*) \longrightarrow \ldots$$

As Z_* and D_* satisfy the conditions of Lemma 14.7 we get a description of $H_*(D_* \otimes C'_*)$ and $H_*(Z_* \otimes C'_*)$ and therefore we can consider δ_{n+1} as a map

$$\begin{array}{c} \bigoplus_{p+q=n+1} H_p(D_*) \otimes H_q(C'_*) = \bigoplus_{p+q=n+1} B_{p-1} \otimes H_q(C'_*) \\ \big\downarrow {\scriptstyle j\otimes \mathrm{id}} \\ \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) = \bigoplus_{p+q=n} H_p(Z_*) \otimes H_q(C'_*) \end{array}$$

which is just induced by the inclusion $j\colon B_p \hookrightarrow Z_p$ (unravelling the definition of the boundary map). We can cut the long exact sequence in homology into short exact pieces and obtain that

$$0 \to \operatorname{coker}(\delta_{n+1}) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow \ker(\delta_n) \to 0$$

is exact. The cokernel of δ_{n+1} is isomorphic to $\bigoplus_{p+q=n}(Z_p/B_p) \otimes H_q(C'_*)$ because the tensor functor is right exact, thus

$$\operatorname{coker}(\delta_{n+1}) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

As $0 \to B_p \longrightarrow Z_p \longrightarrow H_p(C_*) \to 0$ is a free resolution of $H_p(C_*)$ we obtain that

$$\operatorname{Tor}(H_p(C_*), H_q(C'_*)) \cong \ker(j \otimes \mathrm{id}\colon B_p \otimes H_q(C'_*) \to Z_p \otimes H_q(C'_*))$$

and therefore

$$\ker(\delta_n) \cong \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(C_*), H_q(C'_*))$$

noting that the kernel of δ_n is a subspace of $\bigoplus_{p+q=n} B_{p-1} \otimes H_q(C')$ and relabelling indices. This proves the exactness of the Künneth sequence.

We will prove that the Künneth sequence is split in the case where both chain complexes, C_* and C'_*, are free. In that case the sequences

$$0 \to Z_p \to C_p \to B_{p-1} \to 0, \quad 0 \to Z'_q \to C'_q \to B'_{q-1} \to 0$$

are split by Lemma 14.6 and we denote by $r\colon C_p \to Z_p$ and $r'\colon C'_q \to Z'_q$ chosen retractions. Consider the two compositions

$$C_p \xrightarrow{r} Z_p \twoheadrightarrow H_p(C_*), \quad C'_q \xrightarrow{r'} Z'_q \twoheadrightarrow H_q(C'_*)$$

and view $H_*(C_*)$ and $H_*(C'_*)$ as chain complexes with trivial differential. Then these compositions yield a chain map

$$r \otimes r'\colon C_* \otimes C'_* \to H_*(C_*) \otimes H_*(C'_*)$$

This is indeed a chain map as the diagram

$$\begin{tikzcd} C_p \arrow[r, "r"] \arrow[d, "d"] & Z_p \arrow[r] & H_p \arrow[d, "0"] \\ C_{p-1} \arrow[r, "r"] & Z_{p-1} \arrow[r] & H_{p-1} \end{tikzcd}$$

commutes, which follows as r sends boundaries in C_p to boundaries in Z_p, which get sent to 0 in homology.

On homology we get

$$r \otimes r' : H_n(C_* \otimes C'_*) \longrightarrow H_n(H_*(C_*) \otimes H_*(C'_*)) = \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

This map gives the desired splitting; it is easy to check it is left inverse to λ. $\square$

In the cases we are interested in (singular or cellular chains), the complexes will be free.
As we have seen for the universal coefficient theorem the splitting of the Künneth sequence is *not* natural. We have chosen a splitting of the short exact sequences in the proof and usually, there is no canonical choice possible.

15. Künneth theorem in topology

What does the Künneth formula give for two topological spaces and their chain complexes? The Künneth sequence for $C_* = S_*(X)$ and $C'_* = S_*(Y)$ yields that

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(S_*(X) \otimes S_*(Y)) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0$$

is exact. But what is $H_n(S_*(X) \otimes S_*(Y))$? In the following we will show that this group is actually isomorphic to $H_n(X \times Y)$, thus the Künneth Theorem has some geometric content! First of all, we define a map.

LEMMA 15.1. *There is a homomorphism* $\times \colon S_p(X) \otimes S_q(Y) \longrightarrow S_{p+q}(X \times Y)$ *for all* $p, q \geqslant 0$ *with the following properties.*

(a) For all points $x_0 \in X$ viewed as zero chains

$$(x_0 \times \beta)(t_0, \ldots, t_q) = (x_0, \beta(t_0, \ldots, t_q))$$

for $\beta \colon \Delta^q \to Y$. Analogously, for all $y_0 \in Y$ and $\alpha \colon \Delta^p \to X$

$$(\alpha \times y_0)(t_0, \ldots, t_p) = (\alpha(t_0, \ldots, t_p), y_0).$$

(b) The map $\times$ is natural in X and Y, so for $f \colon X \to X'$ and $g \colon Y \to Y'$

$$S_{p+q}(f, g) \circ (\alpha \times \beta) = (S_p(f) \circ \alpha) \times (S_q(g) \circ \beta).$$

(c) The Leibniz rule holds

$$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta + (-1)^p \alpha \times \partial(\beta).$$

The map $\times$ is called the *homology cross product*.

PROOF. For p or q equal to zero, we define $\times$ as dictated by property (a). Therefore we can assume that $p, q \geqslant 1$ and induct on $p+q$. The method of proof that we will apply here is called *method of acyclic models* – you'll see why. Let $X = \Delta^p$, $Y = \Delta^q$, $\alpha = \mathrm{id}_{\Delta^p}$, and $\beta = \mathrm{id}_{\Delta^q}$. If $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}$ were already defined, then property (c) would force

$$\partial(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) = \partial(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^p \mathrm{id}_{\Delta^p} \times \partial(\mathrm{id}_{\Delta^q}) =: R \in S_{p+q-1}(\Delta^p \times \Delta^q).$$

For this element R (which is already defined) we get

$$\partial R = \partial^2(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^{p-1}\partial(\mathrm{id}_{\Delta^p}) \times \partial(\mathrm{id}_{\Delta^q}) + (-1)^p \partial(\mathrm{id}_{\Delta^p}) \times \partial(\mathrm{id}_{\Delta^q}) + (-1)^{2p-1}\mathrm{id}_{\Delta^p} \times \partial^2(\mathrm{id}_{\Delta^q}) = 0$$

so R is a cycle. But $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$ because $p+q-1 \geqslant 1$ and $\Delta^p \times \Delta^q$ is contractible and therefore $S_*(\Delta^p \times \Delta^q)$ has no homology. Thus R has to be a boundary, so there is a $c \in S_{p+q}(\Delta^p \times \Delta^q)$ with $\partial c = R$.

We fix such a c and define

$$\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} := c.$$

Now let X and Y be arbitrary spaces and $\alpha\colon \Delta^p \to X$, $\beta\colon \Delta^q \to Y$. Then $S_p(\alpha)(\mathrm{id}_{\Delta^p}) = \alpha$ and $S_q(\beta)(\mathrm{id}_{\Delta^q}) = \beta$ and therefore binaturality dictates

$$\alpha \times \beta = S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}) = S_{p+q}(\alpha, \beta)(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}).$$

By construction, this definition satisfies all desired properties. $\square$

Note that for spaces X, Y with trivial homology in positive degrees, the Künneth Theorem yields that $H_n(S_*(X) \otimes S_*(Y)) = 0$ for positive n.

Lemma 15.2. *Let C_* and C'_* be two chain complexes which are trivial in negative degrees and such that C_n is free abelian for all n and $H_nC'_* = 0$ for all positive n, then we have*

(a) Any two chain maps $f_*, g_*\colon C_* \to C'_*$ with $f_0 = g_0$ are chain homotopic.
(b) Is $f_0\colon C_0 \to C'_0$ a homomorphism with $f_0(\partial C_1) \subset \partial C'_1$ then there is a chain map $f_*\colon C_* \to C'_*$ extending f_0.

Proof. For (a) we will define a map $H_n\colon C_n \to C'_{n+1}$ for all $n \geqslant 0$ with $\partial H_n + H_{n-1}\partial = f_n - g_n$ inductively. For $n = 0$ we can take zero because $f_0 = g_0$ by assumption. Assume that we have H_k for $k \leqslant n-1$. Let $\{x_i\}$ be a basis of the free abelian group C_n and define

$$y_i := f_n(x_i) - g_n(x_i) - H_{n-1}\partial(x_i) \in C'_n.$$

Then

$$\begin{aligned}\partial y_i =& \partial f_n(x_i) - \partial g_n(x_i) - \partial H_{n-1}\partial(x_i)\\ =& \partial f_n(x_i) - \partial g_n(x_i) - H_{n-2}\partial^2(x_i) - f_{n-1}\partial(x_i) + g_{n-1}\partial(x_i)\\ =& 0.\end{aligned}$$

But C'_* is acyclic by assumption and therefore y_i has to be a boundary and we define $H_n(x_i) = z_i$ for some z satisfying $\partial z_i = y_i$. Then

$$(\partial H_n + H_{n-1}\partial)(x_i) = y_i + H_{n-1}\partial(x_i) = f_n(x_i) - g_n(x_i).$$

For (b) we define $f_n\colon C_n \to C'_n$ inductively with $\partial f_n = f_{n-1}\partial$. Assume that $\{x_i\}$ is a basis of C_n. Then $f_{n-1}\partial(x_i)$ is a cycle and thus there is a y_i with $\partial y_i = f_{n-1}\partial(x_i)$ due to the acyclicity of C'_*. We define $f_n(x_i)$ as y_i. Then

$$\partial f_n(x_i) = \partial y_i = f_{n-1}\partial(x_i). \qquad \square$$

Proposition 15.3. *Any two binatural chain maps $f_{X,Y}, g_{X,Y}$ from $S_*(X) \otimes S_*(Y)$ to $S_*(X \times Y)$ which agree in degree zero and send the zero chain $x_0 \otimes y_0 \in (S_*(X) \otimes S_*(Y))_0 = S_0(X) \otimes S_0(Y)$ to $(x_0, y_0) \in S_0(X \times Y)$ are chain homotopic.*

Here by $f_{X,Y}$ being a binatural chain map we mean that any pair of maps $f : X \to X'$ and $g : Y \to Y'$ we have a commutative diagram

$$\begin{tikzcd} S_*(X) \otimes S_*(Y) \arrow[r, "f_{X,Y}"] \arrow[d, "f_* \otimes g_*"] & S_*(X \times Y) \arrow[d, "(f \times g)_*"] \\ S_*(X') \otimes S_*(Y') \arrow[r, "f_{X',Y'}"] & S_*(X' \times Y') \end{tikzcd}$$

Proof. First we deal with the case $X = \Delta^p$ and $Y = \Delta^q$ for $p, q \geqslant 0$. If $f, g \colon S_*(\Delta^p) \otimes S_*(\Delta^q) \longrightarrow S_*(\Delta^p \times \Delta^q)$ are two chain maps then $S_*(\Delta^p) \otimes S_*(\Delta^q)$ is free abelian and $S_*(\Delta^p \times \Delta^q)$ is acyclic so we can apply Lemma 15.2 and get a chain homotopy $(H'_n)_n$,

$$H'_n \colon (S_*(\Delta^p) \otimes S_*(\Delta^q))_n \longrightarrow S_{n+1}(\Delta^p \times \Delta^q)$$

with $\partial H'_n + H'_{n-1}\partial = f_n - g_n$.

Note that for arbitrary X and Y binaturality implies

$$f_{X,Y} \circ (S_*(\alpha) \otimes S_*(\beta)) = S_*(\alpha, \beta) \circ f_{\Delta^p,\Delta^q}, \quad g_{X,Y} \circ (S_*(\alpha) \otimes S_*(\beta)) = S_*(\alpha, \beta) \circ g_{\Delta^p,\Delta^q}$$

for all $\alpha \colon \Delta^p \to X$, $\beta \colon \Delta^q \to Y$.

We define

$$H_n \colon (S_*(X) \otimes S_*(Y))_n \longrightarrow S_{n+1}(X \times Y)$$

as

$$H_n(\alpha \otimes \beta) = S_{n+1}(\alpha, \beta) \circ H'_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}).$$

This is well-defined and by construction:

$$\begin{aligned} \partial H_n(\alpha \otimes \beta) &= \partial S_{n+1}(\alpha, \beta) \circ H'_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \\ &= S_n(\alpha, \beta)\partial H'_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \\ &= S_n(\alpha, \beta) \circ (-H'_{n-1}\partial(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) + f_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) - g_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})) \\ &= f_n(\alpha \otimes \beta) - g_n(\alpha \otimes \beta) - H_{n-1}\partial(\alpha \otimes \beta). \end{aligned}$$

For the last step use that we can rewrite

$$H_{n-1}\partial(\alpha \otimes \beta) = H_{n-1}(S_*(\alpha) \otimes S_*(\beta_*))\partial(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})$$

as $S_*(\alpha)$ is a chain map, and the left hand side is $S_n(\alpha, \beta)H'_{n-1}(\partial(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})$ by definition. □

Next we need existence and essential uniqueness of a suitable map from $S_*(X \times Y)$ to $S_*(X) \otimes S_*(Y)$.

Proposition 15.4. *(a) There is a chain map $S_*(X \times Y) \longrightarrow S_*(X) \otimes S_*(Y)$ for all spaces X and Y such that this map is natural in X and Y and such that in degree zero this map sends (x_0, y_0) to $x_0 \otimes y_0$ for all $x_0 \in X$ and $y_0 \in Y$.*

(b) Any two such maps are chain homotopic.

Proof. Let $X = \Delta^n = Y$ for $n \geqslant 0$ and set $C_* = S_*(\Delta^n \times \Delta^n)$ and $C'_* = S_*(\Delta^n) \otimes S_*(\Delta^n)$. Set $f_0 \colon C_0 \to C'_0$ as dictated by condition (a). Then by Lemma 15.2 there is a chain map $(f_m)_m$, $f_m \colon S_m(\Delta^n \times \Delta^n) \to (S_*(\Delta^n) \otimes S_*(\Delta^n))_m$. We need to check the condition $f(\partial C_1) \subset \partial C'_1$. Consider a boundary $(x_0, y_0) - (x_1, y_1) \in S_0(X \times Y)$, so there is

$(\sigma, \tau) : \Delta^1 \to X \times Y$ with $\partial\sigma = x_0 - x_1$ and $\partial\tau = y_0 - y_1$. Then one can check that the image $x_0 \otimes y_0 - x_1 \otimes y_1$ is of the form $d_\otimes(\sigma \otimes y_0 + x_0 \otimes \tau)$.

Now for $\alpha\colon \Delta^n \to X \times Y$ we define

$$\tilde{f}_n(\alpha) := (S_*(p_1 \circ \alpha)) \otimes S_*((p_2 \circ \alpha)) \circ f(\Delta_{\Delta^n}).$$

Here, $\Delta_{\Delta^n}\colon \Delta^n \longrightarrow \Delta^n \times \Delta^n$ is the diagonal map viewed as a singular simplex $\Delta_{\Delta^n} \in S_n(\Delta^n \times \Delta^n)$ and the p_i are the projection maps $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$:

$$\begin{tikzcd} S_n(\Delta^n \times \Delta^n) \arrow[r, "f_n"] & (S_*(\Delta^n) \otimes S_*(\Delta^n))_n \arrow[d, "S_*(\alpha)\otimes S_*(\alpha)"] \\ & (S_*(X \times Y) \otimes S_*(X \times Y))_n \arrow[d, "S_*(p_1)\otimes S_*(p_2)"] \\ & (S_*(X) \otimes S_*(Y))_n. \end{tikzcd}$$

It is easy to check that this map sends (x_0, y_0) to $x_0 \otimes y_0$.

Claim (b) follows as in Proposition 15.3. $\square$

Theorem 15.5 (Eilenberg-Zilber). *The homology cross product $\times\colon S_*(X) \otimes S_*(Y) \longrightarrow S_*(X \times Y)$ is a homotopy equivalence of chain complexes.*

Proof. Using Proposition 15.4 let f be any natural chain map $S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ with $f_0(x_0, y_0) = x_0 \otimes y_0$ for any pair of points. Then

$$f \circ (- \times -)\colon S_*(X) \otimes S_*(Y) \to S_*(X) \otimes S_*(Y)$$

and this composition sends $x_0 \otimes y_0$ to itself. We now proceed exactly as in the proof of Proposition 15.3: By Lemma 15.2 for $X = \Delta^p$ and $Y = \Delta^q$ there is a chain homotopy H' between $f \circ (- \times -)$ and the identitiy map on $S_*(\Delta^p) \otimes S_*(\Delta^q)$. We then define a chain homotopy by $H(\alpha \otimes \beta) = S_{n+1}(\alpha, \beta) \circ H'(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})$. Similarly we get that the composition $(- \times -) \circ f$ is homotopic to the identity. $\square$

Corollary 15.6 (Topological Künneth formula). *For any pair of spaces X and Y the following sequence is split short exact*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0.$$

The sequence is natural in X and Y but the splitting is not.

Example 15.7. (a) For the n-torus $T^n = (\mathbb{S}^1)^n$ we get

$$H_i(T^n) \cong \mathbb{Z}^{\binom{n}{i}}$$

where we can identify the rank of the homology in degree i as the coefficient of x^i in $(1 + x)^i$ (the two numbers are given by the same combinatorics).

(b) For a space of the form $X \times \mathbb{S}^n$ we obtain

$$H_q(X \times \mathbb{S}^n) \cong H_q(X) \oplus H_{q-n}(X).$$

There is also a *relative version of the Künneth formula*. The homology cross product in its relative form is a map

$$\times\colon H_p(X,A)\otimes H_q(Y,B)\longrightarrow H_{p+q}(X\times Y, A\times Y\cup X\times B).$$

In particular for A and B a point we get a *reduced Künneth formula* which yields

$$\tilde{H}_p(X)\otimes \tilde{H}_q(Y)\longrightarrow \tilde{H}_{p+q}(X\times Y, X\vee Y)$$

and in good cases (see Proposition 8.4) the latter is isomorphic to $\tilde{H}_{p+q}(X\wedge Y)$ where $X\wedge Y = X\times Y/X\vee Y$.

16. Simplicial homology

Singular homology has a very unwieldy definition which gives good formal properties, but it may only be computed using general theorem.

Cellular homology gives a very small chain complex computing homology, but determining the differentials in terms of degree copmutations is highly non-trivial.

There is a third approach called *simplicial homology* which is also historically the first definition of homology.

It is defined not for arbitrary topological spaces but for simplicial complexes, which are glued out of the standard simplices Δ^n. We restrict ourselves to finite ones.

Recall that an affine simplex, denoted $[v_0,\dots,v_n]$ is a singular simplex of the form $(t_0,\dots,t_n)\mapsto \sum_i t_i v_i$ where $\{v_i\}$ is some set of points. (This makes sense in any affine target space.)

In this section we will mean by a *simplex* an affine simplex in $\mathbb{R}^\infty$ sucht that all v_i are affinely independent. Here $\mathbb{R}^\infty = \operatorname{colim}_n \mathbb{R}^n$, although in pratictice it is enough to consider $\mathbb{R}^N$ for some very large N.

The *faces* of a simplex $[v_0,\dots,v_n]$ are all simplices spanned by a subset of $\{v_i\}$. The *i-th face* of $\sigma=[v_0,\dots,v_n]$, denoted by $d_i\sigma$ is $[v_0,\dots,\widehat{v_i},\dots,v_n]$ where $\widehat{v_i}$ denotes that the vertex v_i is left out.

Definition 16.1. A *finite simplicial complex* is a a collection K of simplices $\{\sigma\}$ such that

(a) if $\sigma\in K$ then so are all the faces of σ,
(b) if $\sigma,\tau\in K$ then $\sigma\cap\tau$ is a face of both σ and τ.

We call the associated topological space $|K|=\cup_K\sigma$ the *polyhedron* of K.

Given a topological space X a homeomorphism $X\cong |K|$ for some simplicial complex K is a *triangulation*.

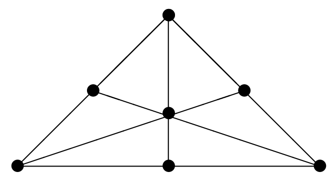
Importantly, any finite simplical complex gives rise to a finite CW complex if we filter $|K|$ it by the dimension of the simplices, noting $\Delta^n\cong\mathbb{D}^n$.

Example 16.2. The torus has a triangulation given by the following simplicial complex with 9 0-simplices, 27 1-simplices and 18 2-simplices. A smaller triangulation would not satisfy that every simplex is determined by its vertices (which is necessary for a simplicial complex).

Recall that the barycenter of a simplex $\sigma=[v_0,\dots,v_n]$ is defiend as $\hat{\sigma}=\frac{1}{n+1}\sum v_i$.

Definition 16.3. The *barycentric subdivison* $K^{(1)}$ of a simplicial complex K has vertices $\hat{\sigma}$ for all $\sigma \in K$ and simplices $[\widehat{\sigma_0}, \ldots, \widehat{\sigma_k}]$ for any sequence of simplices $\sigma_0, \ldots, \sigma_k$ where σ_i is a proper face of σ_{i+1}.

We have met the linear version of this construction in Definition 7.4. The barycentric subdivision of Δ^2 is the following simplicial complex:



One can check that $|K^{(1)}| \cong |K|$.

We will denote iterated barycentric subdivision by $K^{(r)}$.

Remark 16.4. If you are put off by the size of this triangulation you may want to consider Δ-complexes, which are somewhere between CW complexes and simplicial complexes and are used extensively in Hatcher's book.

Remark 16.5. In the beginnings of the subject of topology people assumed any reasonable space could be given a triangulation, and any two triangulations of a space would have some common refinement, thus allowing us to reduce the study of homology to the study of simplicial complexes.

The latter was called the *Hauptvermutung*. It is very false, even for manifolds. In dimensions greater or equal to 4 there are always manifolds with multiple inequivalent triangulations. In dimensions greater or equal to 4 there are also manifolds which do not admit any triangulation at all!

Incidentally, in dimension 4 it is unknown if every manifold is homeomorphic to a CW complex. (This is known to be true in all other dimensions.)

Note that differentiable manifolds always admit a triangulation (and thus a CW structure).

