Suboptimal control strategies for backward facing step flows

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Abstract

In our talk suboptimal distributed and boundary control strategies for fluid flows governed by the Navier-Stokes equation (NSE) together with some aspects of the numerical and practical implementation of the underlying algorithms will be adressed. The objective of our numerical investigations is the development of practical relevant feedback control laws for backward facing step flows. We will present several examples as well as we will adress analytical questions related to the approach sketched in the following.

For what follows denote by $\Omega \subset \mathbb{R}^2(\mathbb{R}^3)$ the region of fluid flow with boundary $\Gamma = \Gamma_c \cup \Gamma_{obs} \cup \Gamma_d$, where in the case of boundary control Γ_c denotes the part of the boundary where control is applied, Γ_{obs} is the observation boundary and Γ_d is the remaining part of the boundary part where fixed Dirichlet data is prescribed. For the case of boundary control and boundary observation the control problem is given by

$$
(P) \quad \left\{ \begin{array}{l}
\min J(\Phi) = J(\Phi, u) = \frac{l}{2} \int\limits_0^T \int_{\Gamma_c} \Phi^2 dS dt + \int\limits_0^T \int_{\Gamma_s} F(\nabla u) dS dt \\
s.t. \\
\begin{array}{rcl}
u_t - \frac{1}{Re}\Delta u + (u\nabla)u + \nabla p & = & f \text{ in } \Omega \times \ (0,T] \\
-div\, u & = & 0 \text{ in } \Omega \times \ (0,T] \\
u & = & \left\{ \begin{array}{ll} \Phi & \text{on } \Gamma_c \times \ (0,T] \\ 0 & \text{on } (\Gamma_d \cup \Gamma_{obs}) \times \ (0,T] \end{array} \right. \\
u(0) & = & u_0,
\end{array}
\end{array} \right. \tag{1}
$$

where $F(z) = F(z_1, \dots, z_4)$ is some sufficiently smooth, scalar valued function, u_0 denotes the initial value of the flow and Φ denotes the control. Applying a Lagrangian approach to problem (P) leads to an optimality system involving the NSE, a backward in time parabolic linear system for the adjoint variables and the optimality condition for Φ (see [1]). To solve (1) numerically for realistic flows necessitates a computing power that is beyond the presently available resources.

Instead of looking for an optimal control for problem (P) and thus looking for an optimal cost reduction, we apply a time integration scheme to the NSE and solve at each time step an optimization problem with an instantaneous cost function related to the one of problem (1). As we shall demonstrate this allows a significant reduction of the costs with readilly available computing power. To be more precise consider f.e. the first order semi implicit time integration scheme

$$
\begin{array}{rll}
u_i - \frac{\delta t}{Re} \Delta u_{i+1} + \nabla p_i = & \delta t(f_i - (u_{i-1}\nabla)u_{i-1}) + u_{i-1} =: R_i & \text{in } \Omega \\
-div\, u_i = & 0 & \text{in } \Omega \\
u_i = & \left\{ \begin{array}{ll} \Phi_i & \text{on } \Gamma_c \\ 0 & \text{on } \Gamma_d \cup \Gamma_{obs}, \end{array} \right. &
\end{array} \tag{2}
$$

where $u_0 = u(0)$. The minimization problem at each time step then is

$$
(P_i) \quad \left\{ \begin{array}{l}
\min J(\Phi_i) = J(\Phi, u_i) = \frac{l}{2} \int_{\Gamma_c} \Phi_i^2 dS + \int_{\Gamma_s} F(\nabla u_i) dS \\
s.t. \ (2).
\end{array} \right. \tag{3}
$$

The Lagrangian corresponding to problem (1) turns out to be (subscripts are dropped)

$$
L(u, p, \phi; \xi, \pi, \mu) =
$$
$$
\frac{l}{2} \int_{\Gamma_c} \Phi^2 \, dS + \int_{\Gamma_s} F(\nabla u) \, dS + \int_\Omega (u - c\Delta u + \delta t \nabla p - R)\xi \, dV - \int_\Omega \pi div u \, dV + \int_{\Gamma_c} (u - \Phi)\mu \, dS,
$$

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where ξ, π, μ denote the adjoint velocity, pressure and boundary control variable, respectively and $c := \frac{\delta t}{Re}$. The optimality system can be deduced from computing formally the derivatives of L w.r.t. u, p and Φ. Having in mind the application of a gradient-type algorithm to problem (3) one needs to be able to compute the derivatives of J at Φ in directions χ. Using the Lagrangian one obtains

$$\nabla J(\Phi)(\chi) = \nabla_\Phi L(\chi) = \int_{\Gamma_c} (l\Phi + c\nabla\xi\eta - \pi\eta)\chi \; dS, \tag{4}$$

where η denotes the outer normal to Ω, ξ is the adjoint velocity and π represents the adjoint pressure. The adjoint variables are determined by the following linear system of equations

$$\begin{aligned}
u - c\Delta u + \delta t \; \nabla p &= R \text{ in } \Omega \\
-div\, u &= 0 \text{ in } \Omega \\
u &= \begin{cases} \Phi & \text{on } \Gamma_c \\ 0 & \text{on } \Gamma_d \cup \Gamma_{obs}, \end{cases} \\
\xi - c\Delta\xi + \nabla\pi &= 0 \text{ in } \Omega \\
-\delta t\, div\, \xi &= 0 \text{ in } \Omega \\
\xi &= \begin{cases} \frac{1}{c} DF(\nabla u)\eta & \text{on } \Gamma_{obs} \\ 0 & \text{on } \Gamma_d \cup \Gamma_c. \end{cases}
\end{aligned} \tag{5}$$

With the gradient of J available the following gradient algorithm can be applied to reduce the value of the cost functional J.

1. Set $k = 0$ and choose Φ^0
2. $\rho^* = \text{argmin}_{\rho \geq 0} H(\rho) := J(\Phi^k - \rho\nabla J(\Phi^k))$
3. $\Phi^{k+1} = \Phi^k - \rho^*\nabla J(\Phi^k)$
4. $k = k + 1$, goto 2.

In this algorithm the expensive part is step 2. Every evaluation $H(\rho)$ involves the solution of system (5) with $\Phi = \Phi^k - \rho\nabla J(\Phi^k)$, so that an efficient inexact line-search should be utilized. At this stage the structure of problem (P) must be exploited. The optimal stepsize in the minimum problem of step 2. can be charactrized via a Lagrange approach similiar to the one above. In practical applications only a few steps of the gradient algorithm are applied at each time step utilizing the optimal stepsize ρ^* (or a suitable estimate thereof).

Further literature concerning this class of problems can be found in [2], [3], [4], [5] [7] and [8].

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