# A short introduction to Categories and Universal Constructions

**For Algebraic Topology - summer semester 2023**

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## Contents

This is a short introduction to category theory with a focus on definitions and examples for the Algebraic Topology masters course at University of Hamburg in summer semester 2023.

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## Literature

Some useful books and lecture notes for further reading are the following.


- J. Goedecke, *Category Theory* (lecture notes), `www.julia-goedecke.de/pdf/CategoryTheoryNotes.pdf`;
- T. Leinster, *Basic Category Theory*, Cambridge University Press 2014;
- S. Mac Lane, *Categories for the Working Mathematician*, Springer 1971 (second edition 1998);
- B. Richter, *From Categories to Homotopy Theory*, Cambridge University Press 2020;
- E. Riehl, *Category Theory in Context*, Courier Dover Publications 2017.

# 1 Categories, Functors and Natural Transformations

## 1.1 Categories

**Definition 1.1.** A **category** $\mathscr{C}$ consists of the following data:

- a class of **objects** $\mathrm{Ob}(\mathscr{C})$,
- for every pair of objects $X, Y \in \mathrm{Ob}(\mathscr{C})$ a class of **morphisms** $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ (also called arrows),
- for every object $X$ a distinguished morphism $\mathbf{1}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$, the **identity**,
- for every three objects $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ a **composition** $\circ : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$,

such that

- compositon is associative: $(f \circ g) \circ h = f \circ (g \circ h)$,
- the identity is an identity for composition: $\mathbf{1}_Y \circ f = f = f \circ \mathbf{1}_X$ for $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Given $f$ in $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ we call $X$ the **source** or **domain** and $Y$ the **target** or **codomain** of $f$.

*Remark* 1.2. Sometimes we say **arrows** or **maps** instead of morphisms.

We will often abuse notation and write $C \in \mathscr{C}$ as a shortcut for "$C$ is an object of $\mathscr{C}$".

*Example* 1.3.

Set The category Set has all sets as objects and the morphisms are given by functions between sets. Note that the collection of all sets is not a set itself, which is why we referred to Ob as a class of objects in our definition.

Top Topological spaces and continuous maps form a category Top.

$\mathsf{Top}_*$ There is also a category $\mathsf{Top}_*$ whose objects are pointed topological spaces $(X, x_0)$ with basepoint $x_0 \in X$ and whose morphisms are basepoint preserving (and continuous) maps, i.e. $f : (X, x_0) \to (Y, y_0)$ is given by $f : X \to Y$ with $f(x_0) = y_0$.

hTop The category of topological spaces and homotopy classes of maps as morphisms.

Grp Groups and group homomorphisms.

Ab Abelian groups and group homomorphisms.

Ch Chain complexes and chain maps.

$\mathsf{Vect}_k$ $k$-vector spaces and $k$-linear maps.

**I** There is a category with one object and one morphism (the identity of the object). In general a category is called **discrete** if the identities are the only morphisms. Every set $I$ can be considered as a discrete category **I** with $\mathrm{Ob}(\mathbf{I}) = I$.

$\mathscr{C}^{\mathrm{op}}$ For every category $\mathscr{C}$ there is an **opposite category** $\mathscr{C}^{op}$ with the same objects, $\mathrm{Hom}_{\mathscr{C}^{op}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$ and $f \circ_{\mathscr{C}^{op}} g := g \circ_{\mathscr{C}} f$. Thus we obtain the opposite category $\mathscr{C}^{op}$ from $\mathscr{C}$ by turning around all arrows. See section 1.5.

*Remark* 1.4. If the objects and morphisms of a category form sets we call it a **small category**. If there may be a class of objects but the morphisms between any two pair of objects form a set we say the category is **locally small**.

Many categories we are interested in, like Top, Set and Grp are not small, but locally small.

*Example* 1.5. A small category in which there is at most one morphism between any two objects and in which any isomorphism is an identity is called a **partial order**. Then the composition is uniquely determined by the morphisms as there is only one function into a set with one element.

An example is the category $\mathbb{N}$ whose objects are the natural numbers and where there is a morphism $i \to j$ if and only if $i \leq j$.

**Definition 1.6.** A morphism $f : C \to D$ in a category $\mathscr{C}$ is called **isomorphism**, if there is a morphism $g : D \to C$ in $\mathscr{C}$ such that $g \circ f = \mathbf{1}_C$ and $f \circ g = \mathbf{1}_D$. In this case we call $C$ and $D$ isomorphic.

In all categories we consider isomorphic object as equivalent and (almost) interchangeable.

*Example* 1.7.

- Set The isomorphisms in Set are the bijective maps.
- Top The isomorphisms in Top are the homeomorphisms.
- hTop The isomorphisms in hTop are the homotopy equivalences.

We can also define generalizations of injective and surjective maps.