Definition 16.6. The *simplicial chain complex* $C_*(K)$ of a simplicial complex K is defined by $C_n(K) = \oplus_{K_n}\mathbb{Z}$ where K_n is the set of n-simplices and the differential is given on generators by $\partial\sigma = \sum_i (-1)^i d_i\sigma$, explicitly given by $\partial[v_0, \ldots, v_n] = \sum_{i=0}^{n}(-1)^i[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$.

Proposition 16.7. *Let K be a finite simplicial complex. The homology of $C_*(K)$ is isomorphic to the homology of the polyhedron $|K|$.*

Proof. There are two reasonable proofs:

The first proof is more geometric. We observe that $C_*(K)$ is nothing but the cellular chain complex of $|K|$ with the induced CW structure. This is clear for the C_n, one has to take some care when considering the differentials (exercise).

The other proof is more systematic. We note that every simplex of K defines a singular simplex of $|K|$, and by construction this is compatible with the differentials and we get a map $C_*(K) \to S_*(|K|)$.

Denoting $\cup i \leqslant nK_n$ by K^n this induces a map of short exact sequences of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(K^{n-1}) & \longrightarrow & C_*(K^n) & \longrightarrow & C_*(K^n)/C_*(K^{n-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_*(|K^{n-1}|) & \longrightarrow & S_*(|K^n|) & \longrightarrow & S_*(|K^n|, |K_{n-1}|) & \longrightarrow & 0
\end{array}$$

This induces a map between long exact sequences on homology:

$$\begin{array}{ccccccccc}
H_{i+1}(C(K^n)/C(K^{n-1}) & \longrightarrow & H_i(C(K^{n-1})) & \longrightarrow & H_i(C(K^n)) & \longrightarrow & H_i(C(K^n)/C(K^{n-1}) & \xrightarrow{\delta} & H_{i-1}(C(K^{n-1})) \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
H_{i+1}(|K^n|, |K^{n-1}|) & \xrightarrow{\delta} & H_i(|K^{n-1}|) & \longrightarrow & H_i(|K^n|) & \longrightarrow & H_i(|K^n|, |K^{n-1}|) & \xrightarrow{\delta} & H_{i-1}(|K^{n-1}|)
\end{array}$$

Here the first and fourth column are isomorphisms as we observe that $C_*(K^n)/C_*(K^{n-1})$ is just the free abelian group on K_n in degree n, and the inclusion induces a natural isomorphism with $H_*(|K^n|, |K^{n-1}|)$ which is $H_n(\vee_{K_n}\mathbb{S}^n)$ in degree 0.

For $n = 0$ we have $H_*(C(K^0)) \cong H_*(|K^0|)$. Thus we may assume the second and fifth columns are isomorphisms by induction assumption.

We find by the 5-Lemma 9.5 that $H_*(|K^n|) \cong H_*(C(K^n))$ for all n. As $K = K^n$ for some large n we are done. $\square$

Simplical complexes form a category whose morphisms are *simplicial maps* $f : K \to L$ which are maps of 0-simplices $K_0 \to L_0$ such that for an $[v_0, \ldots, v_n] \in K$ we have a simplex with vertices $\{f(v_0), \ldots, f(v_n)\}$ in L. (Note that the $f(v_i)$ need not be distinct.) Simplicial maps clearly induce morphism on simplicial chain complexes.

A simplicial map f induces a continuous maps on polyhedra $|f| : |K| \to |L|$ by sending a point $\sum t_i v_i \in |K|$ to $\sum t_i f(v_i) \in |L|$.

Theorem 16.8 (Simplicial approximation theorem). *Let K, L be finite simplicial complexes and $f : |K| \to |L|$ a continuous map. Then there is a simplicial map $g : K^{(r)} \to L$ from an iterated barycenric subdivision of K to L such that f is homotopic to g.*

We introduce some notation and one lemma to organize the proof.

Definition 16.9. Given a simplex σ in a simplicial complex we define its *star* $St(\sigma)$ to be the union of all simplices containing σ.

We define its *open star* $st(\sigma)$ to be the union of all the interiors of the simplices containing σ.

Here the interior of τ is $\tau \setminus \partial\tau$ (and a 0-simplex is equal to its own interior). The open star of σ is an open subset of $|K|$ and $St(\sigma)$ is its closure.

Example 16.10. Consider a vertex σ of the simplicial complex $\partial\Delta^3$. Its star consists of all the faces of Δ^3 except for the one opposite σ. The open star conists of the interor of the star, i.e. the complement of the face opposite σ in $\partial\Delta^3$.

By definition if σ is a face of τ then $St(\tau) \subset St(\sigma)$.

Lemma 16.11. *Let $v_1, \ldots, v_n$ a collection of a simplicial complex K. Then $\cap_i st(v_i)$ is nonempty if and only if $v_1, \ldots, v_n$ are the vertices of a simplex σ in K. In this case $st(\sigma) = \cap st(v_i)$.*

Proof. By definition the intersection consists of all the interiors of simplices containing all v_i. So if the intersection is nonempty there is such a simplex and contains $\sigma = [v_1, \dots, v_n]$ as a face. Moreover these simplices containing all v_i are exactly the simplices containing σ. $\square$

Corollary 16.12. *Let $f : \mathbb{S}^k \to \mathbb{S}^n$ be a continuous map. If $k < n$ then f is homotopic to a constant map.*

Proof. Any $\mathbb{S}^k$ may be triangulated as the boundary of the $(k+1)$-simplex. It follows from Theorem 16.8 that f is homotopic to a simplicial map which must send $\mathbb{S}^k$ to the k-skeleton of $\mathbb{S}^n$, and any such map is null-homotopic. $\square$

In other words $\pi_k(\mathbb{S}^n) = 0$ if $k < n$.

Proof of 16.8. We note that K may be embedded in some large $\mathbb{R}^N$. In fact we can choose $N = \#K_0$ and send the i-th vertex to the standard basis vector e_i. This equips $|K|$ with a metric which restricts to the usual Euclidean metric on every simplex.

Let $\{v\}$ be the set of vertices of L. Then $f^{-1}(st(v))$ is an open cover of K. Let ϵ be its Lebesgue number. We recall that barycentric subdivison reduces the diameter of the simplices from Lemma 7.8. Thus we may take an iterated subdivison $K^{(r)}$ such that each simplex has diameter $< \epsilon/2$ and then the closed star of any vertex $x \in K$ has diameter $< \epsilon$. So we have $f(St(x)) \subset st(v)$ and we set $g(x) = v$.

We claim that this map extends to a simplicial map $g : K \to L$. So consider a simplex $[x_1, \dots, x_n]$ in K. We need $[f(x_1), \dots, f(x_n)]$ to be a simplex in L. Consider any x in the interior of $[x_1, \dots, x_n]$. it lies in every $st(x_i)$, so by definition $f(x)$ lies in every $st(g(x_i))$. Thus by Lemma 16.11 $[g(x_1), \dots, g(x_n)]$ is a simplex in L.

Then $|g|(x)$ is defined by linear interpolation from the $g(x_i)$.

It remains to show f and $|g|$ are homotopic. We embed L into $\mathbb{R}^N$ again and define a linear homotopy $h_t(x) = (1-t)f(x) + tg(x)$. This wis a continuous homotopy between f and $|g|$ in $\mathbb{R}^N$, we just have to check it is contained in $|L|$.

Any x in $|K|$ lies in the interior of some simplex $[x_1, \dots, x_n]$ and $|g|(x)$ lies in $\sigma = [g(x_1), \dots, g(x_n)]$. By construction $f(x)$ lies in $St(\sigma)$, thus there is a simplex τ containing $f(x)$ and $g(x)$ and thus $h_t(x) = (1-t)f(x) + tg(x) \in \tau \subset |L|$. $\square$

17. The Lefschetz fixed point theorem

Simplicial homology has many practical short comings, but it does have some uses. Our next goal is to prove the famed Lefschetz fixed point theorem.

To simplify things a little bit we work over the rational numbers $\mathbb{Q}$, that means instead of abelian groups we consider chain complexes which are given by $\mathbb{Q}$-vector spaces in every degree.

We recall the Euler characteristic of a chain complex from Exercise sheet 6. So from now on let all our chain complexes have $\sum \dim C_i < \infty$. Then $\chi(C) = \sum (-1)^i \dim(C_i)$.

We may also define something like the Euler characteristic of a morphism:

Definition 17.1. Let $f : C \to C$ be a chain map. Then we define $\tau(f)$ to be $\sum_i (-1)^i tr(f_i : C_i \to C_i)$.

In particular $\tau(\mathrm{id}_C) = \chi(C)$.

LEMMA 17.2. *Let $f : C \to C'$. Then $\tau(f) = \tau(H_*(f)) := \sum_i (-1)^i tr(H_i(f))$.*

PROOF. We first consider a short exact sequence $0 \to V \to W \to W/V \to 0$ of chain complexes and an endomorphism $f : W \to W$ with $f(V) \subset V$. Then there is an induced map $f_{W/V} : W/V \to W/V$ and we have $\tau(f) = \tau(f|_V) + \tau(f_{W/V})$. The proof is elementary linear algebra, just note that f has a block upper triangular form and the trace is the sum of the traces of the diagonal blocks.

We can then apply our observation to the short exact sequences $0 \to B_n \to Z_n \to H_n \to 0$ and $0 \to Z_n \to C_n \to B_{n-1}$.

We find

$$\begin{aligned} \tau(H_*(f)) = \sum_i (-1)^i tr(H_i(f)) &= \sum_i (-1)^i \left(tr(Z_i(f)) - tr(B_i(f))\right) \\ &= \sum_i (-1)^i \left(tr(Z_i(f)) + tr(B_{i-1}(f))\right) \\ &= \sum (-1)^i tr(f|_{C_i}) = \tau(f) \quad \square \end{aligned}$$

DEFINITION 17.3. Let X be a topological space and $f : X \to X$. Then we define the *Lefschetz number* $\tau(f)$ to be $\tau(f_* : H_*(X) \to H_*(X))$.

It is clear from the definition that $\tau(f)$ is homotopy invariant.

THEOREM 17.4 (Lefschetz fixed point theorem). *Let K be a finite simplicial complex and $f : |K| \to |K|$ a continuous map with $\tau(f) \neq 0$. Then f has a fixed point.*

PROOF. Assume f has no fixed point. We choose a metric on $|K|$ as in the proof of Theorem 16.8. As $|K|$ is compact we see that $d(x, f(x))$ attans a minimum $\epsilon > 0$. Subdividing K we obtain a simplicial complex K' such that the stars of all simplices have diameter $< \epsilon/3$.

Subdividing K' further we find a simplicial map $g : K'' \to K'$ homotopic to f. By construction $f(x)$ and $|g|(x)$ always lie in the same simplex, so $d(f(x), |g|(x)) < \epsilon/3$.

We claim $\sigma \cap g(\sigma) = \emptyset$. Indeed if $x, y \in \sigma$ then

$$d(y, g(x)) > d(x, f(x)) - d(x, y) - d(f(x), g(x)) > \epsilon/3,$$

so the intersection is empty.

We now note that g does not give a simplicial map $K'' \to K''$ (as subdividing the right hand side means the images of simplices on the left hand side may no longer be simplices). However, it does induce a cellular map on the CW complex associated to K'' as the n-skeleton of $|K'|$ is contained in the n-skeleton of $|K''|$.

By Lemma 17.2 we can compute $\tau(f)$ by computing $\tau(|g|)$ on the cellular chain complex of K''. On the basis given simplices all of the diagonal entries of g are 0 as every n-simplex is moved. This shows $\tau(f) = \tau(|g|) = 0$ and completes the proof. $\square$

One may also work over the integers but has to divide out all torsion subgroups.

EXAMPLE 17.5.
(a) Let f be an endomorphism of the closed disk $\mathbb{D}^n$. As $\mathbb{D}^n$ is contractible $\tau(f)$ is just the trace of f on H_0, which is 1 for every path connected space. Thus f has a fixed point and we have reproven the Brouwer fixed point theorem.

(b) The same argument applies to any space with trivial rational homology. In particular any endomorphism of $\mathbb{R}P^{2n}$ has a fixed point.

(c) Consider a (rotationally symmetric) torus and rotate it by some angle θ around the axis through the hole. This continuous map does not change the homology class of any generator on homology, thus the Lefschetz number is $1-2+1=0$ and the map need not have a fixed point.

(d) Consider a surface Σ of genus 2 that has reflectional symmetry through a plane separating the two holes. The reflection f has trace 1 in degree 0 and trace 0 in degree 1 as the generators of $H_1(\Sigma)$ are permuted. In degree 2 we use $H_2(\Sigma) \cong H_2(\Sigma, \Sigma \setminus \{x\})$ for some x in the fixed plane. Then reflection changes the sign of the fundamental class of $H_2(\Sigma, \Sigma \setminus \{x\}) \cong H_2(\mathbb{S}^2, \mathbb{S}^2 \setminus \{x\})$ and the trace of f on H_2 is -1. Thus the Lefschetz number is $\tau(f) = 1 - 0 + (-1) = 0$. But f clearly has fixed points. Thus there is no converse to the fixed point theorem. However, we may note that $\tau(f)$ is the Euler characteristic of the fixed point set!

The following observation makes the Lefschetz fixed point theorem more powerful:

Corollary 17.6. *Let X be a retract of a finite simplicial complex and $f : X \to X$ has $\tau(f) \neq 0$. Then X has a fixed point.*

Proof. Let $r : |K| \to X$ be the retraction with $ri = \mathrm{id}_X$. Aussume f has no fixed point, then neither does $ifr : |K| \to |K|$. So $\tau(rfi) = 0$ by Theorem 17.4. But as $H_*(X)$ is a direct summand of $H_*(|K|)$ then $\tau(f) = 0$ also. $\square$

Remark 17.7. Any compact manifold and any finite CW complex is a retract of a finite simplicial complex, see Theorem A.7 in [Hatcher].

Corollary 17.8. *Let f be a simplicial homeomorphism of a finite simplicial complex K. Then $\tau(f) = \chi(K^f)$ where K^f is the subspace of fixed points of $|K|$.*

CHAPTER 2

Singular cohomology

1. Definition of singular cohomology

Definition 1.1. A *cochain complex of abelian groups* is a sequence $(C^n)_{n\in\mathbb{Z}}$ of abelian groups C^n together with homomorphisms $\delta\colon C^n \to C^{n+1}$ with $\delta^2 = 0$. The map δ is called *coboundary operator*. The group

$$H^n(C^*) = \frac{\ker(\delta\colon C^n \to C^{n+1})}{\operatorname{im}(\delta\colon C^{n-1} \to C^n)}$$

is the *nth cohomology group of* C^*.

If (C_*, d_C) is a chain complex, then we can define $D^n := C_{-n}$, $d_D = d|C$ and this is a cochain complex. The fact that d_C lowers degree by one gives $d\colon C_{-n} = D^n \to C_{-n-1} = D^{n+1}$, so d_D raises degree by one. We therefore don't need a theory of cochain complexes; it is just often convenient to switch the notation.

Definition 1.2. For two cochain complexes (C^*, δ) and $(\tilde{C}^*, \tilde{\delta})$ a *map of cochain complexes* from C^* to $\tilde{C}^*$ is a sequence of homomorphisms $f^n\colon C^n \to \tilde{C}^n$ with $f^{n+1}\circ\delta = \tilde{\delta}\circ f^n$.

$$\begin{tikzcd} C^{n+1} \arrow[r, "f^{n+1}"] & \tilde{C}^{n+1} \\ C^n \arrow[u, "\delta"] \arrow[r, "f^n"] & \tilde{C}^n. \arrow[u, "\tilde{\delta}"'] \end{tikzcd}$$

Maps of cochain complexes induce maps on cohomology.

Definition 1.3. Let (C_*, d) be a chain complex. Then the *dual cochain complex* $\operatorname{Hom}(C_*, \mathbb{Z})$, often denoted C^*, is defined to be $\operatorname{Hom}(C_n, \mathbb{Z})$ in degree n with differential induced by d, i.e. $\delta(\phi)(\alpha) = \phi(d\alpha)$ for $\alpha \in C_{n+1}$ and $\phi \in \operatorname{Hom}(C_n, \mathbb{Z})$.

The composition $\delta^2(\varphi)(\alpha)$ is $(\delta\varphi)(d\alpha) = \varphi(d^2\alpha) = 0$ for $\alpha \in C_{n+2}$, $\phi \in \operatorname{Hom}(C^n, \mathbb{Z})$.

Definition 1.4. For a topological space X we call the dual of the singular chain complex the *singular cochain complex* $S^*(X, \mathbb{Z}) = \operatorname{Hom}(S_*(X), \mathbb{Z})$.

If G is any abelian group we may similarly define

$$S^*(X; G) = (\operatorname{Hom}(S_*(X), G), \delta)$$

as the *cochain complex of* X *with coefficients in* G.

For $\alpha\colon \Delta^{n+1} \to X$ and $\varphi\colon S_n(X) \to \mathbb{Z}$, $\delta(\varphi)(\alpha) = \varphi(\partial\alpha)$.

$$\begin{tikzcd}
S_n(X) \arrow[r, "\varphi"] & \mathbb{Z} \\
S_{n+1}(X) \arrow[u, "\partial"] \arrow[ur, "\delta\varphi"'] &
\end{tikzcd}$$

Definition 1.5. Let G be an abelian group, then

$$H^n(X;G) = \frac{\ker(\delta\colon S^n(X;G) \to S^{n+1}(X;G))}{\operatorname{im}(\delta\colon S^{n-1}(X;G) \to S^n(X;G))}$$

is the *nth cohomology group of X with coefficients in G.*

Every continuous map $f\colon X \to Y$ induces a map of cochain complexes $S^*(Y;G) \to S^*(X;G)$. Thus $S^* : \mathsf{Top}^{op} \to \mathsf{Ch}$ and $H^n : \mathsf{Top}^{op} \to \mathsf{Ab}$ are contravariant functors from the category of topological spaces and continuous maps to the category of chain complexes, respectively abelian groups.

For a continuous map $f\colon X \to Y$ we denote $S_*(f)$ by f_* and $S^*(f) : S^*(Y;G) \to S^*(X;G)$ by f^*. For $\varphi \in S^*(Y;G)$ and $\alpha \in S_*(X)$,

$$f^*(\varphi)(\alpha) = \varphi(f_*\alpha) \in G.$$

In order to compute cohomology we may again use cellular methods:

Definition 1.6. Given a CW complex X we define the *cellular cochain complex* with coefficients in abelian group G to be the $\operatorname{Hom}(C_*(X), G)$.

Example 1.7. (a) Dualizing the cell complex $\mathbb{Z}e_n \oplus \mathbb{Z}e_0$ we compute that $H^i(\mathbb{S}^n)$ is $\mathbb{Z}$ if $i = n$ or $i = 0$ and 0 otherwise (for $n > 0$).

(b) The cellular cochain complex of $\mathbb{R}P^2$ with its usual CW structure is $\operatorname{Hom}(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}, \mathbb{Z})$, which is $\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$.

Thus we have $H^2(\mathbb{R}P^2) = \mathbb{Z}/2$, $H^1(\mathbb{R}P^2) = 0$, $H^0(\mathbb{R}P^2) = \mathbb{Z}$.

As chains and cochains are dual we may define a pairing:

Definition 1.8.

- For two abelian groups A and G, we define the *Kronecker pairing*
$$\langle -,-\rangle\colon \operatorname{Hom}(A,G) \otimes A \longrightarrow G, \quad \langle \varphi, a\rangle = \varphi(a) \in G$$
where $\varphi \in \operatorname{Hom}(A,G)$, $a \in A$.
- For a homomorphism $f\colon B \to A$ we define $f^*(\varphi) = \varphi \circ f \in \operatorname{Hom}(B,G)$ and have
$$\langle f^*\varphi, b\rangle = \langle \varphi, fb\rangle = \varphi(f(b)).$$
- For a chain complex C_* and $C^n = \operatorname{Hom}(C_n, G)$ we define
$$\langle -,-\rangle\colon C^n \otimes C_n \to G, \varphi \otimes a \mapsto \langle \varphi, a\rangle = \varphi(a).$$
- In particular, for $A = S_n(X)$ we get a Kronecker pairing
$$\langle -,-\rangle\colon S^n(X;G) \otimes S_n(X) \to G.$$
- For $\partial\colon S_{n+1}(X) \to S_n(X)$ and $a \in S_{n+1}(X)$ we get
$$\langle \delta\varphi, a\rangle = \langle \varphi, \partial a\rangle = \varphi(\partial(a)).$$

LEMMA 1.9. *The Kronecker pairing* $\langle -,- \rangle\colon C^n \otimes C_n \to G$ *is well-defined on the level of cohomology and homology,* i.e., *we obtain an induced map*

$$\langle -,- \rangle\colon H^n(C^*) \otimes H_n(C_*) \to G.$$

PROOF. Let φ be a cocycle, then

$$\langle \varphi, a + \partial b \rangle = \langle \varphi, a \rangle + \langle \varphi, \partial b \rangle = \langle \varphi, a \rangle + \langle \delta\varphi, b \rangle = \langle \varphi, a \rangle.$$

Assume that $\varphi = \delta\psi$ and a is a cycle. Then we get

$$\langle \varphi, a \rangle = \langle \delta\psi, a \rangle = \langle \psi, \partial a \rangle = 0.$$

Therefore $\langle \varphi, - \rangle$ is well-defined on $H_n(C_*)$ and $H^n(C^*)$. □

For later use we choose $\nu_n \in H^n(\mathbb{S}^n)$ with $\langle \nu_n, \mu_n \rangle = 1$.

The Kronecker pairing also defines a natural map

$$\kappa\colon H^n(C^*) \longrightarrow \mathrm{Hom}(H_n(C_*), G)$$

via $\kappa[\varphi][a] := \langle \varphi, a \rangle$. How much does the map κ see?

2. Universal coefficient theorem for cohomology

Dual to Tor, we consider a corresponding construction for the functor $\mathrm{Hom}(-,-)$ instead of $(-) \otimes (-)$. For a short exact sequence

$$0 \to A \longrightarrow B \longrightarrow C \to 0$$

the sequence

$$0 \to \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G) \to 0$$

is always exact on the left, but not necessarily on the right.

As an example, consider $0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ for a natural number $n > 1$. Then the sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0 \longrightarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z} \xrightarrow{n} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$$

is exact but multiplication by n isn't surjective, so we cannot prolong this sequence to the right with a zero.

