Calculus – 9. Series, Solutions

1. Compute the limits.

$$\text{(a)}\quad \lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}$$

$$\text{(b)}\quad \lim_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2},\qquad m,n\in\mathbb{N}$$

$$\text{(c)}\quad \lim_{x\to 1}\frac{x^n-1}{x^m-1},\qquad m,n\in\mathbb{N}$$

$$\text{(d)}\quad \lim_{x\to+\infty}\frac{(2x-3)^{20}(3x-1)^{30}}{(2x+1)^{50}}$$

$$\text{(e)}\quad \lim_{x\to 1-0}\frac{1}{1-x}\qquad\qquad \text{(f)}\quad \lim_{x\to 1+0}\frac{1}{1-x}$$

$$\text{(g)}\quad \lim_{x\to 1}\frac{1}{1-x}\qquad\qquad \text{(h)}\quad \lim_{x\to 1}\frac{1}{(1-x)^2}$$

Solution. (a) For $x\neq 0$ we have $\frac{(1+x)(1+2x)(1+3x)-1}{x}=6+11x+6x^2$. By Example 2 (a),

$$\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}=\lim_{x\to 0}(6+11x+6x^2)=(6+11x+6x^2)\mid_{x=0}=6.$$

(b) Let $x\neq 0$. By the binomial formula we have with some polynomials $p(x)$ and $q(x)$

$$\begin{aligned}
&\frac{(1+mx)^n-(1+nx)^m}{x^2}=\\
&=\frac{1+\binom{n}{1}mx+\binom{n}{2}m^2x^2+x^3p(x)-\left(1+\binom{m}{1}nx+\binom{m}{2}n^2x^2+x^3q(x)\right)}{x^2}\\
&=\frac{\frac{n(n-1)}{2}m^2x^2-\frac{m(m-1)}{2}n^2x^2+x^3(p(x)-q(x))}{x^2}\\
&=\frac{mn(n-m)}{2}+x(p(x)-q(x)).
\end{aligned}$$

Taking the limit $x\to 0$ in the above expression we obtain $mn(n-m)/2$.
(c) Since $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$ we have for $x\neq 1$

$$\frac{x^n-1}{x^m-1}=\frac{x^{n-1}+x^{n-2}+\cdots+x+1}{x^{m-1}+x^{m-2}+\cdots+x+1}.$$

Since the rational function on the right hand side is defined at $x=1$ we obtain by Example 2 (a),

$$\lim_{x\to 1}\frac{x^n-1}{x^m-1}=\lim_{x\to 1}\frac{x^{n-1}+x^{n-2}+\cdots+x+1}{x^{m-1}+x^{m-2}+\cdots+x+1}=\frac{n}{m}.$$

(d) Since both polynomials $p(x) = (2x-3)^{20}(3x-1)^{30}$ and $q(x) = (2x+1)^{50}$ are of degree 50, by Example 3 the limit $\lim_{x\to\infty} p(x)/q(x)$ is a_{50}/b_{50} where a_{50} and b_{50} are the leading coefficients of p and q, respectively. Hence,

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \frac{2^{20}3^{30}}{2^{50}} = \left(\frac{3}{2}\right)^{30}.$$

(e) We use the E-δ characterization of the left-hand limit $\lim\limits_{x\to a-0} f(x) = +\infty$ given in Homework 9.2 (b) to show that

$$\lim_{x\to 1-0} \frac{1}{1-x} = +\infty.$$

Given $E > 0$ choose $\delta = 1/E$ then

$$0 < 1 - x < \delta = \frac{1}{E}$$

implies

$$\frac{1}{1-x} > E.$$

This proves the claim.

(f) We use the E-δ characterization of the right-hand limit $\lim\limits_{x\to a+0} f(x) = -\infty$ given in Homework 9.4 (d) to show that

$$\lim_{x\to 1+0} \frac{1}{1-x} = -\infty.$$

Given $E > 0$ choose $\delta = 1/E$ then

$$0 < x - 1 < \delta = \frac{1}{E}$$

implies $\dfrac{1}{x-1} > E$ and finally

$$\frac{1}{1-x} < -E.$$

This proves the claim.

(g) The limit does not exist since the left-hand and the right-hand limits do not coincide.