**Definition 1.8.** A morphism $f : C \to D$ in a category $\mathscr{C}$ is called **monomorphism** if it is left-cancellative, i.e. for all objects $B$ and morphisms $g, h : B \to C$ in $\mathscr{C}$ we have

$$f \circ g = f \circ h \implies g = h.$$

Similarly, $f$ is an **epimorphism** if is right-cancellative, i.e. for all objects $B$ and morphisms $g, h : B \to C$ in $\mathscr{C}$ we have

$$g \circ f = h \circ f \implies g = h.$$

*Example* 1.9.

- Set Monomorphisms in Set are exactly the injections and epimorphisms are the surjections.
- $\mathscr{C}^{\mathrm{op}}$ A monomorphism in a category $\mathscr{C}$ is an epimorphism in $\mathscr{C}^{\mathrm{op}}$. An epimorphism in $\mathscr{C}$ is a monomorphism in $\mathscr{C}^{\mathrm{op}}$.

Every isomorphism is both a monomorphism and an epimorphism. The converse fails in general:

*Remark* 1.10. A monomorphism does not necessarily have a left inverse. In Ab the morphism $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ given by $n \mapsto 2n$ is left-cancellative. But there is no morphism $g : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ such that $g \circ f = \mathbf{1}_{\mathbb{Z}/2\mathbb{Z}}$.

## 1.2 Functors

An important motivation for the study of category theory is the observation that mathematical objects are often better understood through the morphisms between them. The same principle holds for categories.

**Definition 1.11.** A **functor** $F$ between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of the following data:

- a map that associates to any $X \in \mathrm{Ob}(\mathscr{C})$ an object $F(X) \in \mathrm{Ob}(\mathscr{D})$,

- for each pair of objects $X, Y \in \mathrm{Ob}(\mathscr{C})$ a map from $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ to $\mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ which we write as $f \mapsto F(f)$,

such that

- $F$ is compatible with composition: $F(f \circ g) = F(f) \circ F(g)$,
- $F$ preserves the identities: $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$.

*Example* 1.12.

1. For every category $\mathscr{C}$ there is an identity functor $\mathbf{1}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ that maps each object and each morphism to itself.
2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $D$ an object of $\mathscr{D}$. Then there is a constant functor $c_D : \mathscr{C} \to \mathscr{D}$ that sends every object of $\mathscr{C}$ to $D$ and any morphism of $\mathscr{C}$ to $\mathbf{1}_D$.
3. A family of topological spaces $(X_i)_{i \in I}$ is nothing but a functor from $I$, considered as a discrete category, to $\mathsf{Top}$.
4. From every category whose objects have an underlying set, e.g. $\mathsf{Top}, \mathsf{Grp}, \mathsf{Vect}_k$, there is a **forgetful functor** to $\mathsf{Set}$, that forgets all additional structure.
5. There is an abelianization functor $\cdot_{\mathrm{ab}} : \mathsf{Grp} \to \mathsf{Ab}$ that associates to each group $G$ its abelianization $G/[G, G]$. For a homomorphism $f : G \to H$ the map $f_{\mathrm{ab}} : G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is constructed by composing $f$ with the projection to $H_{\mathrm{ab}}$ and showing that it factors through $G/[G, G]$.
6. The fundamental group is a functor
$$\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$$
associating to any pointed topological space $(X, x_0)$ the fundamental group $\pi_1(X, x_0)$ and to any map $f : X \to Y$ the induced map $f_*$. The same construction also defines a functor
$$\pi_1 : \mathsf{hTop}_* \to \mathsf{Grp}, \quad (X, x_0) \mapsto \pi_1(X, x_0)$$
from the category of pointed topological spaces with homotopy classes of maps relative to the basepoint.
7. homology groups are functors
$$H_n : \mathsf{Top} \to \mathsf{Ab}.$$
8. Let $C$ be an onject in a category $\mathscr{C}$. Then the **Hom functor**
$$\mathrm{Hom}_{\mathscr{C}}(C, \cdot) : \mathscr{C} \to \mathsf{Set}, \qquad A \mapsto \mathrm{Hom}_{\mathscr{C}}(C, A), \quad f \mapsto f_*$$
is defined if $\mathscr{C}$ is locally small. In this case there is also a functor $\mathrm{Hom}_{\mathscr{C}}(\cdot, C) : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$. Note that this functor turns around the direction of arrows, which is why we write it as a functor from the opposite category. We also call such functors **contravariant functors** (from $\mathscr{C}$ to $\mathsf{Set}$). See section 1.6.

It is easy to see that functors can be composed, so there is a **category of categories** $\mathsf{Cat}$ whose objects are (small) categories and whose morphisms are functors.

## 1.3 Natural Transformations

Remarkably, there are not just maps between categories (the functors) but also maps between maps between categories.

**Definition 1.13.** Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors. A **natural transformation** $\alpha : F \implies G$ from $F$ to $G$ consists of maps $\alpha_C : FC \to GC$ for every $C \in \mathscr{C}$ such that for every map $f : C \to C'$ in $\mathscr{C}$ there is a commutative diagram:

$$\begin{array}{ccc} FC & \xrightarrow{\alpha_C} & GC \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FC' & \xrightarrow[\alpha'_C]{} & GC' \end{array}$$

*Remark* 1.14. You might think that it is easier to write $\alpha_{C'} \circ Ff = Gf \circ \alpha_C$ instead of drawing the commutative diagram.