DEFINITION 2.1. For two abelian groups A, G and the standard free resolution $0 \to F_1 \to F_0 \to A \to 0$ we define $\mathrm{Ext}(A,G)$ as the cokernel of the map

$$\mathrm{Hom}(i,G)\colon \mathrm{Hom}(F_0,G) \to \mathrm{Hom}(F_1,G).$$

Here, Ext comes from 'extension', because one can describe $\mathrm{Ext}(A,G)$ in terms of extensions of abelian groups.

- As for Tor it is true that $\mathrm{Ext}(A,G)$ is independent of the free resolution of A. We may use essentially the same proof.
- The functor $A, G \mapsto \mathrm{Ext}(A,G)$ is covariant in G and contravariant in A: for homomorphisms $f\colon A \to B$ and $g\colon G \to H$ we get

$$f^*\colon \mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G), g_*\colon \mathrm{Ext}(A,G) \to \mathrm{Ext}(A,H).$$

- It follows from the corresponding properties of Hom that for a family of abelian groups $(G_i, i \in I)$

$$\mathrm{Ext}(A, \prod_{i\in I} G_i) \cong \prod_{i\in I} \mathrm{Ext}(A, G_i)$$

  and

$$\mathrm{Ext}(\bigoplus_{i\in I} G_i, B) \cong \prod_{i\in I} \mathrm{Ext}(G_i, B).$$

- Similarly to Tor the group $\mathrm{Ext}(A, G)$ can be explicitly calculated if A is a finitely generated abelian group (see the exercise sheet).

REMARK 2.2. It is clear that $\mathrm{Ext}(A, G)$ is trivial if A is free. It is also trivial if G is divisible, *i.e.*, for all $g \in G$ and $n \in \mathbb{Z} \setminus \{0\}$ there is a $t \in G$ with $g = nt$. For example this holds if G is isomorphic to $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}/\mathbb{Z}$, or $\mathbb{C}$.

In more general settings, when we replace abelian groups by R-modules over some commutative unital ring R, the properties ensuring that $\mathrm{Ext}_R(A, G)$ disappears are that A is *projective* or G is *injective*. In the special case of $\mathbb{Z}$-modules, i.e. abelian groups, this is equivalent to A being free respectively G being divisible.

THEOREM 2.3. *(Universal coefficient theorem for cochain complexes) For every free chain complex C_* and $C^* = \mathrm{Hom}(C_*, G)$ the following sequence is exact and splits*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C_*), G) \longrightarrow H^n(C^*) \xrightarrow{\kappa} \mathrm{Hom}(H_n(C_*), G) \longrightarrow 0.$$

Setting $C_* = S_*(X)$ we immediately obtain:

COROLLARY 2.4. *(Universal coefficient theorem for singular cohomology) Let X be an arbitrary space. Then the sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \xrightarrow{\kappa} \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

is split exact.

PROOF OF THEOREM 2.3. Let C_* be a free chain complex and $C^* = \mathrm{Hom}(C_*, G)$. Then the sequence $0 \to Z_n \longrightarrow C_n \longrightarrow B_{n-1} \to 0$ is split exact. There is a potential ambiguity here between the dual group $\mathrm{Hom}(B_n, \mathbb{Z})$ and the space of coboundaries $B^n \subset C^n$. But the groups agree: Any coboundary δf comes from a map $f \in C^{n+1}$

As in the case of tensor products this means that the G-dual sequence

$$0 \to B^{n-1} \longrightarrow C^n \longrightarrow Z^n \to 0$$

is short exact. (Note that (contrary to what I said in lectures) B^n here is $\mathrm{Hom}(B_n, G)$, which is not the space of boundaries in C^n!)

As the sequence is compatible with differentials (the trivial differential on B^* and Z^*), we get a short exact sequence of cochain complexes. This yields a long exact sequence on the level of cohomology groups

$$\ldots \longrightarrow Z^{n-1} \xrightarrow{\partial} B^{n-1} \longrightarrow H^n(C^*) \longrightarrow Z^n \xrightarrow{\partial} B^n \longrightarrow \ldots$$

Here, ∂ denotes the connecting homomorphism in the cohomological case. By the very definition of the connecting homomorphism we get that ∂ is the dual of the inclusion $i_n\colon B_n \subset Z_n$, $\partial = i_n^*$. We cut the long exact sequence above into short ones

$$0 \to \operatorname{coker}(i_{n-1}^*) \longrightarrow H^n(C^*) \longrightarrow \ker(i_n^*) \to 0$$

and hence we have to identify the kernel and the cokernel above.

Left exactness of hom gives us the exact sequence

$$0 \longrightarrow \operatorname{Hom}(H_n(C_*), G) \xrightarrow{\pi^*} \operatorname{Hom}(Z_n, G) \xrightarrow{i_n^*} \operatorname{Hom}(B_n, G),$$

which tells us that the kernel of i_n^* is the image of π^* and due to the injectivity of π^* this is isomorphic to $\operatorname{Hom}(H_n(C_*), G)$.

The sequence

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C_*) \longrightarrow 0$$

is a free resolution of $H_{n-1}(C_*)$ and therefore the cokernel of i_{n-1}^* is $\operatorname{Ext}(H_{n-1}(C_*), G)$.

THe splitting is left as an exercise. $\square$

Example 2.5. We know that the homology of $\mathbb{C}P^n$ is free with

$$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}, & 0 \leqslant k \leqslant 2n, k \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore $H^k(\mathbb{C}P^n) \cong \operatorname{Hom}(H_k(\mathbb{C}P^n), \mathbb{Z})$, thus the cohomology is given by the $\mathbb{Z}$-dual of the homology.

3. Axiomatic description of a cohomology theory

We will now give the axiomatic description of singular cohomology. These axioms will be the main results we proved for homology, and that hold equally for cohomology.

We begin by noting the following facts, easy consequences of some of the results we proved for chain complexes.

- For a chain map $f\colon C_* \to C'_*$ (such as the barycentric subdivision) the G-dual map
$$f^* = \operatorname{Hom}(f, G)\colon \operatorname{Hom}(C'_*, G) \longrightarrow \operatorname{Hom}(C_*, G)$$
is a map of cochain complexes.
- If $(H_n\colon C_n \to C'_{n+1})_n$ is a chain homotopy, then the G-dual
$$(H^n := \operatorname{Hom}(H_n, G)\colon \operatorname{Hom}(C'_{n+1}, G) \to \operatorname{Hom}(C_n, G))_n$$
is a cochain homotopy. Thus if $\partial H_n + H_{n-1}\partial = f_n - g_n$, then $H^n\delta + \delta H^{n-1} = f^n - g^n$.
- As we mentioned above, for a split exact sequence $0 \to B_1 \longrightarrow B_2 \longrightarrow B_3 \to 0$ the dual sequence $0 \to \operatorname{Hom}(B_3, G) \longrightarrow \operatorname{Hom}(B_2, G) \longrightarrow \operatorname{Hom}(B_1, G) \to 0$ is exact. For instance, if A is a subspace of X, then the short exact sequence
$$0 \to S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X, A) \to 0$$
is split. We define $r_n\colon S_n(X) \to S_n(A)$ on $\alpha\colon \Delta^n \to X$ via
$$r_n(\alpha) \begin{cases} \alpha, & \text{if } \alpha(\Delta^n) \subset A, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore $0 \to S^*(X,A) \longrightarrow S^*(X) \longrightarrow S^*(A) \to 0$ is a short exact sequence.

With the help of these facts and using the results we have established for singular homology we can show that singular cohomology satisfies the *axioms of a cohomology theory*:

THEOREM 3.1. *Singular cohomology satisfies the following* axioms for cohomology*:*

(a) The assignment $(X,A) \mapsto H^n(X,A)$ is a contravariant functor from the category of pairs of topological spaces to the category of abelian groups.
(b) For any subspace $A \subset X$ there is a natural homomorphism $\partial\colon H^n(A) \to H^{n+1}(X,A)$
(c) If $f,g\colon (X,A) \to (Y,B)$ are two homotopic maps of pairs of topological spaces, then $H^n(f) = H^n(g)\colon H^n(Y,B) \to H^n(X,A)$.
(d) For any subspace $A \subset X$ we get a long exact sequence

$$\ldots \xrightarrow{\partial} H^n(X,A) \longrightarrow H^n(X) \xrightarrow{H^n(i)} H^n(A) \xrightarrow{\partial} \ldots$$

(e) Excision holds, i.e., *for $W \subset \bar{W} \subset \mathring{A} \subset A \subset X$*

$$H^n(i)\colon H^n(X,A) \cong H^n(X \setminus W, A \setminus W), \text{ for all } n \geqslant 0.$$

(f) Let $*$ be the one-point space, then

$$H^n(*) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

This is called the axiom about the coefficients or the dimension axiom*.*
(g) Singular cohomology is additive:

$$H^n(\bigsqcup_{i \in I} X_i) \cong \prod_{i \in I} H^n(X_i).$$

PROOF. We have shown the corresponding theorems for homology and together with our observations above this gives (a)-(f). For (g) note that $S_*(\amalg X_i) \cong \oplus_i S_(X_i)$, $\mathrm{Hom}(\oplus_i S_*(X_i), \mathbb{Z}) = \prod_i \mathrm{Hom}(S_*(X_i), \mathbb{Z})$ and cohomology commutes with direct products of chain complexes. □

For singular cohomology with coefficients in G we have an analoguous set of axioms, replacing the dimension axioms by $H^*(*) = G$ in degree 0.

Remarkably these axioms determine the cohomology groups uniquely, at least if we restrict attention to CW pairs!

THEOREM 3.2. *On the category of CW pairs the singular cohomology groups H^n are the only functors satisfying the above axioms.*

PROOF. See Theorem 4.59 in [Hatcher]. The idea is to use the filtration of CW complexes and compare cellular singular cochains and cellular cochains based on some other cohomology theory. □

One may drop the dimension axiom and a set of functors satisfying all other axioms is called a *generalized cohomology theory*. In particular we may allow the point to have cohomology in nonzero degrees.

There are many important examples of generalizied cohomology theories, like (different flavours of) topological K-theory or cobordism. An important example of a generalized homology theory (defined entirely analogusly) is stable homotopy theory.

4. Cup product

In the following, we fix a commutative ring with unit R and we will consider homology and cohomology with coefficients in R. We will often suppress the R in our notation, so $H_n(X, A)$ will stand for $H_n(X, A; R)$ and similarly $S_n(X)$ is $S_n(X; R)$ etc. We'll use analogous abbreviations for cochains and cohomology. We will introduce $\mu : R \otimes R \to R$ as an explicit name for the multiplication on R.

If we consider cohomology groups with coefficients in a commutative ring then cohomology itself can be equipped with a product.

The key point is that by contravariance of cohomology the diagonal map induces a map $H^*(X \times X) \to H^*(X)$, and by our considerations when proving the Künneth theorem the left hand side receives a map form $H^*(X) \otimes H^*(X)$.

We first recall from Proposition 15.4 that there is an essentially unique natural chain map $S_*(X \times X) \to S_*(X) \otimes S_*(X)$. We will now pick an explicit model for the composition of this map with the diagonal.

Definition 4.1. Let $a\colon \Delta^n \to X$ and let $0 \leqslant q \leqslant n$.

- The $(n-q)$-dimensional *front face* of a is
$$F(a) = F^{n-q}(a) = a \circ i\colon \Delta^{n-q} \xrightarrow{i} \Delta^n \xrightarrow{a} X$$
where i is the inclusion $i\colon \Delta^{n-q} \hookrightarrow \Delta^n$ with $i(e_j) = e_j$ for $0 \leqslant j \leqslant n-q$, explicitly $(t_0, \ldots, t_{n-q}) \mapsto (t_0, \ldots, t_{n-q}, 0 \ldots, 0)$.
- The q-dimensional back or *rear face* of a is
$$R(a) = R^q(a) = a \circ h\colon \Delta^q \xrightarrow{r} \Delta^n \xrightarrow{a} X$$
where $r\colon \Delta^q \hookrightarrow \Delta^n$ is the inclusion with $r(e_0) = e_{n-q}, \ldots, r(e_q) = e_n$, i.e. $r(e_i) = e_{n-(q-i)}$, , explicitly $(t_0, \ldots, t_q) \mapsto (0, \ldots, 0, t_0, \ldots, t_q)$.

We can express the $(n-q)$-dimensional front face of a as
$$F^{n-q}(a) = \partial_{n-q+1} \circ \ldots \circ \partial_n(a).$$
Similarly,
$$R^q(a) = \partial_0 \circ \ldots \circ \partial_0(a)$$
where ∂_0 is repeated $n-q$ times.

Definition 4.2. The *Alexander-Whitney diagonal map* $S_*(X) \to S_*(X) \otimes S_*(X)$ is defined by
$$\mathrm{AW}(a) = \sum_{p+q=n} F^p(a) \otimes R^q(a)$$
for a generating simplex $a : \Delta^n \to X$ in $S_n(X)$.

Proposition 4.3. *The Alexander Whitney map is a chain map and satisfies $AW(x) = x \otimes x$ for $x \in S_0(X)$.*

Proof. The first statement follows by unravelling the definitions (note the convention for the differential on the tensor product). The second statement is immediate. □

Definition 4.4. The *cup product* $\cup : S^p(X) \otimes S^q(X) \to S^{p+q}(X)$ on cochains is defined by

$$\alpha \cup \beta(c) = \mu(\alpha \otimes \beta)AW(a)$$

If $|\alpha| = p$, $|\beta| = q$ then for $c \in S_{p+q}(X)$ we have

$$\alpha \cup \beta(c) = \alpha(F^p(c))\beta(R^q(c)).$$

Remark 4.5. Somebody might object that the sign is not right. I have mentioned before that moving an object of degree p past an object of degree q picks up a sign pq. With this rule we should have

$$\alpha \cup \beta(c) = (-1)^{pq}\alpha(F^p(c))\beta(R^q(c)).$$

as we commute β of degree q past $F^p(c)$ of degree p.

In general, for two elements $x, y \in C_*$ and $\xi, \upsilon \in C^*$ one can define $(\xi \otimes \upsilon)(x \otimes y) = (-1)^{|x||\upsilon|}\xi(x) \otimes \upsilon(y))$. This is an instance of the *Koszul rule of signs.*

But in fact, to be principled the same applies to the sign in the cochain complex, and we want the formula $0 = \partial f(a) = (\delta f)(a) + (-1)^| f|f(\partial a)$ to be true for a cochain f and a chain a. But that implies $\delta f(a) = (-1)^{|f|+1} f(\partial a)$.

All of this is a matter of convention, in some sense Bredon is the most principled source, but it is a bit easier to work with Hatcher's conventions, so this is what we will do.

Lemma 4.6. *The cup product is associative, unital and functorial.*

Proof. We compute that $\alpha \cup (\beta \cup \gamma)(c)$ and $(\alpha \cup \beta) \cup \gamma$ are both given by $\mu(\mu \otimes \mathrm{id})(\alpha \otimes \beta \otimes \gamma)(F^{|\alpha|}(c) \otimes M^{|\beta|}(c) \otimes R^{|\gamma|}(c))$ where $M^{|\beta|}(c)$ is the "middle face" of c, given by the composition with the map $e_i \mapsto e_{i+|\alpha|}$ from $\Delta^{|\beta|}$ to $\Delta^{|c|}$.

The constant cochain with value 1 is the identity.

For the last statement we need to check that $f^*(\alpha) \cup f^*(\beta) = f^*(\alpha \cup \beta)$. But this is immediate as AW is a chain map:

$$\begin{aligned} f^*(\alpha \cup \beta)(c) &= \mu(\alpha \otimes \beta)AW(f(a)) \\ &= \mu(\alpha \otimes \beta)f_*(AW(a)) \\ &= (f^*(\alpha) \cup f^*(\beta))(a) \end{aligned}$$

□

We want to show our product descends to cohomology, and this is a consequence of the following Leibniz formula:

Lemma 4.7. *For $\alpha \in S^p(X)$ and $\beta \in S^q(X)$ we have $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^q \alpha \cup \delta\beta$.*

Proof. We need to check on $c \in S_{p+q+1}$ we compute:

$$(\delta\alpha \cup \beta)(c) = \sum_{i=0}^{p+1} (-1)^i \alpha(\partial_i F^{p+1}(c))\beta(R^q(c))$$

and

$$(-1)^p(\alpha \cup \delta\beta)(c) = \sum_{i=0}^{q+1} (-1)^{p+i} \alpha(F^p(c))\beta(\partial_i R^{q+1}(c))$$

Investigating the summands in turn and repeatedly using that $\partial_j\partial_i = \partial_{i-1}\partial_j$ wo commute the boundary past the front and rear face maps we see that we obtain exactly the summands of

$$(\delta(\alpha \cup \beta)(c) = \sum_{i=0}^{p+q+1} (-1)^i \alpha(F^{p+1}(\partial_i c))\beta(R^q(\partial_i c))$$

except for the terms $(-1)^{p+1}\alpha(\partial_{p+1}F^{p+1}(c))\beta(R^q(c))$ (which is the last summand of the first sum) and $(-1)^p\alpha(F^p(c))\beta(\partial_0 R^{q+1}(c))$ (first summand of second sum). Those two terms cancel, as $\partial_{p+1}F^{p+1}(c) = F^p(c)$ and $\partial_0 R^{q+1}(c) = R^q(c)$. □

As the cup product mixes up different degrees it is best to consider it on all cohomology groups a the same time. We thus consider the category of graded rings.

Definition 4.8. A *graded ring* is a ring R with a decomposition $R = \oplus_{i\in\mathbb{Z}} R^i$ such that $R^i \cdot R^j \subset R^{i+j}$. A homomomorphis of graded rings $f : R_* \to S_*$ is a ring homomorphism $R \to S$ such that $f(R_i) \subset S_i$ for all $i \in \mathbb{Z}$. We denote by $\mathsf{Ring}^{\mathsf{gr}}$ the category of graded rings.

Theorem 4.9. *The direct sum of cohomology groups defines a functor from topological spaces to the category of graded rings $H^* : \mathsf{Top} \to \mathsf{Ring}^{\mathsf{gr}}$ (or the category of graded R-algebras if we consider coefficients in R).*

Proof. As $\delta(\alpha\cup\beta) = \delta\alpha\cup\beta + (-1)^{|\alpha|}\alpha\cup\delta\beta$ by Lemma 4.7 the cup product of cocycles is a cocycle. Setting $\beta = \delta\gamma$ or $\alpha = \delta\gamma$ in this equation shows that the cup product of a cocycle with a coboundary (and vice versa) is a coboundary. Thus there is an induced cup product on cohomology. It is associative, unital and functorial and respects degree as it is on cochains. Note that the constant cochain that takes the value 1 on every 0-chain is a cocycle. □

We may extend the cup product to relative cohomology. We consider $\alpha \in H^p(X, A; R)$ and $\beta \in H^q(X, B; R)$, i.e. α and vanishes on chains taking values in A, and β vanishes on chains with values in B. If A and B are open in $A \cup B$ we can use the following argument: Let some homology class c be represented by a chain take values in $A\cup B$. Using barycentric subdvision we may assume $c = c' + c''$ with c taking values in A and c'' taking values in B. But then $(\alpha \cup \beta)(c) = 0$ as the first factor is 0 on c' and the second factor is 0 on c''. We thus find that

$$\cup : H^p(X, A; R) \otimes H^q(X, B; R) \to H^{p+q}(X, A \cup B; R)$$

is well defined.

In particular $H^*(X, A)$ is a graded ring, but note that it is in general non-unital!

Example 4.10. Many cup products are trivial for degree reasons.

(a) Let $\mathbb{S}^n$ be a sphere of dimension $n \geqslant 1$. We know that $H^0(\mathbb{S}^n) \cong \mathbb{Z} \cong H^n(\mathbb{S}^n)$ and the cohomology is trivial in all other degrees. We have $1 \in H^0(\mathbb{S}^n)$ and $\nu_n \in H^n(\mathbb{S}^n)$. We know that
$$1 \cup \nu_n = \nu_n = \nu_n \cup 1, 1 \cup 1 = 1$$
but $\nu_n \cup \nu_n = 0 \in H^{2n}(\mathbb{S}^n) = 0$. Thus, $H^*(\mathbb{S}^n)$ has the structure of a so-called graded exterior algebra with the generator ν_n, $\Lambda_{\mathbb{Z}}(\nu_n)$.

(b) More generally, if X is a CW complex of finite dimension, then $\alpha \cup \beta = 0$ for all α, β for $|\alpha| + |\beta|$ big enough.

(c) In particular, if X is a finite-dimensional CW complex then every element in $H^{\geqslant 1}(X)$ is nilpotent.

We now compute our first non-trivial cup product.

EXAMPLE 4.11. Consider $H^*(\mathbb{R}P^2, \mathbb{Z}/2)$. This is $\mathbb{Z}/2$ in degree 1 and 2. In fact the only interesting cup product is the product of the generator γ of $H^1(\mathbb{R}P^2)$ as H^0 is generated by 1 and all other products must be zero for degree reasons.

So let us compute $\gamma \cup \gamma$. We recall the presentation of $\mathbb{R}P^2$ as a circle with the two halves of the boundary identified. Fix a base point $*$. Let $*_1$ be the constant 1-simplex and $*_2$ the constant 2-simplex at the base point $*$. Let c be the 1-simplex that is half the boundary of the disk, it is easy to see this is a generator for $\pi_(\mathbb{R}P^2, *)$ and thus for $H_1(\mathbb{R}P^2, \mathbb{Z}/2)$. Finally let s be the 2-simplex that maps onto the disk homeomorphically, with boundary $2\gamma - *_1$. It follows that $s - *_2$ is a generator for $H_2(\mathbb{R}P^2$.

Now we consider a generator γ of $H^1(\mathbb{R}P^2, \mathbb{Z}/2)$. We then must have $\gamma(c) = 1$ and $\gamma(*_1)$ is 0 as $*_1 = \partial *_2$ is a boundary.

We compute $(\gamma \cup \gamma)(s) = \gamma(\partial_2 s) f(\partial_0 s) = 1 \cdot 1$. Similarly $(\gamma \cup \gamma)(*_2) = \gamma(*_1)\gamma(*_1) = 0$. We need to show $\gamma \cup \gamma$ is not a coboundary. But $\delta\beta(s) = \beta(\partial s) = \beta(2c\, *_1) = \beta(*_1)$ using characteristic 2. But $\beta(*_1) = \beta(\partial *_2)$ is a boundary. So any coboundary takes value 0 on s and $\gamma \cup \gamma$ is not a coboundary.