(h) Given $E > 0$ choose $\delta = 1/\sqrt{E}$. Then

$$0 < |\,x - 1\,| < \delta = \frac{1}{\sqrt{E}}$$

implies

$$\left(\frac{1}{1-x}\right)^2 > \left(\sqrt{E}\right)^2 = E,$$

that is

$$\lim_{x\to 1} \frac{1}{(1-x)^2} = +\infty.$$

2. Using inequalities formulate the following statements and give an example for each.

(a) $\lim_{x \to -\infty} f(x) = b$ (b) $\lim_{x \to a-0} f(x) = +\infty$

(c) $\lim_{x \to -\infty} f(x) = +\infty$ (d) $\lim_{x \to a+0} f(x) = -\infty$

Solution. (a)

$$\forall \varepsilon > 0 \ \exists D > 0 \ \forall x \in D(f) : x < -D \Longrightarrow |f(x) - b| < \varepsilon.$$

Example: $f(x) = 1/x$, $b = 0$. Given $\varepsilon > 0$ choose $D = 1/\varepsilon$, then $x < -D = -1/\varepsilon$ implies $|x| = -x > 1/\varepsilon$ and therefore $|1/x| < \varepsilon$.
(b)

$$\forall E > 0 \ \exists \delta > 0 \ \forall x \in D(f) : 0 < a - x < \delta \Longrightarrow f(x) > E.$$

Example: Homework 9.1 (e).
(c)

$$\forall E > 0 \ \exists D > 0 \ \forall x \in D(f) : x < -D \Longrightarrow f(x) > E.$$

Example: $f(x) = -x$. Given $E > 0$ choose $D = E$, then $x < -D = -E$ implies $f(x) = -x > E$.
(d)

$$\forall E > 0 \ \exists \delta > 0 \ \forall x \in D(f) : 0 < x - a < \delta \Longrightarrow f(x) < -E.$$

Example: Homework 9.1 (f).

3. Prove the following three statements.
(a) If $\lim_{x \to a} \dfrac{1}{f(x)} = 0$ and $f(x) > 0$ then

$$\lim_{x \to a} f(x) = +\infty.$$

(b) If $\lim_{x \to a} f(x) = +\infty$ and there exists a real number C such that $g(x) \geq C$ for every real x, then

$$\lim_{x \to a} (f(x) + g(x)) = +\infty.$$

(c)

$$\lim_{x \to 0+0} f(x) = b \quad \text{if and only if} \quad \lim_{z \to +\infty} f\left(\frac{1}{z}\right) = b.$$

Proof. (a) We shall prove that

$$\forall E > 0 \ \exists \delta > 0 \ \forall x \in D(f) : 0 < |x - a| < \delta \Longrightarrow f(x) > E.$$

Given $E > 0$. Since $\lim_{x \to a} \frac{1}{f(x)} = 0$, there exists $\delta > 0$ such that for all $x \in D(f)$, $|x - a| < \delta$ implies

$$\left| \frac{1}{f(x)} \right| < \frac{1}{E}.$$

Taking the reciprocal of the preceding inequality, we have

$$|\, f(x)\,| > E.$$

Since $f(x) > 0$

$$f(x) > E.$$

This completes the proof of (a).
(b) We shall prove that

$$\forall\, E > 0\ \ \exists\, \delta > 0\ \ \forall\, x \in D(f) : |\, x - a\,| < \delta \Longrightarrow f(x) + g(x) > E.$$

Given $E > 0$. Since $\lim\limits_{x \to a} f(x) = +\infty$ there exists $\delta > 0$ such that for all $x \in D(f)$, $|\, x - a\,| < \delta$ implies

$$f(x) > E - C.$$

Therefore,

$$f(x) + g(x) > E - C + C = E,$$

this completes the proof of (b).
(c) Suppose first that $\lim_{x \to 0+0} f(x) = b$. Let $\varepsilon > 0$ be given. By assumption, there exists $\delta > 0$ such that for all $x \in D(f)$

$$0 < x < \delta \quad \text{implies} \quad |\, f(x) - b\,| < \varepsilon.$$

Setting $z = 1/x$ this is equivalent to: For all $1/z \in D(f)$

$$z > \frac{1}{\delta} \quad \text{implies} \quad \left|\, f\left(\frac{1}{z}\right) - b\,\right| < \varepsilon.$$

This proves one direction.
Suppose now $\lim_{z \to +\infty} f\left(\frac{1}{z}\right) = b$. Given $\varepsilon > 0$ choose $D > 0$ such that for all $1/z \in D(f)$

$$z > D \Longrightarrow \left|\, f\left(\frac{1}{z}\right) - b\,\right| < \varepsilon.$$

Setting $x = 1/z$ this is equivalent to: For all $x \in D(f)$

$$0 < x < \frac{1}{D} \Longrightarrow |\, f\,(x) - b\,| < \varepsilon.$$

This completes the proof of (c). ∎