The commutative diagram has the advantage that it keeps track of all the objects as well as the morphisms between them. More importantly, in category theory, algebraic topology and homological algebra there is often a plethora of maps whose compositions we want to compare, and it is much easier to keep track if one arranges them all in a beautiful diagram.

*Example* 1.15.

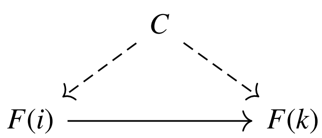
1. For any functor $F : \mathscr{C} \to \mathscr{D}$ there is the identity natural transformation $\mathbf{1}_F$ defined by $(\mathbf{1}_F)_C = \mathbf{1}_{FC}$ for every $C \in \mathscr{C}$.
2. There is a functor $D : \mathsf{Vect}_k \to \mathsf{Vect}_k$ that takes every vector space to its double dual $V \mapsto (V^*)^*$. Then for every vector space there is a map $\iota : V \to DV$ that sends $v \in V$ to the functional $\alpha \mapsto \alpha(v)$. This map is natural, meaning it is compatible with linear maps. In other words, $\iota$ is a natural transformation from the identity functor $\mathbf{1}_{\mathsf{Vect}}$ to the double dual $D$.
3. Fix two categories $I$ and $\mathscr{C}$, where we may think of $I$ as being somehow small.

   We will consider a functor $F : I \to \mathscr{C}$ as a **diagram** in $\mathscr{C}$, given by objects $F(i)$ together with arrows $F(f) : F(i) \to F(j)$ for every morphism $f : i \to j$ in $I$.

   Any object $C$ of $\mathscr{C}$ determines a constant functor $c_C : I \to \mathscr{C}$ that sends any $i$ to $C$ and any $f : i \to j$ to $\mathbf{1}_C$.

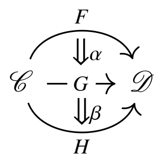
   Then natural transformation from $c_C$ to another functor $F : I \to \mathscr{C}$ is given by maps $\alpha_i : C \to F(i)$ for every $i \in I$ such that $F(f) \circ \alpha_i = \alpha_j$ for every $f : i \to j$.

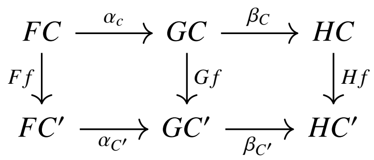
   We call a natural transformation from a constant diagram to $F$ a **cone** over $F$. We think of $C$ as the tip of the cone, and there are arrows going to all the vertices of the diagram, making all the triangles commute.



4. For every topological space $X$ we have a functor which takes the underlying set of $X$ and equips it with the discrete topology, write this as $X^\delta$. Then the identity map from $X^\delta$ to $X$ is continuous. In fact it is a natural transformation from the discretization functor to the identity functor $X^\delta \mapsto X$.

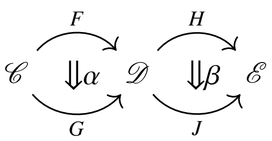
Natural transformations may be composed and form the morphisms in the **category of functors** $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ between two categories: For two natural transformations $\alpha : F \implies G$ and $\beta : G \implies H$ for the functors $F, G, H : \mathscr{C} \to \mathscr{D}$, define $\beta \circ \alpha : F \implies H$ componentwise $(\beta \circ \alpha)_C := \beta_C \circ \alpha_C$.

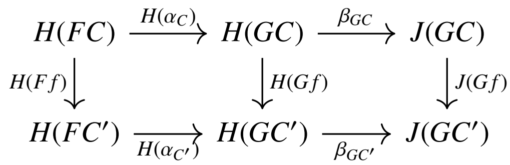




This is called **vertical composition** of natural transformations.

*Remark* 1.16. There is also a **horizontal composition**: For two natural transformations $\alpha : F \implies G$ and $\beta : H \implies J$ for the functors $F, G : \mathscr{C} \to \mathscr{D}$ and $H, J : \mathscr{D} \to \mathscr{E}$, define $\beta * \alpha : H \circ F \to J \circ G$ by $(\beta * \alpha)_C := \beta_{GC} \circ H(\alpha_C) = J(\alpha_C) \circ \beta_{FC}$.





**Definition 1.17.** A natural tranformation $\alpha$, such that all $\alpha_C$ are isomorphisms, is an isomorphism in the category of functors (with vertical composition) and is called a **natural isomorphism**. Two functors, such that there exists a natural isomorphism between them, are called **naturally isomorphic**.

*Remark* 1.18. The fact that we have a category of small categories Cat, where the sets of morphisms $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ form a category again, hints at a higher level concept: (strict) 2-categories. In fact, there a lot of different "higher categories" and their theory is very important in modern Algebra, Topology and Geometry. (But we will not need it for this course.)

## 1.4 Equivalence of Categories

For two categories to be considered the same, looking at isomorphisms in Cat is to strict. Instead, we only require them to be isomorphic "up to natural isomorphism".