As a graded ring $H^*(\mathbb{R}P^2) = \mathbb{Z}[\gamma]/(\gamma^2)$ with $|\gamma| = 1$.

We conclude with two more vanishing results:

LEMMA 4.12. *If X can be covered as $X = X_1 \cup \ldots \cup X_r$ by open and path-connected sets with $H^*(X_i) = 0$ then in $H^*(X)$ all r-fold cup products of elements of positive degree vanish.*

PROOF. We prove the case where $r = 2$; the general claim then follows by induction. So assume $X = X_1 \cup X_2$ such that the X_i have vanishing cohomology groups in positive degrees and let $i_j \colon X_j \hookrightarrow X$ be the inclusion of X_j into X $(j = 1, 2)$. Then for all $\alpha \in H^*(X)$, $i_j^*(\alpha) = 0$. Consider the exact sequence

$$H^*(X, X_j) \longrightarrow H^*(X) \longrightarrow H^*(X_j).$$

Therefore, for all α there is an $\alpha' \in H^*(X, X_1)$ that is mapped isomorphically to α. Similarly, for $\beta \in H^*(X)$ there is an $\beta' \in H^*(X, X_2)$ that corresponds to β. The cup product $\alpha \cup \beta$ then corresponds to $\alpha' \cup \beta'$ but this is an element of $H^*(X, X_1 \cup X_2) = H^*(X, X) = 0$. $\square$

A pointed space $(X, *)$ such that $(X, *)$ is a good pair is also called *well-pointed.*

LEMMA 4.13. *If $X = X_1 \vee X_2$ and X_1, X_2 are well-pointed and connected, then $\tilde{H}^*(X) \cong \tilde{H}^*(X_1) \oplus \tilde{H}^*(X_2))$ as nonunital rings, i.e. for $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ with $\alpha_i, \beta_i \in H^*(X_i)$ in positive degrees, the cup product is*

$$\alpha \cup \beta = (\alpha_1 + \alpha_2) \cup (\beta_1 + \beta_2) = \alpha_1 \cup \beta_1 + \alpha_2 \cup \beta_2.$$

PROOF. As X_i is well-pointed we have $\tilde{H}^*(X_1 \vee X_2) = H^*(X_1 \amalg X_2, * \amalg *) \subset H^*(X_1 \amalg X_2)$ for $* \geqslant 2$ by the long exact sequence.

So any nonzero product in degree $\geqslant 2$ must map to a nonzero product in $H^*(X_1 \amalg X_2)$, but by the definition the product of a cochain in $S^*(X_1 \amalg X_2)$ supported on X_1 and a cochain supported on X_2 is zero. $\square$

5. The cross product

There is another multiplication on cohomology, which relates the cohomologies of two spaces with the cohmology of their product.

Recall from Proposition 15.4 that there is an essentially unique natural chain map $S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$. Fix such a map and call it EZ.

We want to work in a bit more generality, so we state here the relative version. There are some subtleties to consider (and it is perfectly legitimate for you to focus on the absolute case, which I will do next time I lecture this course).

First, the result we expect is false unless we make some assumption on our pairs (X, A) and (Y, B). We could assume A and B are open in X and Y respectively or that one of them is empty.

Then there is a natural map $K : S_*(X, A) \otimes S_*(Y, B) \to S_*(X \times Y, X \times B \cup A \times Y)$ obtained by composing a map $L : S_*(X, A) \otimes S_*(Y, B) \to S_*(X \times Y)/(S_*(X \times B) + S_*(A \times Y))$ with $M : S_*(X \times Y)/(S_*(X \times B) + S_*(A \times Y)) \to S_*(X \times Y, X \times B \cup A \times Y)$ which induces an isomorphism on homology. Some details can be found in Spanier's "Algebraic Topology", Theorem 5.3.9 together with Theorem 4.6.3. Note that Spanier moves from showing the map M induces an isomorphism on homology (by using barycentric subdivision) to declaring it is a chain homotopy equivalence. In fact as K induces an isomorphism on homology it follows by general homological algebra that it is a chain homotopy equivalence as the two complexes are free and bounded below.

So we shall choose a homotopy inverse of our map K and denote it by EZ, as in the absolute case above.

Definition 5.1. Let $A \subset X$ and $B \subset Y$ be open. For $\alpha \in S^p(X, A)$ and $\beta \in S^q(Y, B)$ we define the *cohomology cross product*, $\times$, as

$$\alpha \times \beta := \mu \circ (\alpha \otimes \beta) \circ \mathrm{EZ} \in S^{p+q}(X \times Y, X \times B \cup A \times Y)$$

where EZ is any Eilenberg-Zilber map as above. Thus

$$\begin{tikzcd}
S_n(X \times Y; X \times B \cup A \times Y) \arrow[d, "\mathrm{EZ}"'] \arrow[ddrr, "\alpha\times\beta"] & & \\
\bigoplus_{p+q=n} S_p(X, A) \otimes S_q(Y, B) \arrow[d] & & \\
S_p(X, A) \otimes S_q(Y, B) \arrow[r, "\alpha\otimes\beta"'] & R \otimes R \arrow[r, "\mu"'] & R
\end{tikzcd}$$

We note without proof some useful properties of the cross product, compare similar statements in Lemma 15.1:

- The cohomology cross product is natural, *i.e.*, for maps of pairs of spaces $f\colon (X, A) \to (X', A')$, $g\colon (Y, B) \to (Y', B')$
$$(f, g)^*(\alpha \times \beta) = (f^*\alpha) \times (g^*\beta).$$
- The Leibniz formula holds
$$\delta(\alpha \times \beta) = (\delta\alpha) \times \beta + (-1)^{|\alpha|}\alpha \times (\delta\beta)$$

where $|\alpha|$ denotes the degree of α. Thus the cross product descends to cohomology and gives $H^p(X,A)\otimes H^q(Y,B)\to H^{p+q}(X\times Y, X\times B\cup A\times Y)$.

- For the Kronecker pairing we have for cohomology classes α,β and homology classes a,b of a corresponding degree
$$\langle \alpha\times\beta, a\times b\rangle = \langle\alpha,a\rangle\langle\beta,b\rangle$$
where we use the cross product in homology and in cohomology.
- For $1\in R$ and thus $1\in S^0(X,A)$
$$1\times\beta = p_2^*(\beta),\quad \alpha\times 1 = p_1^*(\alpha)$$
where p_i $(i=1,2)$ denotes the projection onto the ith factor in $X\times Y$.
- The cohomology cross product is associative
$$\alpha\times(\beta\times\gamma) = (\alpha\times\beta)\times\gamma$$
on the level of cohomology groups.

We may use the cohomology cross product to define the cup product on H^*, and conversely the cross product may be defined via the cup product.

We recall the diagonal map $\Delta\colon X\to X\times X$ and the two projections $p_1,p_2 : X\times Y\to X$.

Lemma 5.2. *For $\alpha\in H^p(X,A)$ and $\beta\in H^q(X,B)$, with $A\subset X$ and $B\subset X$ open, the cup-product of α and β is given by*
$$\alpha\cup\beta = \Delta^*(\alpha\times\beta) = \Delta^*(\mu\circ(\alpha\otimes\beta)\circ EZ).$$
Conversely the cross product of α and β is given by $p_1^*(\alpha)\cup p_2^*(\beta)\in H^*(X\times Y, X\times B\cup A\times Y)$.

As a diagram we have:
$$\begin{tikzcd} H^p(X,A)\otimes H^q(X,B) \arrow[r, "\times"] \arrow[dr, "\cup"'] & H^{p+q}(X\times X, X\times B\cup A\times X) \arrow[d, "\Delta^*"] \\ & H^{p+q}(X, A\cup B) \end{tikzcd}$$

Proof. The first statement follows immediately if we recall that our Alexander Whitney diagonal that defines the cup product $\alpha\cup\beta(a) = \mu(\alpha\otimes\beta)AW(a)$ is a model for the chain map $EZ\circ\Delta_*$.

For the second statement let $\alpha\in H^p(X)$, $\beta\in H^q(Y)$.
$$p_1^*(\alpha)\cup p_2^*(\beta) = (\alpha\times 1)\cup(1\times\beta).$$
Here, $\alpha\times 1$ and $1\times\beta$ live in the cohomology of $X\times Y$. By definition, the cup product is the pull-back of the cross product by the diagonal. Here, $\Delta_{X\times Y}\colon X\times Y\to(X\times Y)^2$. Therefore, the above is equal to
$$\Delta^*_{X\times Y}((\alpha\times 1)\times(1\times\beta)) = \alpha\times\beta.$$
□

We prove the following key property of the cross product:

Proposition 5.3. *The cross product induces a graded commutative product on cohomology, i.e. $\alpha\times\beta = (-1)^{|\alpha||\beta|}\beta\times\alpha$.*

PROOF. We consider the twist map $T\colon X\times Y\to Y\times X$ and the swap map $\tau : C_*(X)\otimes C_*(Y)\to C_*(Y)\otimes C^*(X)$ given by $a\otimes b\mapsto (-1)^{|a||b|}b\otimes a$.

Let $EZ : S_*(X\times Y)\to S_*(X)\otimes S_*(Y)$ be map as in Prposition 15.4. Then we consider the map

$$EZ^\tau=\tau\circ EZ\circ T : S_*(X\times Y)\to S_*(X)\otimes S_*(Y).$$

This is also a chain map (note the sign on τ that is needed for compatibility with the tensor differential), and clearly agrees with EZ on S_0. Thus by Proposition 15.3 the two maps are chain homotopic and, setting $X=Y$, we have that the cup product on cohomology may equivalently be defined using EZ or EZ^τ. But the EZ^τ definition gives $\alpha\cup^\tau\beta=(-1)^{pq}\beta\cup\alpha$, proving the proposition. □

COROLLARY 5.4. *The cup product on $H^*(X;R)$ is graded commutative, i.e. $\alpha\cup\beta=(-1)^{|\alpha||\beta|}\beta\cup\alpha$.*

This corollary only holds if R is commutative and this is the reason we always we assume we are working with a commutative coefficient ring.

PROOF. This is immediate from Proposition 5.3 and Lemma 5.2. □

COROLLARY 5.5. *Assume that $\alpha\in H^p(X;R)$ with p odd. Assum that R is a field of characteristic $\neq 2$ or a torsion free ring. Then $\alpha^2=0$.*

PROOF. We compute

$$\alpha^2=(-1)^{p^2}\alpha^2=-\alpha^2.$$

Therefore $2\alpha^2=0$ and if R is a field of characteristic not equal to 2 or if R is a torsionfree commutative ring, then $\alpha^2=0$. □

REMARK 5.6. Our formula for the cup product in terms of the Alexander-Whitney diagonal showed that $\cup$ is associative on the cochain level and not just on the level of cohomology groups (this was not obvious from the EZ map). But note that the explicit formula does not give a (graded) commutative product on singular cochains. The cup product is only homotopy commutative, in fact it is homotopy commutative up to coherent homotopies, it is an E_∞-*algebra*.)

The cross product looks reminiscent of the Künneth theorem. To simplify matters we work over a field k to avoid having to worry about Tor groups.

THEOREM 5.7. *Let X,Y be topological spaces such that Y has finite-dimensional homology groups in each degree. The cross product induces an isomorphism of graded commutative rings $H^*(X;k)\otimes H^*(Y;k)\to H^*(X\times Y;k)$.*

Here the notation means that for each p,q we have a map $H^p\otimes H^q\to H^{p+q}$ such that $\oplus_{p+q=n}H^p\otimes H^q=H^n$, and the left hand side has the product $(\mu\otimes\mu)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})$, which on basis elements is defined by $(a\otimes b).(a'\otimes b')=(-1)^{|a'||b|}aa'\otimes bb'$, $(\mu\otimes\mu)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})$.

PROOF. The natural map $\times : \alpha\otimes\beta\mapsto\alpha\times\beta=p_1^*(\alpha)\cup p_2^*(\beta)$ induces a morphism of graded rings on cohomology groups. This follows as p_i^* is a ring homomorphism by functoriality and a homomorphism from a tensor product of rings is determined by its restriction to the tensor functors. (The tensor product is the coproduct in the category of commutative graded rings.)

It remains to check that $\times$ is an isomorphism. We know from Theorem 15.5 that the map $EZ : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ induces an isomorphism on homology.

Over a field we may use that the cohomology groups are dual to the homology groups. Moreover, the dual of the tensor product is the tensor product of the duals as one of the factor is finite dimensional: we may compute $\mathrm{Hom}(V \otimes W, k) \cong \mathrm{Hom}(\oplus_i k \otimes W, k) \cong \oplus_i \mathrm{Hom}(W, k) \cong V^* \otimes W^*$ if $V \cong \oplus_i k$ is a finite sum.

It follows that EZ also induces an isomorphism on cohomology. But EZ is exactly the map (unique up to chain homotopy) that we used to induce the cross product. $\square$

The Künneth theorem gives us a few more non-trivial cup products:

Example 5.8. Consider a product of spheres, $X = \mathbb{S}^n \times \mathbb{S}^m$ with $n, m \geqslant 1$. By Theorem 5.7 we have

$$H^*(\mathbb{S}^n \times \mathbb{S}^m) \cong H^*(\mathbb{S}^n) \otimes H^*(\mathbb{S}^m).$$

We have three additive generators

$$\alpha_n = \nu_n \times 1, \beta_m = 1 \times \nu_m, \quad \text{and } \gamma_{n+m} = \nu_n \times \nu_m.$$

The square α_n^2 is trivial:

$$\alpha_n^2 = (\nu_n \times 1) \cup (\nu_n \times 1) = (\nu_n \cup \nu_n) \times (1 \cup 1) = 0.$$

Similarly, $\beta_m^2 = 0 = \gamma_{n+m}^2$. But the products

$$\alpha_n \cup \beta_m = \nu_n \times \nu_m = \gamma_{n+m}, \beta_m \cup \alpha_n = (-1)^{mn} \gamma_{n+m}$$

are non-trivial.

This determines the ring structure of $H^*(\mathbb{S}^n \times \mathbb{S}^m)$. In particular, the cohomology ring $H^*(\mathbb{S}^n \times \mathbb{S}^m)$ is *not* isomorphic to the cohomology ring $H^*(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m})$, which has trivial products by Lemma 4.13. Additively, both graded abelian groups are isomorphic, thus the graded cohomology ring is a finer invariant than the cohomology groups.

6. Cap product

The rough idea of the cap product is to digest a piece of a chain with a cochain of smaller or equal degree.

Definition 6.1. Let R be an associative ring with unit. We define

$$\cap \colon S^q(X, A; R) \otimes S_n(X, A; R) = \mathrm{Hom}(S_q(X, A), R) \otimes S_n(X, A) \otimes R \longrightarrow S_{n-q}(X) \otimes R = S_{n-q}(X; R)$$

using the Kronecker pairing and the Alexander-Whitney diagonal as

$$\beta \cap (a \otimes r) := F^{n-q}(a) \otimes \langle \beta, R^q(a) \rangle r$$

for $a \colon \Delta^n \to X$ and extend the definition linearly to $S_n(X, A; R)$.

This definition does indeed make sense: we claim that $\beta \cap a$ is a well-defined chain in $S_*(X)$, not just in $S_*(X, A)$. But if we modify a by adding a chain a' taking values in A it will not affect $\beta \cap a$ as β vanishes on all faces of a'.

Here we recall that $(n-q)$-dimensional front face of a is

$$F^{n-q}(a) = \partial_{n-q+1} \circ \ldots \circ \partial_n(a).$$

Similarly,

$$R^q(a) = \partial_0 \circ \ldots \circ \partial_0(a)$$

where ∂_0 is repeated $n-q$ times.

Analogously with the cup product we may also express this for a general EZ map as $(\mathrm{id} \otimes \beta) \circ EZ \circ \Delta(a)$. The map $\cap$ is well-defined: for $a = a' \in S_n(X,A)$, *i.e.*, $a = a' + b$ with $\mathrm{im}(b) \subset A$ we get

$$\beta \cap (a \otimes r) = \beta \cap ((a' + b) \otimes r) = \beta \cap (a' \otimes r) + F(b) \otimes \langle \beta, R(b) \rangle r.$$

The image of $R(b)$ is contained in A, but $\beta \in \mathrm{Hom}(S_q(X,A), R)$, thus $\beta \colon S_q(X) \to R$ with $\beta|_{S_q(A)} = 0$ and $\langle \beta, R(b) \rangle = 0$.

Proposition 6.2. *There is a Leibniz formula for the cap product, i.e. for $\beta \in S^q(X,A;R)$ and $a \in S_n(X,A)$ we have*

$$\partial(\beta \cap (a \otimes r)) = (-1)^{n-q}(\delta\beta) \cap (a \otimes r) + \beta \cap (\partial a \otimes r)$$

For the proof we suppress the tensor product with R. It just adds to notational complexity.

Proof. We check the equation $\partial(\beta \cap (a \otimes r)) + (-1)^{n-q+1}(\delta\beta) \cap (a \otimes r) = \beta \cap (\partial a \otimes r))$. For this we consider

$$\partial(\beta \cap a) = \partial(F^{n-q}(a) \otimes \langle \beta, R^q(a) \rangle) = \partial(F^{n-q}(a)) \otimes \langle \beta, R^q(a) \rangle \tag{6.1}$$

and

(6.2)
$$(-1)^{n-q+1}(\delta\beta) \cap a = (-1)^{n-q+1} F^{n-(q+1)}(a) \otimes \langle \delta\beta, R^{q+1}(a) \rangle = (-1)^{n-q+1} F^{n-(q+1)}(a) \otimes \langle \beta, \partial R^{q+1}(a) \rangle$$

Finally,

$$\begin{aligned}
\beta \cap \partial a &= \sum_{j=0}^{n} (-1)^j \beta \cap \partial_j a \\
&= \sum_{j=0}^{n} (-1)^j F^{n-1-q}(\partial_j a) \otimes \langle \beta, R^q(\partial_j a) \rangle \\
&= \sum_{j=0}^{n} (-1)^j \partial_{n-q-2} \cdots \partial_{n-1} \circ \partial_j a \otimes \langle \beta, \partial_0^{n-q} \partial_j a \rangle.
\end{aligned}$$

We examine the summands of this last expression in turn and distinguish cases. If $j \leqslant n-q-2$ we use that $\partial_{i-1}\partial_j = \partial_j \partial_i$ for $i > j$ to show that the summand is

$$(-1)^j \partial_j \partial_{n-q} \cdots \partial_n a \otimes \langle \beta, \partial_0^{n-q} a \rangle = (-1)^j \partial_j F^{n-q} a \otimes \langle \beta, R^q(a) \rangle$$

so we recover exactly the summands of equation 6.1.

If $j \geqslant n-q-1$ we use that $\partial_i \partial_j = \partial_{j-1}\partial_i$ as $i < j$, and after relabelling $j' = j - (n-q-1)$ the summand is

$$(-1)^{j'+(n-q-1)} \partial_{n-q-1} \cdots \partial_n(a) \otimes \langle \beta, \partial_{j'} \partial_0^{n-q-1}(a) \rangle = (-1)^{n-q-1}(-1)^{j'} F^{n-q-1}(a) \otimes \langle \beta, \partial_{j'} R^{q+1}(a) \rangle.$$

and thus we find the summands of equation 6.2.

$\square$

Proposition 6.3. *For a map of pairs of spaces $f\colon (X,A) \to (X,B)$ and classes $a \in H_*(X,A)$, $\beta \in H^*(Y,B)$ we have*

$$f_*(f^*(\beta) \cap (a \otimes r)) = \beta \cap (f_*(a) \otimes r)$$

where $f_*\colon S_*(X,A) \to S_*(Y,B)$ and $f^*\colon S^*(Y,B) \to S^*(X,A)$.

Proof. We plug in the definitions (skipping r for legibility) and obtain

$$\begin{aligned} f_*(f^*(\beta) \cap a) &= f_*(F(a) \otimes \langle f^*\beta, R(a)\rangle) \\ &= f_*(F(a) \otimes \langle \beta, f_*R(a)\rangle) \\ &= F(f_*(a)) \otimes \langle \beta, R(f_*(a))\rangle) \\ &= \beta \cap f_*(a) \end{aligned}$$

as F and R are natural. □

Proposition 6.4. *The cap product induces a map*

$$\cap\colon H^q(X,A;R) \otimes H_n(X,A;R) \longrightarrow H_{n-q}(X;R)$$

via

$$[\beta] \cap [a] := [F(a) \otimes \langle \beta, R(a)\rangle]$$

This defines an action of the graded ring $H^*(X,A;R)$ on the graded R-module $H_*(X,A;R)$.

Here a graded module $M = \oplus M^i$ over a graded ring $R = \oplus R^j$ is an R-module M satisfying $R^j.M^i \subset M^{i+j}$. The sign arises as H^* is graded cohomologically while H_* is graded homologically. To put them on the same footing we should consider the degree q cohomology as living in homological degree $-q$.

Proof. From the Leibniz formula we get that the cap product satisfies that

- a cocycle cap a cycle is a cycle,
- a cocycle cap a boundary is a boundary,
- a coboundary cap a cycle is a boundary.

This implies the first result.

Next consider $1 \in S^0(X;R)$, i.e. $1(a) = 1$ for all $a\colon \Delta^0 \to X$. We claim that $1 \cap a = a$. We have $F(a) = a$ because $q = 0$ and $R(a)(e_0) = a(e_n)$. Therefore, $1 \cap a = a \otimes \langle 1, a(e_n)\rangle = a \otimes 1$ and we identify the latter with a.

For the associativity we compute that $(\alpha \cup \beta) \cap c$ and $(\alpha \cap (\beta \cap c))$ are both given by $\alpha(F^p(c))\beta(M^q(c))R^{n-p-q}(c)$ when $|\alpha| = p$, $|\beta| = q$ and $|c| = n$, and $M^q(c)$ denotes the "middle face" again. □

The cap product also interacts well with the Kronecker product:

Proposition 6.5. *Far $\alpha \in H^p(X), \beta \in H^q(X)$ and $c \in H_{p+q}(X)$ we have*

$$\langle \alpha \cup \beta, c\rangle = \langle \alpha, \beta \cap c\rangle.$$

Note that if $\alpha = 1$ this says $\langle \beta, c\rangle = \beta \cap c$.

