Calculus – 7. Series, Solution

1. Prove that the following series converge and compute their sums.

$$\text{(a)} \sum_{k=2}^{\infty} \frac{(-1)^k}{3^{k+1}} \quad \text{(b)} \sum_{k=1}^{\infty} \frac{100}{4^{k+1}} \quad \text{(c)} \sum_{k=0}^{\infty} \frac{7^k+2}{8^k}.$$

Solution. We use the formula

$$\sum_{n=n_0}^{\infty} cq^n = \frac{cq^{n_0}}{1-q} \quad \text{if} \quad |\, q \,| < 1,$$

which was given in the class.
(a) With $q = -1/3$, $c = 1/3$, and $n_0 = 2$ we obtain

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{3^{k+1}} = \frac{\frac{1}{3}\left(-\frac{1}{3}\right)^2}{1-\left(-\frac{1}{3}\right)} = \frac{\frac{1}{27}}{\frac{4}{3}} = \frac{1}{36}.$$

(b) With $q = 1/4$, $c = 100/4 = 25$, and $n_0 = 1$ we obtain

$$\sum_{k=1}^{\infty} \frac{100}{4^{k+1}} = \frac{25 \cdot \frac{1}{4}}{1-\frac{1}{4}} = \frac{25}{3}.$$

(c) The series can be splitted into two converging geometric series with both $n_0 = 0$; the first one with $q = 7/8$ and $c = 1$, the second one with $q = 1/8$ and $c = 2$. Hence,

$$\sum_{k=0}^{\infty} \frac{7^k+2}{8^k} = \frac{1}{1-\frac{7}{8}} + \frac{2}{1-\frac{1}{8}} = 8 + \frac{2}{7} = 8\frac{2}{7}.$$

2. Which of the following series are convergent which are divergent. Give reasons!

$$\text{(a)} \sum_{k=1}^{\infty} k^3 3^{-k} \quad \text{(b)} \sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k+1} \quad \text{(c)} \sum_{k=1}^{\infty} \frac{1}{2k-1}.$$

(*Hint.* Use the quotient and the comparison tests)

Solution. (a) Noting that $(5/4)^3 < 2$, $n \geq 4$ implies

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^3 \cdot 3^n}{3^{n+1} n^3} = \frac{\left(1+\frac{1}{n}\right)^3}{3} < \frac{\left(\frac{5}{4}\right)^3}{3} < \frac{2}{3} < 1;$$

hence the ratio test gives convergence.
(b) Since

$$\frac{2^k+3^k}{4^k+1} < \frac{2 \cdot 3^k}{4^k} = 2\left(\frac{3}{4}\right)^k$$

the comparison test with the geometric series $2\sum_{k=1}^{\infty}(3/4)^k < \infty$ gives convergence.
(c) Since $1/(2k) < 1/(2k-1)$ the comparison test with the harmonic series

$$+\infty = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{2k} < \sum_{k=1}^{\infty} \frac{1}{2k-1}$$

gives divergence.

3. Which of the following series are convergent which are divergent?

(a) $1 + \frac{2}{3} + \frac{3}{5} + \cdots + \frac{n}{2n-1} + \cdots$

(b) $\frac{1}{1001} + \frac{1}{2001} + \frac{1}{3001} + \cdots + \frac{1}{1000n+1} + \cdots$

(c) $\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}} + \frac{1}{3\sqrt{4}} + \cdots + \frac{1}{n\sqrt{n+1}} + \cdots$

(d) $\sum_{k=1}^{\infty} \frac{1000^k}{k!}$

(e) $0.001 + \sqrt{0.001} + \sqrt[3]{0.001} + \cdots + \sqrt[n]{0.001} + \cdots$

Solution. (a) Since $a_n = \frac{n}{2n-1} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}$ the necessary condition for convergence $a_n \to 0$ does not hold; hence the series diverges.
(b) Since for every positive integer n, $1/(1000n+1) > 1/(1001n)$ we have

$$\frac{1}{1001} + \frac{1}{2001} + \frac{1}{3001} + \cdots + \frac{1}{1000n+1} + \cdots > \frac{1}{1001}\sum_{n=1}^{\infty} \frac{1}{n}.$$

Comparing this with the harmonic series gives divergence.
(c) Since

$$\frac{1}{n\sqrt{n+1}} < \frac{1}{n^{\frac{3}{2}}}$$

Example 10 together with the comparison test give convergence.
(d) Since

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1000^{n+1} n!}{(n+1)! 1000^n} = \frac{1000}{n+1}$$

$n \geq 2000$ implies

$$\left| \frac{a_{n+1}}{a_n} \right| \leq \frac{1}{2}$$

such that the ratio test gives convergence.
(e) Since $\sqrt[n]{p} \underset{n\to\infty}{\longrightarrow} 1$ for every positive number $p > 0$ (Proposition 5 (b)), Corollary 19 gives divergence.

4. True or false? The series $\sum_{n=1}^{\infty} a_n$ converges if for every positive integer p

$$\lim_{n\to\infty} (a_{n+1} + a_{n+2} + \cdots + a_{n+p}) = 0.$$

Solution. The statement is false. A counterexample is the harmonic series $\sum_{n=1}^{\infty} 1/n$. Indeed, for every positive integer p we have

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \right) = \lim_{n\to\infty} \frac{1}{n+1} + \cdots + \lim_{n\to\infty} \frac{1}{n+p} = p \cdot 0 = 0.$$

The condition of the statement is fulfilled; however, the harmonic series diverges.

5. Consider the following series.

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots .$$

Compute $\underline{\lim} \frac{a_{n+1}}{a_n}$, $\underline{\lim} \sqrt[n]{a_n}$, $\overline{\lim} \sqrt[n]{a_n}$, and $\overline{\lim} \frac{a_{n+1}}{a_n}$. Apply both the root test and the ratio test!

Solution. We have

$$a_{2n-1} = \frac{1}{2^n} \quad \text{and} \quad a_{2n} = \frac{1}{3^n}.$$

Therefore

$$\begin{aligned}
\frac{a_{2n}}{a_{2n-1}} &= \left(\frac{2}{3}\right)^n \underset{n\to\infty}{\longrightarrow} 0;\\
\frac{a_{2n+1}}{a_{2n}} &= \frac{\frac{1}{2^{n+1}}}{\frac{1}{3^n}} = \frac{3^n}{2^{n+1}} \underset{n\to\infty}{\longrightarrow} +\infty;\\
\sqrt[2n-1]{a_{2n-1}} &= \frac{1}{2^{\frac{1}{2-1/n}}} \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2}};\\
\sqrt[2n]{a_{2n}} &= \frac{1}{\sqrt{3}} \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{3}}.
\end{aligned}$$

Since the odd and the even numbers cover the entire $\mathbb{N}$, we find

$$\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} = 0, \qquad q := \overline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty,$$

$$\underline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\sqrt{3}}, \qquad \alpha := \overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\sqrt{2}}.$$

Since $\alpha < 1$ the root test indicates convergence. The ratio test gives no information, since neither $q < 1$ nor all but finitely many elements of $\left|\frac{a_{n+1}}{a_n}\right|$ are greater than or equal to 1.