**Definition 1.19.** Two categories are **equivalent** if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that $F \circ G$ is naturally isomorphic to $\mathbf{1}_{\mathscr{D}}$ and $G \circ F$ is naturally isomorphic to $\mathbf{1}_{\mathscr{C}}$. We call $F$ (and $G$) an **equivalence of categories**.

We can give a more concrete description, for which we need some definitions.

**Definition 1.20.** functor $F : \mathscr{C} \to \mathscr{D}$ is **full** if it induces surjections on all hom-sets, i.e. every $g : FC \to FC'$ in $\mathscr{D}$ is $F(f)$ for some $f : C \to C'$.

The functor $F$ is **faithful** if it induces injections on all hom-sets, i.e. $F(f) = F(f')$ only if $f = f'$.

$F$ is **fully faithful** if it is both full and faithful.

$F$ is **essentially surjective** if every object in $\mathscr{D}$ is isomorphic to some object $FC$ in the image of $F$.

**Proposition 1.21.** *Let $F : \mathscr{C} \to \mathscr{D}$ be functor and assume the axiom of choice. Then the following are equivalent*

1. *$F$ is an equivalence of categories*
2. *$F$ is fully faithful and essentially surjective.*

(We only need the Axiom of choice for "2 $\implies$ 1")

*Example* 1.22.

1. Let $\mathscr{C}$ be the category with two objects, $C$ and $C'$, and four morphisms: the two identities and one morphism $C \to C'$ and one $C' \to C$. Then $\mathscr{C}$ is equivalent to the category with one object and one morphism.
2. Let $k$ be a field. There is an equivalence of categories from finite-dimensional $k$-vector spaces to its opposite category, given by $V \mapsto V^*$ on objects.
3. Let $\mathsf{Mat}$ be the category whose objects are non-negative integers and whose morphisms from $m$ to $n$ are $(m \times n)$-matrices. Composition is given by matrix multiplication.

   Then there is a natural functor from $\mathsf{Mat}$ to the category of finite-dimensional $\mathbb{R}$-vector spaces, given by $n \mapsto \mathbb{R}^n$ on objects. This is an equivalence of categories, This is the correspondence of matrices and linear maps from a first course in linear algebra.

## 1.5 Opposite categories

We recall the $\mathscr{C}^{\text{op}}$-example in 1.3:

**Definition 1.23.** Let $\mathscr{C}$ be any category. Then its **opposite category** $\mathscr{C}^{\text{op}}$ is defined to have the same objects as $\mathscr{C}$ but $\text{Hom}_{\mathscr{C}^{\text{op}}}(C, D) := \text{Hom}_{\mathscr{C}}(D, C)$ and $f \circ_{\mathscr{C}^{\text{op}}} g := g \circ_{\mathscr{C}} f$.

In words $\mathscr{C}$ is obtained by turning around all the arrows in $\mathscr{C}$.

Clearly any functor $F : \mathscr{C} \to \mathscr{D}$ induces an opposite functor $F^{\text{op}} : \mathscr{C}^{\text{op}} \to \mathscr{D}^{\text{op}}$.

Many natural functors, like cohomology, turn around the order of arrows, i.e. cohomology is a functor $\mathsf{Top}^{\text{op}} \to \mathsf{Ab}$.

**Definition 1.24.** We call a functor $\mathscr{C}^{\text{op}} \to \mathscr{D}$ a **contravariant functor** from $\mathscr{C}$ to $\mathscr{D}$.

By using the opposite of categories and functors, we can dualize all the definitions and results in category theory.

Moreover, whenever we prove a statement about a category $\mathscr{C}$ then the **dual statement** holds for its opposite category.

### 1.6 The Hom functor

We recall example 1.12.8. Forming the hom-sets in a category is actually functorial. Let us explain what this means.

Let $\mathscr{C}$ be a locally small category, i.e. the morphisms between any two objects form a set (rather than a proper class). Let $C$ be an object of $\mathscr{C}$.

**Definition 1.25.** The **Hom functor**, denoted $h_C : \mathscr{C} \to \mathsf{Set}$, sends any object $D$ to $\mathrm{Hom}_{\mathscr{C}}(C, D)$ and any morphism $f : D \to D'$ to the map $f_* : \mathrm{Hom}_{\mathscr{C}}(C, D)$ to $\mathrm{Hom}_{\mathscr{C}}(C, D')$ defined by $g \mapsto f \circ g$.

We can of course also put the object $C$ in the second place of Hom. Then our functor will be contravariant and turn around the order of arrows. We obtain $h^C : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$ which is defined by $D \mapsto \mathrm{Hom}_{\mathscr{C}}(D, C)$ and $f \mapsto f^*$, where $f^*(g) = g \circ f$.

For another level of abstraction, $h_{(-)}$ defines a functor from $\mathscr{C}^{\mathrm{op}}$ to the category of functors $\mathrm{Fun}(\mathscr{C}, \mathsf{Set})$. This is a fully faithful functor that is called the **Yoneda embedding**. Any functor naturally isomorphic to $h_C$ is called **representable**.