Proof. Both sides are equal to $\alpha(F^p(c)).\beta(R^q(c))$. □

Example 6.6. Let us consider a non-trivial example. So let T be a torus, take a 1-chain given by a meridian $b \subset T$ and another 1-chain given by the longitude a. We also consider a 1-cocycle given by $\beta \in H^1(T)$ that is dual to $[b] \in H_1(T)$, so that $\beta(a) = 0$.

Let c be a generator of $H_2(T)$. Our first guess might be a surjection from Δ^2 to the square such that $\partial_0\Delta^2$ is the vertical and $\partial_2\Delta^2$ is the horizontal edge. However, this is not a cycle, as ∂_1 is not equal to $\partial_0 + \partial_2$. So instead we cover the square with two triangles and take their difference as our 2-chain. It is a cycle and for degree reasons cannot be a boundary. So we have $c = x - y$ and $\partial_0(x) = a = \partial_2(y)$ and $\partial_2(x) = b = \partial_0(y)$. (Here $\partial_1(x) = \partial_1(y)$ is the diagonal.)

Then we identify edges to obtain the torus such that the the vertical edge is the meridian and the horizontal edge is the longitude.

Then $\beta \cap c = \beta \cap x - \beta \cap y$ is $\langle \beta, \partial_2(x)\rangle \partial_0(x) - \langle \beta, \partial_2(y)\rangle \partial_0(y) = 1.a - 0$.

Thus $\beta \cap c$ is exactly the longitude, transversal to $b = \beta^*$. The cap notation is reminiscent of the symbol $\pitchfork$ that denotes transversality.

One can compute that similarly $\alpha \cap c = -b$. Thus the computation also takes account of orientation of the intersection.

Remark 6.7. An alternative notation for $F^q(c)$ is $c|_{0\dots q}$, indicating the restriction of the simplex to the subsimplex spanned by the first $q+1$ vertices. Similarly $R^q(c) = c|_{(n-q+1)\dots n}$.

7. Suspensions

We recall the following constructions:

Definition 7.1. Let X be a topological space. Then the *cone on* X, denoted by CX is defined as $X \times [0,1]/X \times \{1\}$.

The *(free) suspension of* X, denoted by SX is defined as $X \times [0,1]/\sim$ where $\sim$ identifies $X \times \{1\}$ to a point and $X \times \{0\}$ to point.

The *reduced suspension* of a pointed space (X, x_0) is defined as

$$(X \times [0,1])/(X \times \{0\} \cup \{x_0\} \times [0,1] \cup X \times \{1\})$$

i.e. it is the quotient of SX where we also identify $\{x_0\} \times [0,1]$ to a point.

If (X, x_0) is a good pair then one can show there is a homotopy equivalence $\Sigma X \simeq SX$.

We will mostly talk about the free suspension in this course. The suspension can also be written as the colimit of $* \amalg * \leftarrow X \amalg X \rightarrow X \times [0,1]$.

It is clear that CX is contractible and that $SX = CX \amalg_X CX$.

Example 7.2. For any n we have $S\mathbb{S}^n \cong \mathbb{S}^{n+1}$.

Theorem 7.3 (Suspension isomorphism). *If $A \subset X$ is a good pair then for all $n > 0$*

$$\widetilde{H}_n(SX, SA) \cong \tilde{H}_{n-1}(X, A), \quad \textit{and} \quad \widetilde{H}^n(SX, SA) \cong \tilde{H}^{n-1}(X, A)$$

Proof. We prove the result for homology, the proof for cohomology is identical.

Picking open neighbourhoods of the two copies of $CX \subset SX$, e.g. the images of $X \times (\frac{1}{3}, 1]$ and $X \times [0, \frac{2}{3})$, we obtain from the Mayer-Vietoris sequence on reduced homology that $\delta : \tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X)$ for all n, i.e. the boundary map provides an isomorphism.

The same is true for $A \subset X$ and for the relative case we apply the 9-Lemma 9.6 to the quotient of the following short exact sequences of complexes

$$\begin{tikzcd} 0 \arrow[r] & C_*A \arrow[r] \arrow[d] & C_*(CA) \oplus C_*(CA) \arrow[r] \arrow[d] & C_*(SA) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_*X \arrow[r] & C_*(CX) \oplus C_*(CX) \arrow[r] & C_*(SX) \arrow[r] & 0 \end{tikzcd}$$

to obtain a short exact sequence of chain complexs

$$0 \to C_*(X, A) \to C_*(CX)/C_*(CA) \oplus C_*(CX)/C_*(CA) \to C_*(SX, SA) \to 0.$$

Here we use CX and CA to mean the open neughbourhoods for better legibility. As $C_*(CX)/C_*(CA)$ is acyclic the result follows from the long exact sequence on homology. $\square$

Remark 7.4. Note, that the corresponding statement is terribly wrong for homotopy groups. We have $S\mathbb{S}^2 \cong \mathbb{S}^3$, but $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, whereas $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, so homotopy groups (unlike homology groups) don't satisfy such an easy form of a suspension isomorphism. There is a Freundenthal suspension theorem for homotopy groups, but that's more complicated. For the above case it yields:

$$\mathbb{Z}/2\mathbb{Z} \cong \pi_{1+3}(\mathbb{S}^3) \cong \pi_{1+4}(\mathbb{S}^4) \cong \ldots =: \pi_1^s$$

where π_1^s denotes the first stable homotopy group of the sphere.

The suspension construction is in fact functorial and if $f \colon \mathbb{S}^n \to \mathbb{S}^n$ is continuous, then $S(f) \colon S\mathbb{S}^n \to S\mathbb{S}^n$ is given as $S\mathbb{S}^n \ni [x, t] \mapsto [f(x), t]$.

Lemma 7.5. *Suspensions leave the degree invariant,* i.e., *for $f \colon \mathbb{S}^n \to \mathbb{S}^n$ we have*

$$\deg(S(f)) = \deg(f).$$

Proof. The suspension isomorphism of Theorem 7.3 is induced by a connecting homomorphism. Using the isomorphism $H_{n+1}(\mathbb{S}^{n+1}) \cong H_{n+1}(S\mathbb{S}^n)$, the connecting homomorphism sends $\mu_{n+1} \in H_{n+1}(\mathbb{S}^{n+1})$ to $-\mu_n \in \tilde{H}_n(\mathbb{S}^n)$ by definition. But then the commutativity of

$$\begin{tikzcd} H_{n+1}(\mathbb{S}^{n+1}) \arrow[r, "\cong"] & H_{n+1}(S\mathbb{S}^n) \arrow[r, "H_{n+1}(Sf)"] \arrow[d, "\delta"'] & H_{n+1}(S\mathbb{S}^n) \arrow[d, "\delta"] & H_{n+1}(\mathbb{S}^{n+1}) \arrow[l, "\cong"'] \\ & \tilde{H}_n(\mathbb{S}^n) \arrow[r, "H_n(f)"] & \tilde{H}_n(\mathbb{S}^n) & \end{tikzcd}$$

ensures that $\deg(f)\delta\mu_{n+1} = \delta\deg(Sf)\mu_n$, which becomes $-\mathrm{def}(f)\mu_n = -\deg(Sf)\mu_n$. $\square$

This gives another proof that for every $k \in \mathbb{Z}$ and $n \geqslant 1$ there is an $f \colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = k$. We just define the k-fold loop on $\mathbb{S}^1$ and suspend it $n - 1$ times.

Finally we note that suspension immediately kills all cup products:

Proposition 7.6. *The cup product structure on SX is trivial for any toplogical space X.*

Proof. This follows immediately from Lemma 4.12 as CX is contractible. $\square$

Note that the cohomology rings of $S(\mathbb{S}^n \times \mathbb{S}^m)$ and $S(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m})$ *are* isomorphic (namely here cup products of elements of positive degree are trivial due to Proposition 7.6. You may wonder if

$$S(\mathbb{S}^n \times \mathbb{S}^m) \simeq S(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m}).$$

8. Orientability of manifolds

DEFINITION 8.1. A topological space X is called *locally euclidean*, if every point $x \in X$ has an open neighborhood U which is homeomorphic to an open subset $V \subset \mathbb{R}^m$.

- A homeomorphism $\varphi\colon U \to V$ is called a *chart*.
- A set of charts is called *atlas*, if the corresponding $U \subset X$ cover X.
- The number m is *the dimension of* X if it is independent of x, for example if X is connected.

EXAMPLE 8.2. Consider the line with two origins, i.e. let

$$X = \{(x,1) | x \in \mathbb{R}\} \cup \{(x,-1) | x \in \mathbb{R}\}/\sim, \quad (x,1) \sim (x,-1) \text{ for } x \neq 0.$$

Then X is locally euclidean, but X is not a particularly nice space. For instance, it is not Hausdorff: you cannot separate the two origins.

DEFINITION 8.3. A topological space X is an *m-dimensional (topological) manifold* (or m-manifold for short) if X is a locally euclidean space of dimension m that is Hausdorff and has a countable basis for its topology.

With this definition, topological manifolds are paracompact: any open cover has a locally finite refinement.

EXAMPLE 8.4.
(a) Let $U \subset \mathbb{R}^m$ an open subset, then U is a topological manifold of dimension m.
(b) The n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is an n-manifold and $\mathbb{S}^n = (\mathbb{S}^n \setminus N) \cup (\mathbb{S}^n \setminus S)$ is an atlas of $\mathbb{S}^n$.
(c) The 2-dimensional torus $T \cong \mathbb{S}^1 \times \mathbb{S}^1$ is a 2-manifold and more generally, the surfaces F_g are 2-manifolds. Charts can be easily given via the $4g$-gon whose quotient F_g is.
(d) The open Möbius strip $[-1,1] \times (-1,1)/\sim$ with $(-1,t) \sim (1,-t)$ is a 2-manifold.
(e) For $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$ let $d = 1, 2, 4$ respectively. The projective space kP^n defined in Example 12.8 is a manifold of dimension dn. The open sets $U_i \subset kP^n$ defined by $[x_0, \ldots, x_n]$ with $x_i \neq 0$ in projective coordinates provide a chart as $U_i \cong k^n \cong \mathbb{R}^{dn}$.

Let M be a connected manifold of dimension $m \geqslant 2$. We denote the open charts by $U_\alpha \subset M$. Without loss of generality we can assume that

$$\varphi\colon U_\alpha \cong \mathring{\mathbb{D}}^m \subset \mathbb{R}^m$$

and for an $x \in M$ we can choose charts with $\varphi(x) = 0$. Excision tells us that for all $x \in M$

$$H_m(M, M \setminus x) \cong H_m(\mathring{\mathbb{D}}^m, \mathring{\mathbb{D}}^m \setminus \{0\}) \cong H_{m-1}(\mathring{\mathbb{D}}^m \setminus \{0\}) \cong \mathbb{Z}$$

for $m \geqslant 2$.

For a triple $B \subset A \subset M$ there are maps of pairs

$$\varrho_{B,A}\colon (M, M \setminus A) \longrightarrow (M, M \setminus B).$$

Definition 8.5. An m-manifold M is *orientable* (with respect to $\mathbb{Z}$) if there is a coherent choice of generators $o_x \in H_m(M, M \setminus x)$, i.e. for all $x \in M$ there is an open neighbourhood U of x and a class $o_U \in H_m(M, M \setminus U)$ such that for all $y \in U$ we have that $(\varrho_{y,U})_* o_U = o_y$.

Note that this implies that for all $x, y \in U$ we have the compatibility condition

$$o_y = \varrho_{xy,U} \circ (\varrho_{x,U})^{-1}(o_x).$$

$$\begin{tikzcd}
 & H_m(M, M \setminus U) \arrow[dl, "\varrho_{x_1,U}"'] \arrow[dr, "\varrho_{y,U}"] & \\
o_x \in H_m(M, M \setminus x) & & H_m(M, M \setminus y) \ni o_y
\end{tikzcd}$$

Definition 8.6. If such a choice is possible, then $(o_x | x \in M)$ is an *orientation of* M.

Note that for an orientation $(o_x | x \in M)$ the family $(-o_x | x \in M)$ is an orientation of M as well.

Example 8.7. Let M be an open Möbius strip and x a point on it. We pick a generator $o_x \in H_2(M, M \setminus x)$ and walk once around the Möbius strip, always picking compatble orientations in $H_2(M, M \setminus y)$ as y moves along the meridian of M. After one circle around the Möbius strip we end up at $-o_x$.

If we choose other coefficients, these problems can disappear. For instance for $G = \mathbb{Z}/2\mathbb{Z}$ there is no choice in local generators, and thus there is automatically a choice of coherent generators for $H_2(M, M \setminus x; \mathbb{Z}/2\mathbb{Z})$ for any manifold M.

Now, we consider integral coefficients again. The easiest way to get an orientation is to have a global class $o_M \in H_m(M; \mathbb{Z}) = H_m(M)$. Then with

$$\varrho_{x,M} =: \varrho_x \colon H_m(M) \to H_m(M, M \setminus x), \quad \varrho_x(o_M) = o_x$$

we have that $(o_x | x \in M)$ is an orientation of M – provided that ρ_x is injective everywhere.

Example 8.8. If $M = \mathbb{R}P^2$, then $H_2(\mathbb{R}P^2) = 0$, but $H_2(\mathbb{R}P^2, \mathbb{R}P^2 \setminus x) \cong \mathbb{Z}$, so here we cannot have such a class. We will show later that in fact there is no orientation on $\mathbb{R}P^2$.

Definition 8.9. Let $K \subset M$ be a compact subset of M. We call an $o_K \in H_m(M, M \setminus K)$ an *orientation of* M *along* K, if the classes $o_x := (\varrho_{x,K})_*(o_K)$ constitute a coherent choice of generators for all $x \in K$. Here $\rho_{x,K} : (M, M \setminus K) \to (M, M \setminus x)$ is the natural restriction.

Of course, if we have a global class $o_M \in H_m(M)$ then we get coherent generators o_x for all $x \in M$ and also a class o_K as above for all compact $K \subset M$.

Lemma 8.10. *Let M be a connected topological manifold of dimension m and assume that M is orientable. Let $K \subset M$ be compact. Then*

(i) $H_q(M, M \setminus K) = 0$ for all $q > m$, and

(ii) if $a \in H_m(M, M \setminus K)$, then a is trivial if and only if $(\varrho_{x,K})_*(a) = 0$ for all $x \in K$.

The following method of proof is a standard method in the theory of manifolds.

Proof.

(a) First, let $M = \mathbb{R}^m$ and let K be convex (and thus in particular contractible) and compact in M. In this case we can assume without loss of generality that $K \subset \mathring{\mathbb{D}}^m$. We calculate

$$H_q(M, M \setminus K) = H_q(\mathbb{R}^m, \mathbb{R}^m \setminus K) \cong H_q(\mathring{\mathbb{D}}^m, \mathring{\mathbb{D}}^m \setminus x) = 0, \text{ for } q > m.$$

All identifications are isomorphisms and this gives the second claim as well.

(b) Let M be again $\mathbb{R}^m$ and let $K = K_1 \cup K_2$ with K_1, K_2 as in (a). In this case the claims follow with the help of the relative Mayer-Vietoris sequence (Theorem 9.7):

$$H_{q+1}(M, M \setminus K_0) \longrightarrow H_q(M, M \setminus K) \xrightarrow{i} H_q(M, M \setminus K_1) \oplus H_q(M, M \setminus K_2) \xrightarrow{\kappa} H_q(M, M \setminus K_0) \longrightarrow \ldots$$

where $K_0 = K_1 \cap K_2$. Here K_1, K_2 and $K_1 \cap K_2$ satisfy the assumptions as in (a) and we can deduce (i) from the exact sequence $0 \to H_q(M, M \setminus K) \to 0$.

To show (ii) consider a class a in $H_m(M, M \setminus K)$. By the exact sequence it is 0 if $\rho_{K_1,K}(a) = \rho_{K_2,K}(a) = 0$, and by (a) this is the case if and only if $\rho_{x,K}(a) = 0$ for all $x \in K$.

(c) An induction shows the case of $M = \mathbb{R}^m$ and $K = K_1 \cup \ldots \cup K_r$ with K_i as in (a).

(d) Let $M = \mathbb{R}^m$ and let K be an arbitrary compact subset and let $a \in H_q(M, M \setminus K)$ with $q > m$. Choose a $\psi \in S_q(\mathbb{R}^m)$ representing the class a. The boundary of ψ, $\partial(\psi)$, has to be of the form

$$\partial(\psi) = \sum_{j=1}^{\ell} \lambda_j \tau_j$$

with $\tau_j \colon \Delta^{q-1} \to \mathbb{R}^m \setminus K$. As Δ^{q-1} is compact, the union

$$\bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1}) \subset \mathbb{R}^m \setminus K$$

is compact.

There exists an open neighborhood U of K in $\mathbb{R}^m$ with

$$\bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1}) \cap U = \varnothing.$$

Therefore ψ gives a cycle in $S_*(\mathbb{R}^m, \mathbb{R}^m \setminus U)$ and we let $a' \in H_q(\mathbb{R}^m, \mathbb{R}^m \setminus U)$ be the corresponding class. Thus

$$(\varrho_{K,U})_*(a') = a.$$

Choose closed balls $B_1, \ldots, B_r \subset \mathbb{R}^m$ with $B_i \subset U$ for all i and $K \cap B_i \neq \varnothing$ such that $K \subset \bigcup_{i=1}^r B_i$. Consider the restriction maps

$$(\mathbb{R}^m, \mathbb{R}^m \setminus U) \xrightarrow{\varrho_{\bigcup B_i, U}} (\mathbb{R}^m, \mathbb{R}^m \setminus \textstyle\bigcup_{i=1}^r B_i) \xrightarrow{\varrho_{K, \bigcup B_i}} (\mathbb{R}^m, \mathbb{R}^m \setminus K).$$

Define a'' as $a'' := (\varrho_{\bigcup B_i, U})_*(a')$. Note that $(\varrho_{K, \bigcup B_i})_*(a'') = a$.

The B_i are convex and compact and therefore

$$(\varrho_{\bigcup B_i, U})_*(a') = 0 = a'', \text{ for all } q > m$$

and hence $a = 0$, showing (i).

To show (ii) let $q = m$ and assume that $(\varrho_{x,K})_*(a) = 0$ for all $x \in K$. We have to show that a is trivial. We express $(\varrho_{x,K})_*(a)$ as above as

$$(\varrho_{x,K})_*(a) = (\varrho_{x,K})_* \circ (\varrho_{K,\bigcup B_i})_*(a'') = (\varrho_{x,\bigcup B_i})_*(a'') = 0$$

for all $x \in K$. For every $x \in B_i \cap K$ the above composition is equal to $(\varrho_{x,B_i})_* \circ (\varrho_{B_i,\bigcup B_i})_*(a'')$, but $(\varrho_{x,B_i})_*$ is an isomorphism and hence $(\varrho_{B_i,\bigcup B_i})_*(a'') = 0$. This implies $(\varrho_{y,B_i})_* \circ (\varrho_{B_i,\bigcup B_i})_*(a'') = 0$ for all $y \in B_i$ and in addition $(\varrho_{y,\bigcup B_i})_*(a'') = 0$ for all $y \in \bigcup B_i$. According to case (c) this implies that $a'' = 0$ and therefore $a = (\varrho_{K,\bigcup B_i})_*(a'')$ is trivial as well.

(e) Now let M be arbitrary and suppose that K is contained in a domain of a chart $K \subset U_\alpha \cong \mathbb{R}^m$. Therefore

$$H_q(M, M \setminus K) \cong H_q(U_\alpha, U_\alpha \setminus K) \cong H_q(\mathbb{R}^m, \mathbb{R}^m \setminus \mathrm{im}(K)).$$

As the image of K is compact in $\mathbb{R}^m$, the claim follows from (d).

(f) If M and K are arbitrary, then $K = K_{\alpha_1} \cup \ldots \cup K_{\alpha_r}$ with $K_{\alpha_i} \subset U_{\alpha_i}$. (Proving this decomposition is an exercise in non-algebraic topology.) An induction as in (c) then proves the claim. □

PROPOSITION 8.11. *Let $K \subset M$ be compact and assume that M is connected and oriented with $(o_x \in H_m(M, M \setminus x) \mid x \in M)$. Then there is a unique orientation of M along K, which is compatible with the orientation of M,* i.e., *there is a class $o_K \in H_m(M, M \setminus K)$ such that $(\varrho_{xK})_*(o_K) = o_x$ for all $x \in K$.*

PROOF. First we show uniqueness. Let o_K and $\tilde{o}_K$ be two orientations of M along K. By assumption we have that

$$(\varrho_{xK})_*(o_K) - (\varrho_{xK})_*(\tilde{o}_K) = (\varrho_{xK})_*(o_K - \tilde{o}_K) = 0.$$

According to Lemma 8.10 this is only the case if $o_K - \tilde{o}_K = 0$.

In order to prove existence we first consider the case where $K \subset U_\alpha \cong \mathring{\mathbb{D}}^m$ and hence $M \setminus U_\alpha \subset M \setminus K$. Let $x \in K$. We denote the isomorphism $H_m(M, M \setminus U_\alpha) \cong H_m(M, M \setminus x)$ by ϕ.

We define o_K as

$$o_K := (\varrho_{K,U_\alpha})_*((\phi^{-1})(o_x)).$$

For $K = K_1 \cup K_2$ with K_i contained in the source of a chart we get that o_{K_1} and o_{K_2} exist. Let $K_0 = K_1 \cap K_2$ and consider the Mayer-Vietoris sequence

$$0 \longrightarrow H_m(M, M \setminus K) \xrightarrow{i} H_m(M, M \setminus K_1) \oplus H_m(M, M \setminus K_2) \xrightarrow{\kappa} H_m(M, M \setminus K_0) \longrightarrow \ldots$$

The uniqueness of the orientation along K_0 implies that

$$\kappa(o_{K_1}, o_{K_2}) = (\varrho_{K_0,K_1})_*(o_{K_1}) - (\varrho_{K_0,K_2})_*(o_{K_2}) = 0.$$

Therefore there is a unique class $o_K \in H_m(M, M \setminus K)$ with $i(o_K) = (o_{K_1}, o_{K_2})$.

