Calculus – 2. Series, Solutions

1. (a) Let
$$M = \left\{ \frac{1}{2^n} + \frac{1}{m} \;\middle|\; m, n \in \mathbb{N} \right\}.$$
Compute $\max M$, $\min M$, $\sup M$, and $\inf M$ if they exist.

A one-line induction proof shows that $2^n \geq 2$ for all positive integers n ($2^1 \geq 2$ and $2^{n+1} = 2 \cdot 2^n \underset{\text{Ind.hyp.}}{\geq} 2 \cdot 2 \geq 2$). Hence $m \geq 1$ and $2^n \geq 2$ for all $m, n \in \mathbb{N}$. Proposition 9 (e) then implies
$$\frac{1}{2^n} + \frac{1}{m} \leq \frac{1}{2} + 1 = \frac{3}{2}.$$
Hence, $3/2$ is an upper bound for M. Since $3/2 \in M$ we have $\max M = 3/2$. By the remark after Definition 4, $\max E = \sup E$ if the maximum exists. Hence $\sup M = 3/2$.
Since m and 2^n are positive for all positive integers m, n, Proposition 9 (e) gives $1/m$ and $1/2^n$ are also positive. Hence
$$0 < \frac{1}{2^n} + \frac{1}{m},$$
and 0 is a lower bound of M. We will show that $0 = \inf M$. For, let $\varepsilon > 0$. Our aim is to show that ε is not a lower bound, and we are done. First note that $2^n > n$ for $n \geq 3$ (proof by induction on n: $2^3 = 8 > 3$ and $2^{n+1} = 2 \cdot 2^n \underset{\text{Ind.hyp.}}{>} 2n > n + 1$). The Archimedean property of the real numbers furnishes positive integers m and $n \geq 3$ with
$$m\varepsilon > 2 \quad \text{and} \quad n\varepsilon > 2.$$
The first inequality implies $1/m < \varepsilon/2$ while the second inequality yields $2^n\varepsilon > n\varepsilon > 2$ and therefore, $1/2^n < \varepsilon/2$. Summing up both inequalities we arrive at
$$\frac{1}{m} + \frac{1}{2^n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Hence, ε is not a lower bound for M, so that $\inf M = 0$. Since $0 \notin M$, $\min M$ does not exist.

(b) Let $A \subset \mathbb{R}_+$ be a set with $\inf A > 0$. Prove that
$$\sup A^{-1} = \frac{1}{\inf A},$$
where $A^{-1} = \{1/a \mid a \in A\}$.

Proof. Put $t = \inf A > 0$. Then $0 < t \leq a$ for all $a \in A$. Proposition 9 (e) gives $0 < 1/a \leq 1/t$; hence $1/t$ is an upper bound for A^{-1}. Suppose s is positive with $s < 1/t$. We will show that s is not an upper bound for A^{-1}.
First $0 < s < 1/t$ implies $t < 1/s$. Since t is the greatest lower bound of A, $1/s$

is not a lower bound for A. Hence, there is some $a \in A$ with $a < 1/s$. Since all a are positive, this implies $1/a > s$. Hence s is not an upper bound for A^{-1}. This completes the proof. ■

2. Solve for $x \in \mathbb{R}$

$$|\,2x-4\,| < |\,x-1\,|. \tag{1}$$

Solution. Case 1: $2x-4 \geq 0$. This implies $x \geq 2$, in particular, $x \geq 1$. Therefore, $|\,2x-4\,| = 2x-4$ and $|\,x-1\,| = x-1$. Inequality (1) reads as

$$2x-4 < x-1 \iff x < 3.$$

Our partial solution in Case 1 is the interval $[2,3)$.
Case 2: $2x-4 < 4$ *and* $x-1 \geq 0$. This implies $1 \leq x < 2$, $|\,2x-4\,| = -2x+4$, and $|\,x-1\,| = x-1$. Inequality (1) now reads as

$$-2x+4 < x-1 \iff \frac{5}{3} < x.$$

Our partial solution in Case 2 is $(\frac{5}{3}, 2)$.
Case 3: $x-1 < 0$. This implies $x < 1$ and $2x-4 < 0$. Therefore, $|\,2x-4\,| = -2x+4$, and $|\,x-1\,| = -x+1$. Inequality (1) reads as

$$-2x+4 < -x+1 \iff 3 < x.$$

There is no solution in Case 3. The inequality (1) is fulfilled if and only if x belongs to the open interval $(5/3, 2)$.

3. Solve for $x \in \mathbb{R}$

$$\frac{1}{x-1} + \frac{1}{x+1} \geq 1. \tag{2}$$

Solution.Case 1: $x > 1$. This implies $x-1 > 0$ and $x+1 > 0$ so that multiplication of (2) by $(x-1)(x+1)$ does not change the relation sign. We can make the following equivalent transformations.

$$x+1+x-1 \geq (x-1)(x+1) = x^2-1 \iff 0 \geq x^2-2x-1$$
$$0 \geq (x-1)^2-2 \iff \sqrt{2} \geq |\,x-1\,|$$
$$1-\sqrt{2} \leq x \leq 1+\sqrt{2}.$$

Taking care of our assumption $x > 1$ we find $1 < x \leq 1+\sqrt{2}$.
Case 2: $x < -1$. This implies $x-1 < 0$ and $x+1 < 0$ so that multiplication of (2) by $(x-1)(x+1)$ does not change the sign. As in Case 1 we get

$$1-\sqrt{2} \leq x \leq 1+\sqrt{2}.$$

Taking care of $x < -1$, there is no solution in this case.
Case 3: $-1 < x < 1$. This implies $x - 1 < 0$ and $x + 1 > 0$ so that multiplication of (2) changes the relation sign. We obtain

$$\sqrt{2} \leq |\, x - 1 \,|$$

which is equivalent to

$$x \geq 1 + \sqrt{2} \quad \text{or} \quad x \leq 1 - \sqrt{2}.$$

Taking care of our assumption $-1 < x < 1$ we find $-1 < x < 1 - \sqrt{2}$.
The inequality (2) is fulfilled, if and only if x belongs to one of the intervals $(-1, 1 - \sqrt{2}]$ or $(1, 1 + \sqrt{2}]$.