*Example* 1.26. The forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$ is representable by the group of integers.

Unravelling our definition this means that there for every group $G$ there is an isomorphism $\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G) \cong U(G)$, and these isomorphisms are compatible with group homomorphisms.

But this just says that the set of morphisms from $\mathbb{Z}$ to $G$ is exactly the set of elements of $G$, the isomorphism is given by sending $f : \mathbb{Z} \to G$ to $f(1) \in G$.

*Remark* 1.27. A key result in category theory is the **Yoneda lemma**. It states that natural transformations from $h^C$ to some other functor $F : \mathscr{C} \to \mathsf{Set}$ are in natural bijection with $F(C)$. It's not hard, but very consequential. (Although we won't need it.)

## 2 Universal constructions

One of the most important aspects of category theory is that it allows us to unify many constructions in mathematics via **universal properties**. There is a way to make the notion of universal property precise via the Yoneda Lemma. For our purposes, however, it is more important to understand examples. Before we go to the general definition of limits and colimits we first discuss some important special cases.

### 2.1 Terminal and initial objects

**Definition 2.1.** Let $\mathscr{C}$ be a category.

An object $T \in C$ is **terminal** if for every $C \in \mathscr{C}$ there is exactly one morphism $C \to T$.

An object $I \in C$ is **initial** if for every $C \in \mathscr{C}$ there is exactly one morphism $I \to C$.

An object $O \in C$ that is both terminal and initial is called a **zero object**.

*Example* 2.2.

$\mathsf{Set}$ The empty set is an initial object. Every singleton set (one element set) is a terminal object. Note that there is a unique isomorphism between two singleton sets.

**Top** The empty space is initial and any singleton space is terminal.

**$\mathsf{Top}_*$** Any singleton space is both initial and terminal, i.e. a zero object.

**Ab** The trivial group is a zero object.

**Ch** The zero complex is a zero object.

**Field** The category Field of fields and field homomorphisms has neither initial nor terminal objects. (There are no field homomorphisms between fields of different characteristics).

(Partial order) Recall example 1.5. A partial order may or may not have initial and/or terminal objects. This depends on whether the corresponding partially ordered set has least and/or maximal elements.

We can see that initial objects and terminal objects need not exist in a category. But if they do, they are unique up to unique isomorphism:

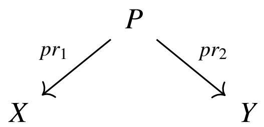
If $I$ and $I'$ are both initial objects in $\mathscr{C}$, then there is exactly one arrow $f : I \to I'$ and one arrow $g : I' \to I$. Since $I$ is initial the only morphism $I \to I$ is $\mathbf{1}_I$ and therefore $g \circ f = \mathbf{1}_I$. Similarly, we have $f \circ g = \mathbf{1}_{I'}$ and we have shown that both $f$ and $g$ are isomorphisms.

This is what it means for the property "initial" to be a universal property.

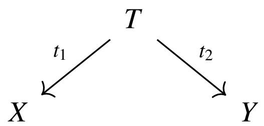
*Remark* 2.3. An terminal object in a category $\mathscr{C}$ is a initial object in $\mathscr{C}^{\text{op}}$ and vice versa.

## 2.2 Products and coproducts

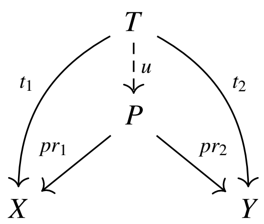
**Definition 2.4.** Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$. The product of $X$ and $Y$ consists of an object $P$ and morphisms




with the universal property that for all other objects and morphisms




there is a unique morphism $u : T \to P$ such that

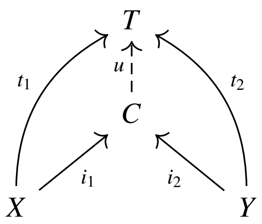



commutes. The morphisms $pr_1$ and $pr_2$ are called **projections**.

Note that a product of $X$ and $Y$ is $(P, pr_1, pr_2)$ and not just the object $P$, although we will often refer to the product as $P$ with the morphisms being implicit. We write $P$ as $X \times Y$ or $X \prod Y$.

We can also define a dual notion.

**Definition 2.5.** Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$. The coproduct of $X$ and $Y$ consists of an object $C$ and maps $i_1 : X \to C$ and $i_2 : Y \to C$ with the universal property that for all other objects $T$ and morphisms $t_1 : X \to T$ and $t_2 : Y \to T$ there is a unique morphism $u : C \to T$ such that




commutes. The morphisms $i_1$ and $i_2$ are called **inclusions**.

We write $C$ as $X \coprod Y$. In some categories like Ab and Ch we often call the coproduct **direct sum** and write $X \oplus Y$.

*Remark* 2.6. A product in a category $\mathscr{C}$ is a coproduct in $\mathscr{C}^{\text{op}}$ and vice versa.