For the last case we consider a compact subset K and we know that $K = K_1 \cup \ldots \cup K_r$ with $K_i \subset U_{\alpha_i}$. An induction then finishes the proof. □

THEOREM 8.12. *Let M be a connected and compact manifold of dimension m. The following are equivalent*

(a) M is orientable,

(b) there is an orientation class $o_M \in H_m(M;\mathbb{Z})$*,*
(c) $H_m(M;\mathbb{Z}) \cong \mathbb{Z}$*.*

PROOF. Proposition 8.11 yields that (a) implies (b). Now assume that (b) holds, thus there is a class $o_M \in H_m(M)$ restricting to the local orientation classes o_x. Then the class o_M satisfies, that o_M is not trivial, because its restriction $(\varrho_{x,M})_* o_M = o_x$ is a generator and hence non-trivial. Furthermore, o_M cannot be of finite order: if $ko_M = 0$, then this would imply $ko_x = 0$ for all $x \in M$ contradicting the generating property of the o_x. Let $a \in H_m(M)$ be an arbitrary element. Thus $(\varrho_{x,M})_*(a) = ko_x$ for some integer k. As the o_x are coherent in x, this k has to be constant and if we set $b := ko_M - a$ then $(\varrho_{x,M})_* b = 0$ for all x and this implies that $b = 0$. Therefore $a = ko_M$, thus every element in $H_m(M)$ is a multiple of o_M and $H_m(M) \cong \mathbb{Z}$.

Assuming (c) there are two possible generators in $H_m(M)$. Choose one of them and call it o_M. Then we claim that $((\varrho_{x,M})_* o_M \mid x \in M)$ is an orientation of M. To show this we first need to know that ρ_x is an injection. From the long exact sequence in relative homology the kernel is given by $H_n(M \setminus x;\mathbb{Z})$ and must be 0 or $\mathbb{Z}$. But it follows from Corollary 14.6 below that $H_n(M \setminus x;\mathbb{Z}/2) = 0$, and this is impossible if $H_n(M \setminus x;\mathbb{Z}) = \mathbb{Z}$. Note that we will use the first implication of this theorem to prove that corollary, but not this implication!

Next we need surjectivity, so let us consider $(\varrho_{x,M})_* o_M) \mid x \in M) = (k_x o_x) \mid x \in M)$ for some collection k_x. But k_x is locally constant on M and thus constant, and if there is a coherent choice $(ko_x \mid x \in M)$ it follows that $(o_x \mid x \in M)$ is a coherent choice of orientation. □

The o_M as in Theorem 8.12 is also called *fundamental class of* M and is often denoted by $[M] = o_M$.

EXAMPLE 8.13. For the m-sphere, $M = \mathbb{S}^m$ we can choose $\mu_m \in H_m(\mathbb{S}^m)$ as a generator, thus

$$[\mathbb{S}^m] = o_{\mathbb{S}^m} = \mu_m.$$

All results about orientations can be transferred to a setting with coefficients in a commutative ring R with unit 1_R.

- Then M is called R-orientable if and only if there is a coherent choice of generators $H_m(M, M \setminus x; R)$ for all $x \in M$.
- The results we had have formulations relative R: Lemma 8.10 goes through, and if M has an R-orientation $(o_x^R | x \in M)$, then for all compact $K \subset M$ there is an R-orientation of M along K, *i.e.*, a class $o_K^R \in H_m(M, M \setminus K; R)$ that restricts to the local classes. The R-version of Theorem 8.12 yields a class $o_M^R \in H_m(M;R)$ restricting to the o_x^R. The class o_M^R is then called the *fundamental class of* M *with respect to* R and is denoted by $[M;R]$.

If the manifold M is triangulated then there is a different way of viewing orientation: It is a coherent choice of orientation of all the n-simplices making up the manifold. An orientation of an n-simplex is a sign given by an ordering of all the vertices, and swapping two vertices changes the orientation.

Coherence just means that if we consider the simplicial n-chain given by the sum of all the n-simplices the boundary will contain each $(n-1)$-simplex twice. If it appears twice with opposite sign and cancels the sum of all n-simplices is an n-cycle, and as it can't be a boundary this generates simplicial $H_n(M)$.

If we have such a triangulated manifold with an orientation we may construct a dual cell decomposition: Each n-simplex becomes a 0-cell (the barycenter), each $(n-1)$-simplex that is a common face of two n-simplices becomes a 1-cell connecting the barycenters, an $(n-2)$-simplex becomes a 2-cell (things are already harder to visualize here, unless $n=2$). Note that this dual cell decomposition is not necessarily a triangulation.

What happens to the boundary operation? As we dualize the decomposition the homological operator $C_k \to C_{k-1}$ must become some operator from $(n-k)$-chains to $(n-k+1)$-chains, i.e. a coboundary operator.

One can thus turn the simplicial chain complex upside down and obtain a cellular cochain complex of the same manifold with a different cell decomposition. With some more work, this shows that if M is a compact R-oriented connected manifold then

$$H_p(M;R) \cong H^{n-p}(M;R).$$

This is *Poincaré duality.* We will prove it in a different way, that avoids triangulations (which are not unique and may not even exist) and allows for a number of generalizations.

9. Cohomology with compact support

Our setting for Poincaré duality is as follows: if M is a compact connected oriented manifold of dimension m, then taking the cap product with $[M] = o_M$ gives a map

$$(-) \cap o_M \colon H^q(M;R) \to H_{m-q}(M;R).$$

We want to show this is an isomorphism.

One of our best strategies for proving theorems has been to chop manifolds into open pieces and prove the result locally first. However, Poincaré duality as stated is visibly wrong for non-compact manifolds. Thus if we want to prove a local version of Poincaré duality, we first need to extend the statement to non-compact M. To this end we define the following notion.

Definition 9.1. Let X be an arbitrary topological space and let R be a commutative ring with unit 1_R. Then the *singular n-cochains with compact support of X* are

$$S_c^n(X;R) = \{\varphi \colon S_n(X) \to R \mid \exists K_\varphi \subset X \text{ compact} \colon \forall \sigma \colon \Delta^n \to X, \sigma(\Delta^n) \cap K_\varphi = \varnothing \quad \varphi(\sigma) = 0.\}$$

The *nth cohomology group with compact support of X with coefficients in R* is

$$H_c^n(X;R) := H^n(S_c^*(X;R)).$$

Note that $S_c^*(X;R) \subset S^*(X;R)$ is a sub-complex. This inclusion of complexes induces a map on cohomology

$$H_c^n(X;R) \longrightarrow H^n(X;R).$$

If X is compact, then we may pick $K_\phi = X$ for all ϕ and $H_c^n(X;R) \cong H^n(X;R)$ for all n.

Is there a map from singular cohomology to singular cohomology with compact support? Not in general, but there is a map in a relative setting. Let $K \subset X$ be compact. The restriction map

$$\varrho_{K,X} \colon (X, X \setminus X) = (X, \varnothing) \longrightarrow (X, X \setminus K)$$

induces a map

$$\varrho^n_{K,X}\colon S^n(X, X\setminus K; R) \longrightarrow S^n(X; R)$$

whose image is contained in $S^n_c(X; R)$: for a φ in the image there is a $\psi \in S^n(X, X\setminus K; R)$ with $\varrho^n_{K,X}(\psi) = \varphi$. The functional ψ is trivial on all simplices $\sigma\colon \Delta^n \to X$ with $\sigma(\Delta^n)\cap K = \varnothing$. Therefore,

$$\varphi(\sigma) = \varrho^n_{K,X}(\psi)(\sigma) = 0$$

for such σ.

LEMMA 9.2. *(a) For all compact $K \subset X$ the map $\varrho^*_{K,X}$ is a cochain map $S^*(X, X\setminus K; R) \longrightarrow S^*_c(X; R)$ and in particular we get an induced map*

$$H^*(\varrho_{K,X})\colon H^*(X, X\setminus K; R) \longrightarrow H^*_c(X; R).$$

(b) For compact subsets $K \subset L \subset X$ we have

$$\varrho_{K,L} \circ \varrho_{L,X} = \varrho_{K,X}$$

and therefore

$$\begin{tikzcd}
S^*(X, X\setminus K; R) \arrow[d, "\varrho^*_{K,L}"'] \arrow[dr, "\varrho^*_{K,X}"] & \\
S^*(X, X\setminus L; R) \arrow[r, "\varrho^*_{L,X}"'] & S^*_c(X; R)
\end{tikzcd}$$

commutes.

Lemma 9.2 says that there is a functor from the poset of compact subsets of K to the category of cochain complexes.

For $K \subset L \subset L'$ we have

$$\varrho^*_{K,L'} = \varrho^*_{L,L'} \circ \varrho^*_{K,L}.$$

Our index category also has the property that for compact K and L we can consider the inclusions $K \subset K \cup L$ and $L \subset K \cup L$, thus these maps meet again.

A poset with the special property that any two elements have a common bound is called a *directed set*. A functor from a directed set, viewed as a category, is called a *direct system*. So this is nothing but a special kind of diagram and we may take the colimit of this diagram, it is called the *direct limit* (even though it is a colimit), and denoted by $\varinjlim M_i$.

We recall some facts about direct limits of R-modules and (co)chain complexes of R-modules.

First we spell out the definitions: Let I be a partially ordered set which we consider as a diagram, i.e. for all $i < j$ there is a unique map $f_{ji}\colon i \to j$ and for $i = j$ we have $f_{ii} = \mathrm{id}_i$. all $i, j \in I$ there is a $k \in I$ with $i, j \leqslant k$.

Consider a functor from I to R-modules. Unravelling the definitions this means: Let M_i for $i \in I$ be a family of R-modules together with maps $f_{ji}\colon M_i \to M_j$ with $f_{kj} \circ f_{ji} = f_{ki}$ for $i \leqslant j \leqslant k$. Then we call $(M_i)_{i\in I}$ a *direct system*.

The direct limit is then the R-module that is determined (up to canonical isomorphism) by the following universal property: there are R-linear maps $h_i\colon M_i \to \varinjlim M_i$ such that for every family of R-module maps $g_i\colon M_i \to M$ that satisfy $g_j \circ f_{ji} = g_i$ for all $i \leqslant j$, there is a unique morphism of R-modules $g\colon \varinjlim M_i \to M$ such that $g \circ h_i = g_i$ for all $i \in I$.

For a direct system $(M_i, i \in I)$ of R-modules we can explicitly construct $\varinjlim M_i$ as

$$\varinjlim M_i = \left(\bigoplus_{i\in I} M_i\right)/U$$

where U is the submodule of $\bigoplus_{i\in I} M_i$ generated by all $m_i - f_{ji}(m_i), i \leqslant j$.

For (co)chain complexes the construction is similar. For a direct system of chain complexes $((C_i)_*)_{i\in I}$ we set

$$(\varinjlim(C_i))_n := \varinjlim((C_i)_n).$$

The boundary operators $d_i\colon (C_i)_n \to (C_i)_{n-1}$ induce a boundary map

$$d\colon (\varinjlim(C_i))_n \longrightarrow (\varinjlim(C_i))_{n-1}.$$

Let $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ be three direct systems of R-modules. If

$$0 \longrightarrow A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \longrightarrow 0$$

is a short exact sequence for all $i \in I$ and if $f^B_{ji} \circ \phi_i = \phi_j \circ f^A_{ji}$, $f^C_{ji} \circ \psi_i = \psi_j \circ f^B_{ji}$ for all $i \leqslant j$, then we call

$$0 \longrightarrow (A_i) \xrightarrow{(\phi_i)} (B_i) \xrightarrow{(\psi_i)} (C_i) \longrightarrow 0$$

a short exact sequence of direct systems.

Lemma 9.3. *(a) If*

$$0 \longrightarrow (A_i) \xrightarrow{(\phi_i)} (B_i) \xrightarrow{(\psi_i)} (C_i) \longrightarrow 0$$

is a short exact sequence of directed systems of R-modules, then the sequence of R-modules

$$0 \to \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \to 0$$

is short exact.

(b) If $(A_i)_{i\in I}$ is a directed system of chain complexes, then

$$\varinjlim H_m(A_i) \cong H_m(\varinjlim A_i).$$

Proof. The maps $\phi_i\colon A_i \to B_i$ give – via composition with $h_i\colon B_i \to \varinjlim B_i$ – maps $A_i \to \varinjlim B_i$ and by the universal property this yields a unique map

$$\phi\colon \varinjlim A_i \longrightarrow \varinjlim B_i.$$

One has to show that i) ϕ is injective, ii) the kernel of ψ is the image of ϕ and iii) ψ is surjective.

We show i) and leave ii) and iii) as an exercise.

Let $a \in \varinjlim A_i$ with $\phi(a) = 0 \in \varinjlim B_i$. Write $a = [\sum_{j=1}^n \lambda_j a_j]$ with $a_j \in A_{i_j}$. Choose $k \geqslant i_1, \ldots, i_n$, then $a = [a_k]$ for some $a_k \in A_k$, using the definition of the direct limit as a quotient. (The inedex k exists as I is directed.). By assumption $\phi(a) = [\phi_k(a_k)] = 0$. Thus there is an $N \geqslant k$ with $f_{Nk}\phi_k(a_k) = 0$ and by the coherence of the maps ϕ_k we have $0 = f_{Nk} \circ \phi_k(a_k) = \phi_N \circ f_{Nk}(a_k)$. But ϕ_N is a monomorphism and therefore $f_{Nk}(a_k) = 0 \in \varinjlim A_i$, hence $a = [a_k] = [f_{Nk}(a_k)] = 0$.

For (b) we fix m and apply (a) to the short exact sequences $0 \to B_m(A_i) \to Z_m(A_i) \to H_m(A_i) \to 0$. $\square$

We can use this algebraic result to approximate singular cohomology with compact support via relative singular cohomology groups.

PROPOSITION 9.4. *For all spaces X we have*

$$\varinjlim S^*(X, X \setminus K; R) \cong S_c^*(X; R)$$

and hence

$$\varinjlim H^*(X, X \setminus K; R) \cong H_c^*(X; R).$$

Here the directed system runs over the poset of compact subsets $K \subset X$.

PROOF. A cochain $\varphi \in S^n(X; R)$ is an element of $S_c^n(X; R)$ if and only if there is a compact $K = K_\varphi$ such that $\varphi(\sigma) = 0$ for all σ with $\sigma(\Delta^n) \cap K = \varnothing$ and this is the case if and only if $\varphi \in S^n(X, X \setminus K; R)$. We obtain a well-defined map

$$S_c^n(X; R) \to \varinjlim S^n(X, X \setminus K; R)$$

in this way. But by Lemma 9.2 $S_c^n(X; R)$ is a cocone under the direct system and the maps to the colimit factor through it. By the universal property of the colimit this is only possible if $S_c^n(X; R)$ is isomorphic to the colimit.

The second statement now follows from Lemma 9.3 (b). $\square$

To the eyes of compact cohomology $\mathbb{R}^m$ looks like a sphere:

PROPOSITION 9.5.

$$H_c^*(\mathbb{R}^m; R) \cong H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R) \cong \begin{cases} R, & * = m, \\ 0, & * \neq m. \end{cases}$$

PROOF. If $K \subset \mathbb{R}^m$ is compact, then there is a closed ball of radius r_K around the origin, $B_{r_K}(0)$, with $K \subset B_{r_K}(0)$. Without loss of generality we can assume that r_K is a natural number.

Instead of considering the colimit over all compact K it now suffices to consider the colimit over all closed balls with integer radius. To see this note that there are natural maps between the colimits of the two diagrams and it is not hard to see they are inverse (we say the diagram $H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R)_{r \in \mathbb{N}}$ is *cofinal* in $H^*(\mathbb{R}^m, \mathbb{R}^m \setminus K; R)_{K \text{ compact}}$. Thus we have)

$$\varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus K; R) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R)$$

where the direct system on the right runs over all natural numbers r. But

$$H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R) \cong H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R)$$

for all r and the diagrams

$$\begin{array}{ccc} H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R) & \longrightarrow & H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_{r+1}(0); R) \\ \downarrow & & \downarrow \\ H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R) & \longrightarrow & H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R) \end{array}$$

commute. Therefore we have an isomorphism of direct systems and this induces an isomorphism of direct limits:

$$\varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R).$$

But the system on the right is constant and therefore

$$H_c^*(\mathbb{R}^m; R) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m \setminus B_r(0); R) \cong H^*(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; R).$$

□

10. Poincaré duality

Let M be a connected m-dimensional manifold with an R-orientation $(o_x \mid x \in M)$. For a compact $L \subset M$ let $o_L^R = o_L \in H_n(M, M \setminus L)$ be the R-orientation of M along L. (We omit R from the notation.) For $K \subset L$ compact we have that

$$(\varrho_{K,L})_*(o_L) = o_K$$

because $(\varrho_{x,K})_*(o_K) = o_x = (\varrho_{x,L})_*(o_L) = (\varrho_{x,K})_* \circ (\varrho_{K,L})_*(o_L)$ and o_K is unique with this property. Consider

$$(-) \cap o_K \colon H^{m-p}(M, M \setminus K; R) \longrightarrow H_p(M; R), \quad \alpha \mapsto \alpha \cap o_K = F(o_K) \otimes \langle \alpha, R(o_K) \rangle.$$

For $K \subset L$ and $\alpha \in H^{m-p}(M, M \setminus K; R)$ we have $(\varrho_{K,L})^*(\alpha) \in H^{m-p}(M, M \setminus L; R)$ and

$$(\varrho_{K,L})^*(\alpha) \cap o_L = \alpha \cap (\varrho_{K,L})_*(o_L) = \alpha \cap o_K.$$

because the cap product is natural, see Proposition 6.3. (There is no $(\rho_{K,L})_*$ on the left hand side as the cap product takes values in $H_p(M; R)$ regardless which compact set we start with.)

This compatibility ensures that the cap product yields a map

$$\varinjlim(- \cap o_K) \colon \varinjlim H^{m-p}(M, M \setminus K; R) = H_c^{m-p}(M; R) \longrightarrow H_p(M; R)$$

where the colimit goes over all the compact subsets K of M.

Definition 10.1. The map

$$\varinjlim(- \cap o_K^R) \colon H_c^{m-p}(M; R) \to H_p(M; R)$$

is called *Poincaré duality map* and is denoted by PD or PD_M.

Theorem 10.2 (Poincaré Duality). *Let M be a connected m-manifold with R-orientation $(o_x \mid x \in M)$. Then PD is an isomorphism $\mathsf{PD} \colon H_c^{m-p}(M; R) \longrightarrow H_p(M; R)$ for all $p \in \mathbb{Z}$.*

Corollary 10.3 (Poincaré duality for compact manifolds). *Let M be a connected compact manifold of dimension m with an R-orientation $(o_x \mid x \in M)$ and let $[M] = o_M$ be the fundamental class of M, then*

$$\mathsf{PD} = (-) \cap [M] \colon H^{m-p}(M; R) \longrightarrow H_p(M; R)$$

is an isomorphism for all $p \in \mathbb{Z}$.

EXAMPLE 10.4. Any connected compact manifold of dimension m possesses a $\mathbb{Z}/2\mathbb{Z}$-orientation and a fundamental class $o_M^{\mathbb{Z}/2\mathbb{Z}} \in H_m(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus for all p

$$(-) \cap o_M^{\mathbb{Z}/2\mathbb{Z}} \colon H^{m-p}(M; \mathbb{Z}/2\mathbb{Z}) \cong H_p(M; \mathbb{Z}/2\mathbb{Z}).$$

For instance the cohomology of $\mathbb{R}P^n$ and its homology satisfy Poincaré duality with $\mathbb{Z}/2\mathbb{Z}$-coefficients regardless of the parity of n.

PROOF OF THEOREM 10.2. (a) First we consider the case of $M = \mathbb{R}^m$ and we know that

$$H_c^{m-p}(\mathbb{R}^m) \cong \begin{cases} R, & p = 0, \\ 0, & p \neq 0 \end{cases}$$

and this is isomorphic to $H_p(\mathbb{R}^m; R)$. Therefore, abstractly, both R-modules are isomorphic. Let B_r be the closed r-ball centered at the origin. We have to understand

$$(-) \cap o_{B_r} \colon H_c^m(\mathbb{R}^m) \to H_0(\mathbb{R}^m; R).$$

We know that $\langle 1, \alpha \cap o_{B_r} \rangle = \langle \alpha, o_{B_r} \rangle$ for all $\alpha \in H^m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r; R)$. But

$$\langle -, o_{B_r} \rangle \colon H^m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r; R) \longrightarrow R, \quad u \mapsto \langle u, o_{B_r} \rangle$$

is bijective because the Kronecker pairing induces the first map in the isomorphism from the universal coefficient theorem

$$H^m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r; R) \cong \mathrm{Hom}(H_m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r), R) \oplus \mathrm{Ext}(H_{m-1}(\mathbb{R}^m, \mathbb{R}^m \setminus B_r), R).$$

The second summand is trivial because $H_{m-1}(\mathbb{R}^m, \mathbb{R}^m \setminus B_r) = 0$. Thus we obtain that for all r the map $(-) \cap o_{B_r}$ is bijective and therefore its direct limit

$$\varinjlim (-) \cap o_{B_r} \colon \varinjlim H^m(\mathbb{R}^m, \mathbb{R}^m \setminus B_r; R) \longrightarrow H_0(\mathbb{R}^m; R)$$

is an isomorphism as well.

(b) Now assume that $M = U \cup V$ such that the claim holds for the open subsets U, V and $U \cap V$, i.e. the maps $\mathsf{PD}_U, \mathsf{PD}_V$ and $\mathsf{PD}_{U \cap V}$ are isomorphisms and each of them uses the orientation that is induced from the orientation of M.

On the example sheet you can show there is a Mayer-Vietoris sequence for compactly supported cohomology. We use it together with the Mayer-Vietoris sequence to build the following diagram

$$\begin{tikzcd}
H_c^{m-p}(U\cap V;R) \arrow[r, "\mathsf{PD}_{U\cap V}=\cap o_{U\cap V}"] \arrow[d] & H_p(U\cap V;R) \arrow[d] \\
H_c^{m-p}(U;R)\oplus H_c^{m-p}(V;R) \arrow[r, "\cap o_U\oplus\cap o_V"] \arrow[d] & H_p(U;R)\oplus H_p(V;R) \arrow[d] \\
H_c^{m-p}(M;R) \arrow[r, "\cap o_M"] \arrow[d] & H_p(M;R) \arrow[d] \\
H_c^{m-p+1}(U\cap V;R) \arrow[r, "\cap o_{U\cap V}"] \arrow[d] & H_{p-1}(U\cap V;R) \arrow[d] \\
H_c^{m-p+1}(U;R)\oplus H_c^{m-p+1}(V;R) \arrow[r, "\cap o_U\oplus\cap o_V"] & H_{p-1}(U;R)\oplus H_{p-1}(V;R).
\end{tikzcd}$$

I claim this diagram commutes up to signs and then the five lemma proves Poincaré duality in the case $M = U \cup V$. Note here that commutativity up to sign is enough to apply the five lemma: just change the signs of the horizontal maps to make everything commute on the nose, and if $-\cap o_M$ is an isomorphism then so is $-\cap o_M$.