4. Define a relation $\prec$ on $\mathbb{R}^2 = \{(x, y) \mid x,\, y \in \mathbb{R}\}$ by

$$(x, y) \prec (x', y') \quad \text{if} \quad (x < x' \quad \text{or} \quad (x = x' \quad \text{and} \quad y < y')).$$

Prove that $(\mathbb{R}^2, \prec)$ is an ordered set. Is $(\mathbb{R}^2, \prec)$ order complete?

Proof. First we show that any two elements p and q of $\mathbb{R}^2$ are comparable, i. e. $\mathbb{R}^2$ has the property in Definition 1 (i). Let $p = (x, y)$ and $q = (x', y')$, and suppose first $p \not\prec q$. That is, $x > x'$ or both $x = x'$ and $y \geq y'$. In the first case, $q \prec p$. If $x = x'$ and $y = y'$ then $p = q$, and finally if $x = x'$ and $y > y'$ again $q \prec p$. We have seen that one and only one of the relations $p \prec q$, $p = q$, and $q \prec p$ is true.
We will show transitivity. Suppose further $r = (x'', y'')$, $p \prec q$, and $q \prec r$. Then, $x < x'$ and $x' \leq x''$ implies $x < x''$ so that $p \prec r$ in this case. If $x = x'$ and $y < y'$, then $x' < x''$ implies $x < x''$ which means $p \prec r$. We are left with the case $x = x'$, $y < y'$, $x' = x''$, and $y' < y''$. We conclude $x = x''$ and $y < y''$ which also means $p \prec r$. This completes the proof; $(\mathbb{R}^2, \prec)$ is an ordered set. ∎

$(\mathbb{R}^2, \prec)$ is not order complete.

Proof. We will construct a subset E of $\mathbb{R}^2$ which is bounded above; however E has not a least upper bound.
(a) The subset $E := \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ is bounded above by $(0, 0)$ since $x < 0$ for all $(x, y) \in E$ and therefore $(x, y) \prec (0, 0)$.

Suppose to the contrary that $p = (a, b) = \sup E$. Since p is the least upper bound and $(0, 0)$ is an upper bound by (a), $p \preccurlyeq (0, 0)$. Hence $a < 0$ or both $a = 0$ and $b \leq 0$. The first case is impossible since otherwise $(a, b) \prec (a/2, b) \in E$, and p is not an upper bound of E. Hence $p = (0, b)$.
Putting $q = (0, b - 1)$, q is also an upper bound of E (since again $x < 0$ for all $(x, y) \in E$). Moreover, $q = (0, b - 1) \prec (0, b) = p$ since $0 = 0$ and $b - 1 < b$. This shows that p is not the least upper bound; a contradiction! Hence, $(\mathbb{R}^2, \prec)$ is not order complete. ∎

5. Let n be a positive integer and $x_1, \ldots, x_n$ real numbers. Prove that

$$\left| \sum_{k=1}^{n} x_k \right| \leq \sum_{k=1}^{n} |\, x_k \,|, \tag{3}$$

$$\left| \prod_{k=1}^{n} x_k \right| = \prod_{k=1}^{n} |\, x_k \,|. \tag{4}$$

Proof. We prove the statements using induction on n. The case $n = 1$ is obvious for both since we have $|\, x_1 \,| = |\, x_1 \,|$. (a) Suppose (3) is true for some fixed positive integer n and all $x_1, \ldots, x_n \in \mathbb{R}$. We will prove that the statement is true for arbitrary $n + 1$ real numbers $x_1, \ldots, x_n, x_{n+1}$. Using the triangle inequality (Lemma 14 (d)) and the induction hypothesis, we compute

$$\left| \sum_{k=1}^{n+1} x_k \right| = \left| \sum_{k=1}^{n} x_k + x_{n+1} \right| \underset{\text{L. 14}}{\leq} \left| \sum_{k=1}^{n} x_k \right| + |\, x_{n+1} \,| \underset{\text{i.h.}}{\leq} \sum_{k=1}^{n} |\, x_k \,| + |\, x_{n+1} \,| = \sum_{k=1}^{n+1} |\, x_k \,| \,.$$

This proves the induction assertion.

(b) Suppose (4) is true for some fixed positive integer n and all $x_1, \ldots, x_n \in \mathbb{R}$. We will prove that the statement is true for arbitrary $n+1$ real numbers $x_1, \ldots, x_n, x_{n+1}$. Using Lemma 14 (c) and the induction hypotesis, we compute

$$\left| \prod_{k=1}^{n+1} x_k \right| = \left| \prod_{k=1}^{n} x_k \cdot x_{n+1} \right| \underset{\text{L. 14}}{=} \left| \prod_{k=1}^{n} x_k \right| \cdot |\, x_{n+1} \,| \underset{\text{ind. hyp.}}{=} \prod_{k=1}^{n} |\, x_k \,| \cdot |\, x_{n+1} \,| = \prod_{k=1}^{n+1} |\, x_k \,| \,.$$

This proves the induction assertion. ∎