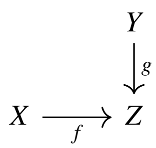
*Remark* 2.7. As with initial and terminal objects, products and coproducts unique in the sense that if there are two products (coproducts), then there is a unique isomorphism between them.

*Example* 2.8.

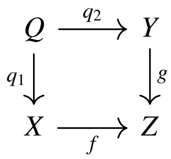
- • • The discrete category with two objects $X$ and $Y$ has no product and no coproduct of $X$ and $Y$.
- Set Products are given by cartesian products and coproducts are disjoint unions.
- Top Products are given by cartesian products of the underlying sets equipped with the product topology. Coproducts are disjoint unions with the disjoint union topology.
- Ab Products are direct products of groups. The coproduct of two groups is given by the direct sum which is the same as the direct product (but with the inclusion maps rather then projections). We will later see that there are products and coproducts with more than two terms. Although the product and coproduct of two abelian groups is isomorphic, this is in general no longer the case for infinite products and coproducts.
- Grp The product of two groups is again the direct product, but the coproduct is the free product of groups.

## 2.3 Pullbacks and pushouts

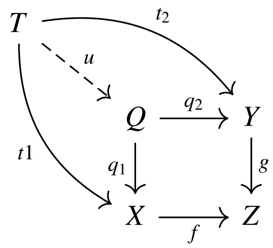
**Definition 2.9.** Let $\mathscr{C}$ be a category. The **pullback** of a diagram



is an object $Q$ with morphisms $q_1 : Q \to X$ and $q_2 :\to Q \to Y$, such that the square

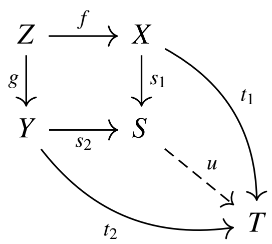



commutes and with the following universal property: For every other object $T$ with morphisms $t_1 : T \to X$ and $t_2 : T \to Y$, that satisfy $f \circ t_1 = g \circ t_2$, there is a unique morphism $u : T \to Q$, such that the diagram




commutes. We denote the pullback with $Q = X \times_Z Y$ or $X \prod_Z Y$.

Dually (with all arrows reversed), we define the **pushout**



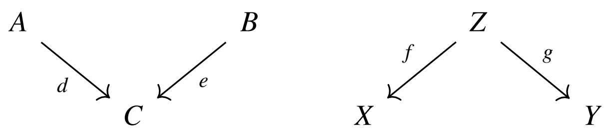

and write $S = X \coprod_Z Y$.

*Remark* 2.10. A pullback square in a category $\mathscr{C}$ is a pushout square in $\mathscr{C}^{\mathrm{op}}$ and vice versa.

Pullbacks and pushouts are unique up to isomorphism.

*Example* 2.11.

∨ ∧ In the category with 6 objects and morphisms given by




(we did not draw identities), $(d, e)$ have no pullback and $(f, g)$ have no pushout.

Set The pullback of $f : X \to Z$ and $j : Y \to Z$ is the subset of the cartesian product $X \times Y$ consisting of pairs $(x, y)$ such that $f(x) = g(y)$ holds. The pushout of $h : Z \to X$ and $j : Z \to Y$ is the disjoint union of $X$ and $Y$, where elements with a common preimage in $Z$ are identified.

- Top The pullback of two continuous maps $f : X \to Z$ and $j : Y \to Z$ is the subspace of $X \times Y$ such that $f(x) = g(y)$ holds. $X \times Y$ consisting of pairs $(x, y)$ such that $f(x) = g(y)$ holds. The pushout of $h : Z \to X$ and $j : Z \to Y$ is the quotient space of the disjoint union of $X$ and $Y$, with the equivalence relation generated by eqivalences of elements with a common preimage in $Z$. A special case of pushouts are adjunction spaces where we "glue" one space onto another.
- Grp The pushout of groups is given by the free product with amalgamation. We have used this to compute fundamental groups with Seifert-Van Kampen.

### 2.4 Limits

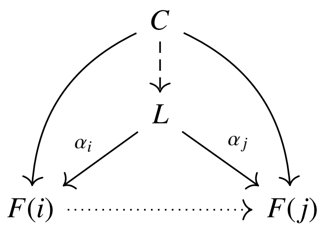
Terminal objects, products and pullbacks are special cases of a limit.

**Definition 2.12.** Let $I$ be a small category and $\mathscr{C}$ any category. A **diagram of shape** $I$ in $\mathscr{C}$ is just a functor $I \to \mathscr{C}$.

We will often write $F_i$ for the objects $F(i)$ for $i \in I$.

**Definition 2.13.** A **limit** of the diagram $F : I \to \mathscr{C}$ is an object $L$ of $\mathscr{C}$ together with a natural transformation $\alpha : c_L \Rightarrow F$ that is **universal** in the sense that any natural transformation from a constant functor $c_C$ to $F$ factors uniquely through $c_L$.

In other words, $L$ and $\alpha$ have the property that whenever we have $C$ in the following diagram there is exactly one dashed arrow $C \to L$ making the diagram commute.