The fact that the first two squares of this diagram are in fact commutative follows by unravelling the definitions. Commutativity of the third square, involving the boundary maps, is significantely more involved. We prove it in Lemma 10.5 below.

By induction this extends to unions of finitely many open sets such that PD is an isomorphism on the sets and their intersections.

(c) Now assume $M = \bigcup_{i=1}^{\infty} U_i$ with open U_i such that $U_1 \subset U_2 \subset \ldots$. We will show that if the claim holds for all U_i with the orientation induced by the one of M, then the claim holds for M. To that end, let $U \subset M$ be an arbitrary open subset and let $K \subset U$ be compact. Excision gives us

$$H^p(M, M\setminus K; R) \cong H^p(U, U\setminus K; R)$$

and we denote by φ_K the inverse of this map. The direct limit of these φ_K induces a map

$$\varphi_U^M := \varinjlim \varphi_K \colon H_c^p(U;R) \longrightarrow H_c^p(M;R).$$

In general, this map is *not* an iso (U is 'too small'), but now we let U vary. For $U \subset V \subset W$ we get

$$\varphi_U^W = \varphi_V^W \circ \varphi_U^V, \quad \varphi_U^U = \mathrm{id}.$$

As the excision isomorphism is induced by the inclusion $(U, U\setminus K) \hookrightarrow (M, M\setminus K)$, we get that the following diagram commutes:

$$\begin{tikzcd}
H_c^{m-p}(U;R) \arrow[r, "\varphi_U^M"] \arrow[d, "\mathsf{PD}_U"] & H_c^{m-p}(M;R) \arrow[d, "\mathsf{PD}_M"] \\
H_p(U;R) \arrow[r, "(i_U^M)_*"] & H_p(M;R)
\end{tikzcd}$$

and hence the corresponding diagram

$$\begin{tikzcd}
\varinjlim H_c^{m-p}(U_i;R) \arrow[r, "\varinjlim \varphi_{U_i}^M"] \arrow[d, "\varinjlim \mathsf{PD}_{U_i}"] & H_c^{m-p}(M;R) \arrow[d, "\mathsf{PD}_M"] \\
\varinjlim H_p(U_i;R) \arrow[r, "\varinjlim (i_{U_i}^M)_*"] & H_p(M;R)
\end{tikzcd}$$

commutes as well. Now the limit of the $(i_{U_i}^M)_*$ is an isomorphism. To show this note that chains on $\cup U_i$ are the direct limits of chains on the U_i as the n-simplex is compact. Then we apply Lemma 9.3 to deduce the isomorphism on homlogy. (This should have been a lemma in the last section.)

The map $\varinjlim \varphi_{U_i}^M$ is an isomorphism as every K lands in some U_i eventually, so by excision $H^{m-p}(M, M \setminus K) = H^{m-p}(U_i, U_i \setminus K)$ and taking the direct limit over $K \subset M$ or simultaneously over K and i gives the same result.

By assumption, each PD_{U_i} is an isomorphism and so is their limit by Lemma 9.3. Putting all this together PD_M is also an isomorphism.

(d) We show that the claim is valid for arbitrary open subsets $M \subset \mathbb{R}^m$. We express M as a union $M = \bigcup_{r=1}^{\infty} \mathring{B}_r$ where the B_r are m-balls. This is possible because $\mathbb{R}^m$ has a countable basis of its topology. Set $U_i := \bigcup_{r=1}^{i} \mathring{B}_r$, then of course

$$U_1 \subset U_2 \subset \dots$$

The claim holds for the U_i and because of (c) it then holds for M. (Note that the U_i may be disconnected, but applying (b) in the special case where the intersection is empty the claim still holds in this case.)

(e) Finally we assume that M is as in the theorem with some fixed R-orientation. Every point in M has a neighborhood which is homeomorphic to some open subset of $\mathbb{R}^m$ and we can choose the homeomorphism in such a way that it preserves the orientation. We know that M has a countable basis for its topology and thus there are open subsets $V_1, V_2, \dots \subset M$ such that $V_i \cong W_i \subset \mathbb{R}^m$ and the V_i cover M. Define $U_i := \bigcup_{j=1}^{i} V_j$, thus $M = \bigcup_i U_i$. The claim holds for the V_j by (a), and it holds for their intersections (which are open subsets of $\mathbb{R}^n$ not necessarily homeomorphic to balls) by (d). Therefore the claims holds for the U_i by (b) and thus for M by (c).

$\square$

Lemma 10.5. *The following diagram is commutative up to sign:*

$$\begin{tikzcd}
H_c^{m-p}(M;R) \arrow[r, "\mathsf{PD}_M"] \arrow[d, "\delta"] & H_p(M;R) \arrow[d, "\delta"] \\
H_c^{m-p+1}(U \cap V;R) \arrow[r, "\mathsf{PD}_{U\cap V}"] & H_{p-1}(U \cap V;R)
\end{tikzcd}$$

Proof. It suffices to prove commutativity befor passinng to the limit, so we consider the diagram:

$$\begin{tikzcd}
H^{m-p}(M, M \setminus K \cup L) \arrow[rrr, "\cap o_{K\cup L}"] \arrow[d, "\partial"] & & & H_p(M) \arrow[d, "\delta"] \\
H^{m-p+1}(M, M \setminus K \cap L) \arrow[r, "\cong"] & H^{m-p+1}(U \cap V, U \cap V \setminus K \cap L) \arrow[rr, "\cap o_{K\cap L}"] & & H_{p-1}(U \cap V; R)
\end{tikzcd}$$

where the unlabelled isomorphism comes from excision.

We represent $o_{K\cup L}$ by a chain $\alpha = \alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K} \in C^*(M)$, where the summands lie in $C^*(U \setminus L)$, $C^*(U \cap V)$ and $C^*(V \setminus K)$ respectively. We use that those three opens form a cover of M and use barycentric subdivison to ensure the decomposition of α.

We observe that $\alpha_{U\cap V}$ represents $o_{K\cap L}$ as the other two summands vanish in $H_n(M, M \setminus U \cap V)$. Similarly $\alpha_{U\cap V} + \alpha_{U\setminus L}$ represents o_K.

To compute the boundary map in the relative Mayer-Vietoris sequence on cohomology we take a cocycle ϕ and represent it as $\phi_K + \phi_L$ with $\phi_K \in C^*(M, M \setminus K)$, $\phi_L \in C^*(M, M \setminus L)$. Then we find by definition that $\partial[\phi]$ is represented by $\delta\phi_K = \delta\phi_L \in C^*(M, M \setminus K \cap L)$.

So we can compute $(\partial\phi) \cap o_{K\cap L} = \delta\phi_K \cap \alpha_{U\cap V}$ in homology. Then we use the Leibniz formula

$$\partial(\phi_K \cap \alpha_{U\cap V}) = (-1)^{n-p}(\delta\phi_K) \cap \alpha_{U\cap V} + \phi_K \cap (\partial\alpha_{U\cap V})$$

and as $\phi_K \cap \alpha_{U\cap V}$ is a chain on $U \cap V$ the left hand side is zero on homology and we find

$$(\partial\phi) \cap o_{K\cap L} = (-1)^{n-p-1}\phi_K \cap \partial\alpha_{U\cap V}$$

in homology

Then we compute $\delta(\phi \cap o_{K\cup L}) = \delta(\phi \cap (\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K})$. We may compute the boundary map by applying ∂ to the first summand and obtain $\partial(\phi \cap \alpha_{U\setminus L})$.

As ϕ is a cycle this is $\phi \cap \partial\alpha_{U\setminus L}$ by the Leibniz formula.

Again we wirte $\phi = \phi_K + \phi_L$ and note that as ϕ_L is zero on chains in $M \setminus L$ it sends $\partial\alpha_{U\setminus L}$ to zero.

Thus we are left with $\phi_K \cap \partial\alpha_{U\setminus L}$. We are close now. We recall that $\alpha_{U\setminus L} + \alpha_{U\cap V}$ represents o_K, thus its boundary is a chain in $M \setminus K$ and and by construction ϕ_K vanishes on chains in $M \setminus K$. Thus we havve

$$\delta(\phi \cap o_{K\cup L}) = -\phi_K \cap \partial\alpha_{U\cap V}$$

in homology and the diagram commutes up to the sign $(-1)^{n-p}$. □

The following corollary holds for general coefficients, but we only need this version:

Corollary 10.6. *Let M be a non-compact connected manifold. Then $H_n(M; \mathbb{Z}/2) = 0$.*

Proof. As M is orientable with respect to $\mathbb{Z}/2$ we may apply Poincaré duality and find that $H_n(M; \mathbb{Z}/2) \cong H^0_c(M; \mathbb{Z}/2)$. But unravelling definitions $H^0_c(M)$ are exactly functions with compact support that are constant along any continuous path. But on a non-compact manifold there are no compactly supported constant functions. □

11. Duality and cup products

Let M be a connected closed m-manifold with an R-orientation for some commutative ring R. We consider the composition

$$\begin{tikzcd} H^k(M;R) \otimes_R H^{m-k}(M;R) \arrow[r, "\cup"] & H^m(M;R) \arrow[d, "(-)\cap o_M^R"] \\ & H_0(M;R) \cong R \end{tikzcd}$$

Definition 11.1. For $\alpha \in H^k(M;R)$, $\beta \in H^{m-k}(M;R)$ the map

$$(\alpha, \beta) \mapsto \langle \alpha \cup \beta, o_M^R \rangle$$

is called *cup product pairing of M*.

Proposition 11.2. *The cup product pairing is non-singular if R is a field or if $R = \mathbb{Z}$ and all homology groups of M are torsion-free.*

Here, non-singular means that the induced maps

$$H^k(M;R) \to \mathrm{Hom}_R(H^{m-k}(M;R), R) \text{ and } H^{m-k}(M;R) \to \mathrm{Hom}_R(H^k(M;R), R)$$

are both isomorphisms.

Proposition 11.2 holds as long as one restricts attention to the free part of the cohomology groups: let $FH^k(M;R)$ denote the free part of $H^k(M;R)$ then there is a non-singular pairing

$$FH^k(M;R) \otimes_R FH^{m-k}(M;R) \to R.$$

In geometric applications the ground ring is often $R = \mathbb{R}$, so then you are dealing with a pairing over the real numbers and methods of linear algebra apply.

Proof. The Kronecker pairing yields a map

$$\kappa \colon H^k(M;R) \to \mathrm{Hom}_R(H_k(M;R), R)$$

and Poincaré duality tells us that capping with o_M^R is an isomorphism between $H_k(M;R)$ and $H^{m-k}(M;R)$. The composite is

$$H^k(M;R) \to \mathrm{Hom}_R(H_k(M;R), R) \cong \mathrm{Hom}_R(H^{m-k}(M;R), R), \ \alpha \mapsto \langle \alpha, (-) \cap o_M^R \rangle.$$

Over a field κ is an isomorphism, and then so is the composite. In the torsion-free setting κ is an isomorphism as well. $\square$

Remark 11.3. Note that we have not assumed finite rank of homology groups anywhere. In fact, we can deduce it from our results. Let's make the sattement over a field k: If M is a compact connected orientable manifold then $H_i(M;k)$ is finite-dimensional in each degree. Suppose $H_i(M;k) \cong \oplus^\infty k$. Then by Poincaré duality $H^{n-i}(M) \cong \oplus^\infty k$ and by the Universal coefficent theorem this means that $H_{n-i}(M)$ is some vector space whose linear dual is $\oplus^\infty k$. But this is impossible.

Dual to the cup product pairing there is the *intersection form*:

$$H_p(M) \otimes H_{m-p}(M) \to \mathbb{Z}$$

with $a \otimes b \mapsto \langle \mathsf{PD}^{-1}(a) \cup \mathsf{PD}^{-1}(b), o_M \rangle$. For even-dimensional manifolds we may restrict to $p = \frac{m}{2}$. Then the signature of this form is an important invariant in differential topology.

For instance one can show that for a compact oriented manifold W such that $\partial W = M$ with a $4n$-dimensional manifold M the signature of the intersection form on M is trivial. One can also show that up to homeomorphism there is exactly one simply connected smooth 4-manifold with a given unimoduler symmetric bilinear form as its intersection form on H_2.

EXAMPLE 11.4. In the case of a torus with meridian a and longitude b we find $(a, b) = \langle \alpha \cup \beta, o_{T^2} \rangle = 1$ and $(b, a) = -1$. (The overall sign may change if we change the orientation.)

Thus the intersection does indeed count the signed points of intersection of two cycles. This holds in general, but it is not easy to prove. One approach can be found in the book "Differential forms in algebraic topology" by Bott & Tu.

LEMMA 11.5. *Let M be as in 11.2 with torsion-free homology groups. If $H^p(M) \cong \mathbb{Z} \cong H^{m-p}(M)$ and if $\alpha \in H^p(M)$, $\beta \in H^{m-p}(M)$ are generators, then $\alpha \cup \beta$ is a generator of $H^m(M) = \mathbb{Z}$.*

PROOF. For α there exists a $\beta' \in H^{m-p}(M)$ with

$$\langle \alpha \cup \beta', o_M \rangle = 1.$$

As β is a generator we know that $\beta' = k\beta$ for some integer k and hence

$$1 = \langle \alpha \cup \beta', o_M \rangle = \langle \alpha \cup k\beta, o_M \rangle = k \langle \alpha \cup \beta, o_M \rangle.$$

But $\langle \alpha \cup \beta, o_M \rangle$ is an integer, so k has to be ± 1 and therefore $\alpha \cup \beta$ generates $H^m(M)$. $\square$

We will use this result to calculate the cohomology rings of projective spaces.

LEMMA 11.6. *If $\alpha \in H^2(\mathbb{C}P^m)$ is a generator, then $\alpha^q \in H^{2q}(\mathbb{C}P^m)$ is a generator as well for $q \leqslant m$.*

PROOF. We have to show by induction that α^{q-1} is an additive generator of $H^{2q-2}(\mathbb{C}P^m)$ and we do that by induction over m because we will use the argument in this proof later again.

For $m = 1$ there is nothing to prove because $\mathbb{C}P^1 \cong \mathbb{S}^2$ and there $\alpha^2 = 0$.

Consider the inclusion $i\colon \mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^m$. The CW structure of $\mathbb{C}P^m$ is $\mathbb{C}P^{m-1} \cup_f \mathbb{D}^{2m}$. For $m > 1$ $i^*\colon H^{2i}(\mathbb{C}P^m) \to H^{2i}(\mathbb{C}P^{m-1})$ is an isomorphism for $1 \leqslant i \leqslant m-1$ and $i^*(\alpha)$ generates $H^2(\mathbb{C}P^{m-1})$. Induction over m then shows that $(i^*(\alpha))^q$ generates $H^{2q}(\mathbb{C}P^{m-1})$ for all $1 \leqslant q \leqslant m-1$. But $(i^*(\alpha))^q = i^*(\alpha^q)$ and therefore α^q generates $H^{2q}(\mathbb{C}P^m)$ for $1 \leqslant q \leqslant m-1$. Lemma 11.5 then shows that $\alpha \cup \alpha^{m-1} = \alpha^m$ generates $H^{2m}(\mathbb{C}P^m)$. $\square$

COROLLARY 11.7. *As a graded ring*

$$H^*(\mathbb{C}P^m) \cong \mathbb{Z}[\alpha]/\alpha^{m+1} \text{ with } |\alpha| = 2.$$

Similarly,

$$H^*(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\beta]/\beta^{m+1} \text{ with } |\beta| = 1$$

and

$$H^*(\mathbb{H}P^m; \mathbb{Z}) \cong \mathbb{Z}[\gamma]/\gamma^{m+1} \text{ with } |\gamma| = 4.$$

PROOF. The first statement follows immediately from the lemma, the other two statements are shown by first proving analogous lemmas in the same way. $\square$

There are two geometric consequences that follow from this calculation.

COROLLARY 11.8. *For $0 < m < n$ the inclusion $j \colon \mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$ is not a weak retract.*

PROOF. Let us assume that there is an $r \colon \mathbb{C}P^n \to \mathbb{C}P^m$ with $r \circ j \simeq \mathrm{id}$. On second cohomology groups j induces an isomorphism

$$j^* \colon H^2(\mathbb{C}P^n) \to H^2(\mathbb{C}P^m).$$

Let $\alpha \in H^2(\mathbb{C}P^m)$ be a generator, then $\beta := r^*(\alpha)$ is a generator as well. As $\alpha^{m+1} = 0$ we get

$$\beta^{m+1} = r^*(\alpha)^{m+1} = r^*(\alpha^{m+1}) = r^*(0) = 0.$$

But $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[\beta]/\beta^{n+1}$ and hence $\beta^{m+1} \neq 0$. $\square$

COROLLARY 11.9. *The attaching map of the $2n$-cell in $\mathbb{C}P^n$ is not null-homotopic.*

PROOF. Let $\varphi \colon \mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$ be the attaching map, thus

$$\mathbb{C}P^n = C_\varphi = \mathbb{C}P^{n-1} \cup_\varphi \mathbb{D}^{2n}.$$

If φ were null-homotopic, then

$$\mathbb{C}P^{n-1} \cup_\varphi \mathbb{D}^{2n} \simeq \mathbb{C}P^{n-1} \vee \mathbb{S}^{2n}$$

since homotopic attaching maps give rise to homotopy equivalent CW complexes, see Proposition 0.18 in [Hatcher].

Thus $\mathbb{C}P^{n-1}$ would be a weak retract of $\mathbb{C}P^n$, contradicting Corollary 11.9. (Or we note there is a direct contradiction to the strucutre of the cohomology rings.) $\square$

EXAMPLE 11.10. A famous example of this phenomenon is the Hopf fibration $\varphi = \eta \colon \mathbb{S}^3 \to \mathbb{C}P^1 = \mathbb{S}^2 = \mathbb{C} \cup \infty$. Consider $\mathbb{S}^3 \subset \mathbb{C}^2$ and send $\mathbb{S}^3 \ni (u, v)$ to

$$\eta(u,v) := \begin{cases} \frac{u}{v}, & v \neq 0, \\ \infty, & v = 0. \end{cases}$$

Then this map is not null-homotopic, $\eta \colon \mathbb{S}^3 \to \mathbb{S}^2$, and in fact it generates $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

A similar consideration for the attachment map $\mathbb{S}^7 \to \mathbb{H}P^1 \cong \mathbb{S}^4$ shows that $\pi_7(\mathbb{S}^4)$ is non-trivial.

12. Further applications

The product structure on $H^*(\mathbb{R}P^n)$ has some interesting consequences.

A famous application of topology to algebra is the classification of finite dimensional division algebras.

DEFINITION 12.1. A *n-dimensional division algebra over* $\mathbb{R}$ is a bilinear multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, denoted $(a, b) \mapsto ab$ satisfying

(a) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for $a, b, c \in \mathbb{R}^n$ (distributivity),
(b) $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for $a, b \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ (scalar associativity)

such that $ax = b$ and $xa = b$ always have a solution for $a, b \in \mathbb{R}^n$ and $a \neq 0$.

We de not assume commutativity or associativity!

You have already met the division algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ in deminsions 1, 2, 4. These are the only associative ones. There is also the non-associative divison algebra of octonions in dimension 8.

THEOREM 12.2. *Any finite-dimensional division algebra over $\mathbb{R}$ has dimension 2^k for some nonnegative integer k.*

PROOF. The multiplication induces a map $m : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ by $(x, y) \mapsto \frac{xy}{|xy|}$ which is continuous (as a bilinear map is continuous) and well-defined (as there are no zero-divisors in a divison algebra). Moreover $m(x, -y) = -m(x, y) = m(-x, y)$ by scalar associativity, thus there is an induced map $h : \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$.

We now investigate h^* on cohomology with coefficients in $\mathbb{Z}/2$. We have $H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}, \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha, \beta]/(\alpha^n = \beta^n = 0)$ where the generators are pulled back from the generators of cohomology from the two factors. Let γ generate $H^*(\mathbb{R}P^{n-1}, \mathbb{Z}/2)$. The key claim now is that $h^*(\gamma) = \alpha + \beta$.

We investigate the map h_* on π_1 first. We may assume $n > 2$ so that $\pi_1(\mathbb{R}P^n, *) \cong H_1(\mathbb{R}P^n, \mathbb{Z}/2)$. So consider a loop w in $\mathbb{R}P^{n-1}$ representing the generator of $\pi_1(\mathbb{R}P^{n-1}, x)$ with base point $x = w(0) \in \mathbb{R}P^{n-1}$. The loops (w, c_x) and (c_x, w) that together generate $\pi_1(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}, (x, x))$ map to $x.w$ and $w.x$, respectively. We claim that $x.w$ and $w.x$ are non-trivial, and then they must be homotopic as there is only one non-trivial element in the fundamental group.

For this we consider that w lifts to a path $\tilde{w}$ in $\mathbb{S}^{n-1}$ connecting a point to its antipode. Multiplication with $\tilde{w}(0)$ gives a continuous self map of $\mathbb{S}^{n-1}$, and we have $\tilde{w}(0).\tilde{w}(0) \neq \tilde{w}(1).\tilde{w}(0)$ as there are no zero divisors. Thus $w.x$ also lifts to a path in $\mathbb{S}^{n-1}$ connecting two antipodel points, and thus it is non-trivial in the fundamental group. The same argument applies to $x.w$.

(The argument from lectures does not work as it was tacitly assuming the existence of a unit in our division algebra.)

Similarly the map on $H_1(\mathbb{R}P^n, \mathbb{Z}/2)$ sends both generators $(0, [w])$ and $([w], 0)$ to $[w]$. If we dualize this stament we obtain that $h^*(\gamma) = \alpha + \beta$.

With the claim in hand we consider $(\alpha + \beta)^n = h^*(\gamma^n) = 0$ and thus

$$\sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^j = 0.$$

This can only be zero if all coefficients for $i \neq 0, n$ vanish, thus $\binom{n}{i}$ is even for all i. It is an exercise in basic number theory to show that this implies $n = 2^k$.