As before, the universal property ensures that if there are two limits $L$ and $L'$ there is a unique isomorphism between them. We thus also speak of **the limit** and denote it by $\lim_I F$ or $\lim F_i$.

Limits in general need not exist.

*Example* 2.14.

1. Let $I$ be the empty set considered as a discrete category without objects. The limit of the unique functor $I \to \mathscr{C}$ is the terminal object of $\mathscr{C}$.
2. The limit of a discrete category with two objects is the product. In general, for any set $I$, considered as a discrete category, the limit of $F : I \to \mathscr{C}$ is called the product of the $F(i)$, often written $\prod_{i \in I} F_i$.

   In particular, the terminal object is the empty product.
3. Let $I$ be the category with two objects and two arrows in the same direction $\bullet \rightrightarrows \bullet$. The limit of $F : I \to \mathscr{C}$ is called **equalizer**.

4. Let $I$ be the category with three objects $\bullet \to \bullet \leftarrow \bullet$. The limit of $F : I \to \mathscr{C}$ is the pullback of the images of the two morphisms in $F$.

   If a pullback diagram takes the form $X \to * \leftarrow Y$, i.e. the middle object goes to the terminal object of $\mathscr{C}$, then the limit is the product $X \times Y$.

*Example* 2.15. The equalizer of two maps $f, g : A \to B$ in Set is exactly the subset of $A$ given by all elements $a$ with $f(a) = g(a)$, this explains the name.

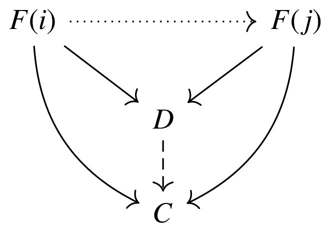
## 2.5 Colimits

We now apply the idea of **dualizing** categorical notions by turning around all the arrows to the previous section. We already saw this idea, when we defined initial objects, coproducts and pushouts corresponding to terminal objects, products and pullbacks.

We change the orientation of all the arrows in the definition of a limit to get the dual notion of a limit, called the colimit.

**Definition 2.16.** A **colimit** of the diagram $F : I \to \mathscr{C}$, denoted by $\mathrm{colim}_I F$, is an object $D$ of $\mathscr{C}$ together with a natural transformation $\alpha : F \Rightarrow c_D$ that is **universal**, in the sense that any natural transformation from $F$ to a constant functor $c_C$ factors uniquely through $c_D$.

The corresponding diagram looks like this:




*Remark* 2.17. We have that $(D, \alpha)$ is a colimit of the diagram $F : I \to \mathscr{C}$ exactly if $(D, \alpha^{\mathrm{op}})$ is a limit of the diagram $F^{\mathrm{op}} : I^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$. Here $\alpha^{\mathrm{op}} : c_D^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$ is the natural transformation corresponding to $\alpha : F \Rightarrow c_D$ under the correspondence of morphisms in $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$.

*Example* 2.18.

1. The colimit of the empty diagram is the initial object of $\mathscr{C}$.
2. The colimit over a discrete category with two objects is the coproduct. In general, for any set $I$, considered as a discrete category, the colimit of $F : I \to \mathscr{C}$ is called the coproduct of the $F(i)$, often written $\coprod_{i \in I} F_i$.
3. Let $I$ The colimit of a diagram of shape $\bullet \leftleftarrows \bullet$ is called **coequalizer**.
4. The colimit of the diagram $\bullet \leftarrow \bullet \to \bullet$ is the **pushout**.

From the definition of limit and colimits it is not hard to obtain the following extremely useful result:

**Proposition 2.19.** *Let $F : I \to \mathscr{C}$ and $G : J \to \mathscr{C}$ be diagrams. Then we have natural isomorphisms*

$$\mathrm{Hom}_{\mathscr{C}}(C, \lim_I F_i) \cong \lim_I \mathrm{Hom}_{\mathscr{C}}(C, F_i)$$

*and*

$$\mathrm{Hom}_{\mathscr{C}}(\operatorname*{colim}_J G_i, C) \cong \lim_J \mathrm{Hom}_{\mathscr{C}}(G_j, C)$$

## 2.6 Existence of (co)limits

We say a category $\mathscr{C}$ **has all small limits**, or that $\mathscr{C}$ is **complete**, if every diagram $I \to \mathscr{C}$ has a limit. Similarly, we say $\mathscr{C}$ **has all small colimits**, or that $\mathscr{C}$ is **cocomplete**, if every diagram $I \to \mathscr{C}$ has a colimit.

This may seem extremely difficult to check, but in fact one can build any limit from just two types of limit:

Recall that an equalizer is a limit for a diagram of the shape $\bullet \rightrightarrows \bullet$ and a product is a diagram whose shape is a discrete category.

We say a category $\mathscr{C}$ has all equalizers if any equalizer diagram has a limit, and similarly for products (and other shapes of diagrams).