Let $n = \sum_i 2^{k_j}$ for some integers k_j and consider $(1 + x)^n = \sum_i \binom{n}{i} x^i$ modulo 2. This may be rewritten $\prod_j (1 + x)^{2^{k_j}} = \prod_j (1 + x^{2^{k_i}})$. But multiplying this out there can be no cancellation as the powers are too far apart. Thus all the binomial coefficients can only vanish if there is only one factor and $n = 2^{k_1}$. □

Next we turn to the Borsuk-Ulam theorem. It has several formulations, several proofs and many applications. One of the most natural proofs uses cohomology of projective space.

LEMMA 12.3. *For $n > m$ there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ that is nontrivial on $H^1(-, \mathbb{Z}/2)$*

PROOF. Assum f is such a map and let β generater $H^*(\mathbb{R}P^m, \mathbb{Z}/2)$. Then $f(\beta) \neq 0$ by assumption and thus it generates $H^*(\mathbb{R}P^n)$. In particular $f^*(\beta)^n = f^*(\beta^n) = 0$ generates $H^n(\mathbb{R}P^n)$, which is a contradiction. □

Theorem 12.4 (Borsuk-Ulam theorem). *Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be an odd map, i.e. $f(-x) = -f(x)$ for all $x \in \mathbb{S}^n$. Then f has a zero.*

The following alternatiev formulation is useful: Given an arbitrary continuous map $g : \mathbb{S}^n \to \mathbb{R}^n$ we may consider $g(x) - g(-x)$ which is always odd. Thus it has a zero and g takes the same value on two antipodal points.

Proof. We only prove the case $n \geqslant 3$ and leave $n = 1, 2$ as exercises (solvable with the methods of a first topology course). Suppose the theorem is false. Then we may define $x \mapsto f(x)/||f(x)|| : \mathbb{S}^n \to \mathbb{S}^{n-1}$ which is also odd. Thus it induces $h : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$. This map induces an isomorphism on fundamental groups. To see this consider that the lift of a generator of $\pi_1(\mathbb{R}P^n, *)$ is a path from north to south pole of $\mathbb{S}^n$, which maps to a path from south to north pole of $\mathbb{S}^{n-1}$, which is the lift of a non-trivial loop in $\mathbb{R}P^{n-1}$. It follows that h is also an isomorphism on H^1 and by Lemma 12.3 we have the desired contradiction. □

The Borsuk-Ulam theorem has many applications, there is a whole book about them. Possibly the most famous one is the "Ham sandwich theorem", named after the special case of $n = 3$, where it expresses the (theoretical) possibility that the two slices of bread and the ham in a sandwich may be sliced into equal parts with a singl stroke of a knife.

Theorem 12.5. *Lett $\{\mu_1, \dots, \mu_n\}$ be a collection of finite Borel measures on $\mathbb{R}^n$ such that all hyperplanes have measure 0. Then there is a single hyperplane that bisects each μ_i, i.e. the opposite half spaces defined by the hyperplane have equal measure.*

A finite Borel measure is a measure on $\mathbb{R}^n$ such that all opens are measurable and the meausure of $\mathbb{R}^n$ itself is positive and finite. An example would be the restriction of the usual Lebesgue measure to some compact subset of $\mathbb{R}^n$.

Proof. The proof in its proper generality needs some topology, which we have available now, and a little bit of analysis.

We write an arbitrary point $u \in \mathbb{S}^n$ as (u_0, u') with $u_0 \in \mathbb{R}$ and $u' \in \mathbb{R}^n$. Then we define the half-space

$$h^+(u) := \{x \in \mathbb{R}^n \mid u'.x \leqslant u_0\}.\}$$

One sees that $h^+(-u)$ is the opposite half space of $h^+(u)$. The only exception is $u = (\pm 1, 0, \dots, 0)$ when $h^+(u) = \mathbb{R}^n$, respectively $\emptyset$.

Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be given in coordinates by $f_i(u) = \mu_i(h^+(u))$ where μ is the Borel measure. (In fact, instead of specifying A_i we could specify an arbitrary finite measure on $\mathbb{R}^n$ such that no hyperplane has positive measure!)

It is an exercise in analysis to rigorously show that the f_i are continuous (see e.g. Theorem 3.1.1 of Matoušek 'Using the Borsuk-Ulam Theorem').

Now if $f(u) = f(-u)$ for some u then the corresponding hyperplane $u'.x = h_0$ exactly bisects all A_i. (We cannot have $f(1, 0, \dots, 0) = f(-1, 0, \dots, 0)$.) But such a poin must exist as $h(u) = f(u) - f(-u)$ is antipodal, so it has a zero by Theorem 12.4. □

13. Lefschetz duality

A number of topological applications can be deduced from a relative version of Poincaré duality. W'll give a very general statement and then prove two special cases and give their applications.

A space X is a Euclidean neighbourhood retract if it is homeomorphic to subspace of $\mathbb{R}^n$ which is a retract of a neighbourhood.

THEOREM 13.1 (Alexander-Lefschetz duality). *Let M be a connected m-dimensional manifold and let $K \subset L \subset M$ be compact subspaces that are Euclidean neighbourhood retracts. Let M be oriented along L with respect to R. Then there is a well-defined map*

$$\mathsf{PD} = (-) \cap o_L \colon H^q(L, K; R) \longrightarrow H_{m-q}(M \setminus K, M \setminus L; R)$$

which is an isomorphism for all integers q.

REMARK 13.2. The statement remains to true without the Euclidean neighbourhood assumption, but one has to replace cohomology by *Čech cohomology* which we won't have time to introduce.

The first special case is if M is a manifold with boundary.

Let

$$\mathbb{R}^m_- := \{(x_1, \dots, x_m), x_i \in \mathbb{R}, x_1 \leqslant 0\}$$

be an m-dimensional half-space. Its topological boundary is

$$\partial\mathbb{R}^m_- = \{(x_1, \dots, x_m), x_i \in \mathbb{R}, x_1 = 0\}.$$

DEFINITION 13.3. An m-dimensional *topological manifold with boundary* is a Hausdorff space M with a countable basis of its topology together with an open cover $\{U_i\}_{i \in I}$ and homeomorphisms $h_i \colon U_i \to V_i$ with $V_i \subset \mathbb{R}^m_-$ open.

An $x \in M$ is a boundary point of M if it has an open neighbourhood U with a homeomorphism $h \colon U \to V$ with V open in $\mathbb{R}^m_-$ and $h(x)$ in $\partial\mathbb{R}^m_-$. The set of boundary points of M is its *boundary*, denoted by ∂M.

EXAMPLE 13.4.
(a) The closed n-dimensional ball is a manifold with boundary $\mathbb{S}^{n-1}$. In this case the boundary agrees with the boundary in the sense of elementary topology if we embed the ball into $\mathbb{R}^n$, but in general this is not possible!
(b) Removing two open disks with disjoint closures from a closed disk produces a disk with two holes, whose boundary is $\mathbb{S}^1 \amalg \mathbb{S}^1 \amalg \mathbb{S}^1$. This manifold with boundary is called the *pair of pants*.

Sometimes the word manifold is used to mean a manifold with boundary. A *closed manifold* is a compact manifold which has empty boundary.

An orientation of a manifold M with boundary is just orientation of the interior $M \setminus \partial M$.

PROPOSITION 13.5 (Lefschetz duality). *Let M be a a compact connected orientable m-manifold with boundary, then there is a natural isomorphism*

$$H^q(M, \partial M) \cong H_{m-q}(M).$$

PROOF. We may glue a collar along ∂M to M, i.e. consider

$$W := M \amalg_{\partial M} (\partial M \times [0, 1)).$$

Then one can check that W is an oriented m-manifold (without boundary) which is homotopy equivalent to M. Thus Poincaré duality applies and we obtain

$$H^q_c(W) \cong H_{m-q}(W) \cong H_{m-q}(M).$$

It remains to show that $H^q_c(W) \cong H^q(M, \partial M)$. By homotopy equivalence $H^q(M, \partial M) \cong H^q(W, \partial M \times [0,1))$. As $M \cup_{\partial M} (\partial M \times [0,d])$ for $0 < d < 1$ is an exhaustion of W by compact subsets which are homotopy equivalent we see that

$$H^q_c(W) \cong \varinjlim H^q(W, W \setminus M \amalg_{\partial M} \partial M \times [0,d)) \cong \varinjlim H^q(W, \partial M \times (d,1))$$

and moreover the inverse limit computing compactly supported cohomology becomes constant. The last group is then isomorphic to $H^q(M, \partial M)$ by homotopy equivalence. $\square$

Proposition 13.6. *Let M be a compact connected and orientable m-manifold and let β_i be the i*th Betti number *of M, $\beta_i = \dim_{\mathbb{Q}} H_i(M; \mathbb{Q})$. Then*

$$\beta_i = \beta_{m-i}.$$

In particular the Euler characteristic $\chi(M) = \sum_{i=0}^m (-1)^i \beta_i$ of M vanishes if the dimension of M is odd.

Proof. Note that orientability implies $\mathbb{Q}$-orientability. Theorem 10.2 then tells us that

$$\dim_{\mathbb{Q}} H_{m-i}(M; \mathbb{Q}) = \dim_{\mathbb{Q}} H^i(M; \mathbb{Q})$$

As $\mathbb{Q}$ is divisible, there is no Ext-term arising in the universal coefficient theorem and thus

$$\dim_{\mathbb{Q}} H^i(M; \mathbb{Q}) = \dim_{\mathbb{Q}}(\mathrm{Hom}(H_i(M), \mathbb{Q}))$$

but this is equal to the dimension of the vector space of the homomorphisms from the free part of $H_i(M)$ to $\mathbb{Q}$ which is equal to the rank of $H_i(M)$ and this in turn is equal to β_i.

The second statement is immediate. $\square$

Corollary 13.7. *If M is a compact connected oriented manifold then the Euler characteristic of ∂M is always even.*

Proof. We consider the collared version W of M again. As $M \simeq W$ we have $\chi(M) = \chi(W)$ and the long exact sequence of the pair $W \setminus M \subset W$ gives

$$\chi(W) = \chi(W \setminus M) + \chi(W, W \setminus M)$$

as Euler characteristic is additive on long exact sequences. Here the relative Euler characteristic is $\chi(W, W \setminus M) = \sum (-1)^i \dim_{\mathbb{Q}} H_i(W, W \setminus M; \mathbb{Q})$. Homotopy invariance yields $\chi(W \setminus M) = \chi(\partial M)$ and Proposition 13.5 guarantees that $\chi(W, W \setminus M) = (-1)^m \chi(M)$. Therefore

$$\chi(\partial M) = (1 + (-1)^{m-1}) \chi(M)$$

and this is always an even number. $\square$

We compute

$$\chi(\mathbb{C}P^{2m}) = \sum_{i=0}^{2m} (-1)^{2i} = 2m + 1$$

and

$$\chi(\mathbb{H}P^{2m}) = \sum_{i=0}^{2m} (-1)^{4i} = 2m + 1$$

by recalling the cell structure of complex and quaternionic projective space.

Thus no even complex or quaternionic projective spaces can occur as the boundary of a connected compact orientable manifold.

It also follows that $\mathbb{R}P^{2m}$ can never be a boundary of a compact connected oriented manifold, because its Euler characteristic is 1. However, this is less interesting, as the boundary of an oriented manifold always inherits an orientation. However, we may adapt Corollary 13.7 to coefficients in $\mathbb{Z}/2$ and deduce that $\mathbb{R}P^{2m}$ which has Euler characteristic 1 cannot be the boundary of any compact connected manifold.

This is important in *bordism theory*: one can introduce an equivalence relation on manifolds by saying that two m-manifolds M and N are *cobordant*, if there is an $(m+1)$-manifold W whose boundary is the disjoint union of M and N, $\partial W = M \sqcup N$. We then call W a *cobordism* from W to M.

One can then define, for example, the *(oriented) bordism groups* Ω_i (respectively Ω_i^{SO}), freely generated by closed (oriented) manifolds of dimension i, up to (oriented) cobordism.

Thus Ω_0 is $\mathbb{Z}/2$ generated by a point. It has order 2 as a pair of points is cobordant to the empty set via a line.

By contrast Ω_0^{SO} is the group of integers generated by a point.

Ω_1 and Ω_1^{SO} on the other hand are trivial, as the only closed 1-manifold is the circle which is bordant to the empty set via a disk! In low degrees one finds the following unoriented bordism groups:

	Group	Generators
Ω_0	$\mathbb{Z}/2$	$*$
Ω_1	0	$\emptyset$
Ω_2	$\mathbb{Z}/2$	$\mathbb{R}P^2$
Ω_3	0	$\emptyset$
Ω_4	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{R}P^4, \mathbb{R}P^2 \times \mathbb{R}P^2$
Ω_5	$\mathbb{Z}/2$	$SU(3)/SO(3)$
Ω_6	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{R}P/6, \mathbb{R}P^2 \times \mathbb{R}P^4, \mathbb{R}P^2 \times \mathbb{R}P^2 \times \mathbb{R}P^2$

and the following oriented bordism groups:

	Group	Generators
Ω_0^{SO}	$\mathbb{Z}$	$*$
$\Omega_{1,2,3}^{SO}$	0	$\emptyset$
Ω_4^{SO}	$\mathbb{Z}$	$\mathbb{C}P^2$
Ω_5^{SO}	$\mathbb{Z}/2$	$SU(3)/SO(3)$
$\Omega_{6,7}^{SO}$	0	$\emptyset$
Ω_8^{SO}	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{C}P^2 \times \mathbb{C}P^2, \mathbb{C}P^4$

In fact the cartesian product gives a natural ring structure to Ω_* and Ω_*^{SO}. The unoriented bordism ring was computed by Thom: It is a polynomial algebra $\mathbb{Z}/2\mathbb{Z}[p_i]$ over $\mathbb{Z}/2$ with generators p_i in degree i for each $i \neq 2^i - 1$.

The oriented bordism ring is more complicated, but also known.

14. Alexander Duality

This is another special case of Alexander-Lefschetz duality.

Proposition 14.1 (Alexander duality). *Let $K \subset M$ be a compact, locally contractible, nonempty, proper subspace of a orientable n-manifold M. Then*

$$\tilde{H}_i(M, M \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i}(K; \mathbb{Z}).$$

Corollary 14.2. *For K as above a subspace of $\mathbb{S}^n$ we have*
$$\tilde{H}_i(\mathbb{S}^n \setminus K) \cong \tilde{H}^{n-i-1}(K;\mathbb{Z}).$$

Remark 14.3. This tells us that the homology of the complement is unaffecetd by how we embed our copy of K into M. In particular, we cannot study knots (i.e. homeomorphic impages of $\mathbb{S}^1$ in $\mathbb{R}^3$) by the homology of the complement. The fundamental group of the knot complement does a better job. Here the un-knot gives the integers, but for instance the complement of the trefoil knot has a fundamental group that is *not* isomorphic to the integers, but is isomorphic to the group $\langle a, b | a^2 = b^3\rangle$. This group is actually isomorphic to the braid group on three strands.

Proposition 14.4. *Let M be a compact oriented connected m-manifold and let $K \subset M$ be nonempty, proper, compact, locally contractible subspace. If $H_1(M)$ is trivial, then $H^{m-1}(K)$ is free abelian and $M \setminus K$ has* $\operatorname{rank}(H^{m-1}(K)) + 1$ *components.*

Proof. Let $k = |\pi_0(M \setminus K)|$ be the number of components of the complement of K in M. Therefore
$$k = \operatorname{rank} H_0(M \setminus K) = 1 + \operatorname{rank} \tilde{H}_0(M \setminus K).$$
By assumption $H_1(M) = 0 = \tilde{H}_0(M)$ and therefore we know from the long exact sequence and Alexander duality 14.1 that
$$\tilde{H}_0(M \setminus K) \cong H_1(M, M \setminus K) \cong H^{m-1}(K).$$
□

We have the following famous corollary:

Corollary 14.5 (Jordan curve theorem). *Let C be a simple curve in $\mathbb{R}^2$, i.e. a subset homeomorphic to $\mathbb{S}^1$. Then $\mathbb{R}^2 \setminus C$ has two components.*

Proof. Add a point to turn $\mathbb{R}^2$ into $\mathbb{S}^2$ and apply Proposition 14.4 for $M = \mathbb{S}^2$. □

As a historical aside, Jordan proved this theorem in 1887, without any use of algebraic topology. Brouwer then proved the n-dimensional version in 1910, an early triumph of topology. There was some consensus that Jordan's proof was incomplete or even flawed, but later authors (notably Thomas Hales) declared the proof essentially correct. Nevertheless, the topological proof is much more concise and generalizes easily. All of this is at the cost of introducing some serious machinery.

Preparing the last proposition I noticed some gaps in the previous notes: In Theorem 8.12 we claimed that if M is a connected and compact manifold of dimension m with $H_m(M;\mathbb{Z}) \cong \mathbb{Z}$ then M is orientable.

There are two possible generators in $H_m(M)$ we choose one of them and call it o_M. Then we claim that $((\varrho_{x,M})_* o_M \mid x \in M)$ is an orientation of M. But this needs proof!

To show this we need to know that ϱ_x is an injection. From the long exact sequence in relative homology the kernel is given by $H_n(M \setminus x; \mathbb{Z})$ and must be 0 or $\mathbb{Z}$. But it follows from Corollary 14.6 below that $H_n(M \setminus x; \mathbb{Z}/2) = 0$, and this is impossible if $H_n(M \setminus x;\mathbb{Z}) = \mathbb{Z}$. Note that we used the first implication of Theorem 8.12 to prove that corollary, but not this implication.

Corollary 14.6. *Let M be a non-compact connected manifold. Then $H_n(M;\mathbb{Z}/2) = 0$.*

Proof. As M is orientable with respect to $\mathbb{Z}/2$ we may apply Poincaré duality and find that $H_n(M;\mathbb{Z}/2) \cong H_c^0(M;\mathbb{Z}/2)$. But unravelling definitions $H_c^0(M)$ are exactly functions with compact support that are constant along any continuous path. But on a non-compact manifold there are no nonzero compactly supported constant functions. □

Proposition 14.7. *Let M be a non-orientable compact manifold. Then $H_n(M;\mathbb{Z}) = 0$ and $H^n(M;\mathbb{Z})$ has torsion.*

Proof. This is true in general, but we will only prove the case that M has a finite cell structure, e.g. a triangulation. In this case it is immediate from cellular homology that $H_n(M;\mathbb{Z})$ is torsion-free, so it is isomorphic to $\mathbb{Z}^r$ and r cannot be 1, else M would be orientable by Theorem 8.12. But if $r > 1$ the universal coefficient theorem implies that $H_n(M;\mathbb{Z}/2)$ has rank greater than 1, contradicting Theorem 8.12 for $\mathbb{Z}/2$-coefficients.

Thus $H_n(M;\mathbb{Z}) = 0$. But then $\mathrm{Tor}(H_{n-1}(M;\mathbb{Z});\mathbb{Z}/2) = \mathbb{Z}/2$ by $\mathbb{Z}/2$-orientability. This implies that $H_{n-1}(M;\mathbb{Z})$ has $\mathbb{Z}/2$-torsion and then $\mathrm{Ext}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}/2) \neq 0$. (In fact it equals $H_{n-1}(M;\mathbb{Z}) = \mathbb{Z}/2$.) □

Proposition 14.8. *If M is a compact connected orientable m-manifold and if the first homology group of M with integral coefficients vanishes, then all compact submanifolds without boundary of dimension $m-1$ are orientable.*

Proof. A submanifold $N \subset M$ satisfies the assumptions of Alexander duality, thus we have
$$H^{m-1}(N) \cong H_1(M, M \setminus N) \cong \tilde{H}_0(M \setminus N)$$
and $H^{m-1}(N)$ is free abelian. This implies that the components of N are orientable by Corollary 14.7 □

Corollary 14.9. *It is not possible to embed $\mathbb{R}P^2$ or K into $\mathbb{R}^3$.*

Proof. If one could, then one could also embed $\mathbb{R}P^2$ or K into $\mathbb{S}^3$ as the one-point compactification of $\mathbb{R}^3$. Due to $H_1(\mathbb{S}^3) = 0$, the 2-manifold $\mathbb{R}P^2$ would be orientable, but we know that it's not. □

At the math institute in Oberwolfach there is a model of the Boy surface, see Figure 1. That is a model of an immersion of $\mathbb{R}P^2$ into three-space.

FIGURE 1. Photo credit: Florian-TFW, CC BY-SA 3.0

APPENDIX A

Some background

A.1. Quotient homotopies

We recall the following key result about the compact open topology on spaces of maps between topological spaces, details are for example in the Appendix of [Hatcher], startin on p. 529.

LEMMA A.1.1. *Fix three topological spaces X, Y, Z. Denote by* $\mathrm{Map}(X, Y)$ *the set of continous maps from X to Y equipped with the* compact open topology. *Whenever Y is locally compact* $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Map}(Y, Z))$.

LEMMA A.1.2. *Let $i : A \to U$ be a deformation retract, i.e. there is $r : U \to A$ such that $ri = \mathrm{id}_A$ and $ir \simeq \mathrm{id}_U$ via a homotopy H fixing A. Then $\bar{i} : A/A \to U/A$ is also a deformation retract.*

PROOF. The canonical projection gives $\bar{r} : U/A \to A/A$ and necessarily $\bar{r} \circ \bar{i} = \mathrm{id}_{A/A}$. It remains to show that $\bar{i} \circ \bar{r}$ is homotopic to $\mathrm{id}_{U/A}$. Let $H : U \times [0, 1] \to U$ be the homotopy from ir to id_U. We want to define $\bar{H} : U/A \times [0, 1] \to U/A$. For any fixed $t \in [0, 1]$ we have $\bar{H}_t : U/A \to U/A$ by the properties of the quotient topology, but it is not at all clear that these maps are continuous in t!

Let $q : U \to U/A$ be the projection and consider $q \circ H \in \mathrm{Hom}(U \times [0, 1], U/A)$. As $[0, 1]$ is locally compact we may apply Lemma A.1.1 and rewrite our map as $H' \in \mathrm{Hom}(U, \mathrm{Map}([0, 1], U/A))$. As $H'(a)$ is the constant function with value A/A for all a H' factors through U/A, and we obtain $\bar{H}' \in \mathrm{Hom}(U/A, \mathrm{Map}([0, 1], U/A))$, which gives rise to $\bar{H} \in \mathrm{Hom}(U/A \times [0, 1], U/A)$ by Lemma A.1.1 again. This is the desired homotopy. $\square$