**Proposition 2.20.** *A category $\mathscr{C}$ has all limits if and only if it has all products and equalizers. It has all colimits if and only if it has all coproducts and coequalizers.*

## 2.7 Adjunctions

It is rare that categories are equivalent, but a weaker notion is extremely fruitful.

**Definition 2.21.** We say $F : \mathscr{C} \to \mathscr{D}$ is **left adjoint** to $G : \mathscr{D} \to \mathscr{C}$, in symbols $F \dashv G$, if for all $C \in \mathscr{C}$ and $D \in \mathscr{D}$ there are natural isomorphisms

$$\phi_{C,D} : \mathrm{Hom}_{\mathscr{C}}(C, GD) \cong \mathrm{Hom}_{\mathscr{D}}(FC, D)$$

Here naturality means that for every map $C \to C'$ in $\mathscr{C}$ the natural diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(C', GD) & \xrightarrow{\phi_{C',D}} & \mathrm{Hom}_{\mathscr{D}}(FC', D) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle Ff^*} \\ \mathrm{Hom}_{\mathscr{C}}(C, GD) & \xrightarrow{\phi_{C,D}} & \mathrm{Hom}_{\mathscr{D}}(FC, D) \end{array}$$

and a similar diagram commutes for $g : D \to D'$ in $\mathscr{D}$.

If $\mathscr{C}$ and $\mathscr{D}$ are locally small we can also phrase naturality as saying that the two functors $\mathrm{Hom}_{\mathscr{C}}(-, G(-))$ and $\mathrm{Hom}_{\mathscr{D}}(F(-), -)$ from $\mathscr{C}^{\mathrm{op}} \times \mathscr{D}$ to $\mathsf{Set}$ are naturally isomorphic.

*Example* 2.22.

1. Throughout algebra there are adjunctions between free and forgetful functors. For example the forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$ has a left adjoint given by taking a set $X$ to the free group with set of $X$ as set of generators.

2. The forgetful functor $\mathsf{Top} \to \mathsf{Set}$ has a left adjoint given by equipping any set with the discrete topology. It also has a right adjoint given by equipping any set with the indiscrete topology.

Left and right adjoints are naturally dual: If $F : \mathscr{C} \to \mathscr{D}$ is left adjoint to $G$, then $F^{\text{op}} : \mathscr{C}^{\text{op}} \to \mathscr{D}^{\text{op}}$ is right adjoint to $G^{\text{op}}$.

Let $F \dashv G : \mathscr{C} \rightleftarrows \mathscr{D}$ and $C \in \mathscr{C}$. By the adjunction the identity map $\mathbf{1}_{FC} : FC \to FC$ corresponds to a map $\epsilon_C : C \to GFC$. By naturality in the definition of an adjunction the $\epsilon$ assemble into a natural transformation $\epsilon : \mathbf{1}_{\mathscr{C}} \Rightarrow GF$. This is called the **unit** of the adjunction.

Simlarly there is a natural transformation $\eta : FG \Rightarrow \mathbf{1}_{\mathscr{D}}$, called the **counit** of the adjunction.

**Proposition 2.23.** *Let $F \dashv G$. Then unit and counit satisfy the following identities of natural transformations: For every $C \in \mathscr{C}$ we have*

$$\eta_{FC} \circ F(\epsilon_C) = \mathbf{1}_{FC}$$

*and for every $D \in \mathscr{D}$ we have*

$$G(\eta_C) \circ \epsilon_{GD} = \mathbf{1}_{GD}.$$

Put a little differently, we have the following identities of natural transformations: $G\eta \circ \epsilon_G = \mathbf{1}_G$ and $\eta_F \circ F\epsilon = \mathbf{1}_F$.

In fact, adjoints may be equivalently characterized by the existence of unit and counit.

*Remark* 2.24. An adjunction induces an equivalence of categories if and only if unit and counit are natural isomorphisms.

One can also show that adjoints are given by a universal property and are thus unique up to unique natural isomorphism.

Adjoints are closely related to limts:

**Proposition 2.25.** *Let $F$ be a left adjoint. Then $F$ preserves colimits, i.e. whenever $(D, \alpha)$ is a colimit of a diagram $G : I \to \mathscr{C}$ then $(FD, F\alpha)$ is a colimit for $F \circ G : I \to \mathscr{D}$.*

*Dually, if $G$ is a right adjoint then $G$ preserves limits.*

*Remark* 2.26. Under some assumption on the categories $\mathscr{C}$ and $\mathscr{D}$ there is even a converse to the lemma: Any functor preserving all colimits has a left adjoint. There are different theorems, depending on the precise assumptions made, but they are all called **adjoint functor theorems**.

We can even characterize limits using adjoints.

**Proposition 2.27.** *Consider the category* $\text{Fun}(I, \mathscr{C})$ *of $I$-shaped diagrams in $\mathscr{C}$. There is a diagonal functor $\Delta : \mathscr{C} \to$* $\text{Fun}(I, \mathscr{C})$ *sending any object $C$ to the constant functor $c_C$. Then taking the limit of a diagram is right adjoint to $\Delta$, and taking the colimit is left adjoint.*