Calculus – 1. Series, Solutions

1. (a) Prove by induction. For all positive integers n and k with $1 \le k \le n$

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

Proof. We use induction over n. If $n = 1$, there is only one pair $(n,k) = (1,1)$ with $1 \le k \le n$. In this case we obtain

$$\sum_{m=1}^{1} \binom{m}{1} = \binom{1}{1} = 1 = \binom{2}{2} = \binom{n+1}{k+1};$$

hence the induction start is done.
Suppose the claim is true for some fixed n and all k, $1 \le k \le n$. We will show that the claim is true for $n+1$ and all k, with $1 \le k \le n+1$.
First fix k with $1 \le k \le n$. Then we have

$$\sum_{m=k}^{n+1} \binom{m}{k} = \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k} \underset{\text{Ind.hypothesis}}{=} \binom{n+1}{k+1} + \binom{n+1}{k} \underset{\text{Lemma 2}}{=} \binom{n+2}{k+1}$$

which proves the assertion in this case. We have to consider separately the case $k = n+1$ since it is not covered by our induction hypothesis. In this case

$$\sum_{m=n+1}^{n+1} \binom{m}{n+1} = \binom{n+1}{n+1} = 1 = \binom{n+2}{n+2} = \binom{n+2}{k+1}.$$

This completes the induction proof. ∎

(b) Find a positive integer n_0 such that all positive integers n, $n \ge n_0$ implies

$$3^n > 10n^2. \tag{1}$$

Prove your statement by induction.

A possible choice is $n_0 = 6$ or any $n_0 \ge 6$ because $3^6 = 729 > 360 = 10 \cdot 6^2$. We will show by induction that

$$n \ge 6 \quad \text{implies} \quad 3^n > 10n^2.$$

Proof. The induction start is satisfied in case $n_0 = 6$. Suppose (1) is fulfilled for some fixed $n \geq 6$; we will show it for $n+1$, i. e. $3^{n+1} > 10(n+1)^2$. First we compute

$$\begin{aligned} & n \geq 6 \\ \Longrightarrow\ & n - \frac{1}{2} \geq 5\frac{1}{2} > 1 \\ \Longrightarrow\ & \left(n - \frac{1}{2}\right)^2 > 1 \\ \Longrightarrow\ & n^2 - n + \frac{1}{4} > 1 \\ \Longrightarrow\ & n^2 - n + \frac{1}{4} - 1 = n^2 - n - \frac{3}{4} > 0 \quad | \cdot 2 \\ \Longrightarrow\ & 2n^2 - 2n - 1 > 2n^2 - 2n - \frac{3}{2} > 0 \quad | +n^2 + 2n + 1 \\ \Longrightarrow\ & 3n^2 > n^2 + 2n + 1 = (n+1)^2. \end{aligned} \tag{2}$$

Now we use our induction hypothesis:

$$3^{n+1} = 3 \cdot 3^n \underset{\text{ind.hyp.}}{>} 3 \cdot 10n^2 \underset{(2)}{>} 10(n+1)^2.$$

This proves the induction assertion. ■

2. Prove that $\sqrt{12}$ is irrational.

   Proof. Suppose to the contrary that $\sqrt{12} = m/n$ with positive integers $m \in \mathbb{N}$ and $n \in \mathbb{N}$ which do not have a prime factor in common (otherwise we can cancel this factor in the enumerator and denominator of the fraction m/n). We obtain $12 = m^2/n^2$ and $12n^2 = m^2$. Since the left hand side of this equation is divisible by 3, we have $3 \mid m^2$; hence $3 \mid m$. Therefore, $m = 3m_1$ for some $m_1 \in \mathbb{N}$. Inserting this into our equation yields

   $$12n^2 = (3m_1)^2 = 9m_1^2.$$

   Dividing this by 3 gives $4n^2 = 3m_1^2$. Since the right hand side is divisible by 3, we have $3 \mid 4n^2$. We conclude $3 \mid n^2$ since 3 and 4 have no factors in common. Finally $3 \mid n$; which contradicts our choice of m and n (both are divisible by 3). ■

3. (a) Let $E := [0, 1)$. Show that $\min E = 0$ whereas E has no maximum.
   Since $0 \leq x < 1$ for all $x \in E$, 0 is a lower bound. Since $0 \in E$, $0 = \min E$.
   Suppose to the contrary that $M = \max E$ exists. Then $M < 1$ and $M < \frac{M+1}{2} < 1$. This inequality shows that M is not an upper bound of E since $\frac{M+1}{2} \in E$; a contradiction. Hence $\max E$ does not exist.

   (b) $F := \{1/n \mid n \in \mathbb{N}\}$. Show that $\max F = 1$ whereas F has no minimum.
   For any positive integer n we have $n \geq 1$. Using Proposition 9 (e) we have $1/n \leq 1$

for all n. Hence 1 is an upper bound of F. Since $1 = 1/1 \in F$, $1 = \max F$. Suppose to the contrary that F has a minimum, say $1/m$. Again, Proposition 9 (e) shows that

$$0 < m < m+1 \quad \text{implies} \quad 0 < \frac{1}{m+1} < \frac{1}{m}.$$

Hence $1/m$ is not a lower bound of F. A contradiction!

4. (a) If $E \subset \mathbb{R}$ is bounded above and $\alpha = \sup E$ exists, prove that $-E$ is bounded below and $\inf(-E) = -\sup E$.

   Proof. Sorry, I've forgotten to mention the notion $-E := \{-x \mid x \in E\}$! Since $\alpha \geq x$ for all $x \in E$, we obtain $-\alpha \leq -x$ for all $x \in E$. Hence, $-E$ is bounded below by $-\alpha$. We will show that $-\alpha$ satisfies the second property in the definition of the infimum. Let $-\alpha < \beta$ for some β. We have to show that β is not a lower bound for $-E$. Equivalently, there exists some $-x \in -E$ such that $-x < \beta$.
   $-\alpha < \beta$ implies $\alpha > -\beta$. Since α is the least upper bound of E, $-\beta$ is not an upper bound. Hence, there is an $x \in E$ with $-\beta < x$. This shows $\beta > -x$ and we are done. ∎

   (b) Suppose that $M \subset N \subset \mathbb{R}$ are bounded. Prove that $\sup M \leq \sup N$ and $\inf M \geq \inf N$.

   Proof. In this exercise we must assume the existence of $\sup M$ and $\sup N$ (which is guarantied by axiom (C)).

   Let $\alpha = \sup N$. Then $\alpha \geq x$ for all $x \in N$. Since N contains M, $\alpha \geq x$ is trivially true for all $x \in M$. Hence, α is an upper bound for M; and therefore $\alpha \geq \sup M$.
   Let $\beta = \inf N$. Then $\beta \leq x$ for all $x \in N$. Since $M \subset N$, $\beta \leq x$ for all $x \in M$. Hence β is a lower bound for M; and therefore $\beta \leq \inf M$. ∎

5. Prove the laws of fractions ($a, b, c, d \in \mathbb{R}$, $b \neq 0$, $d \neq 0$):
   (a) $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
   (b) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

   Proof. (a) We multiply the equation $\frac{a}{b} = \frac{c}{d}$ by bd and obtain on the left hand side

$$\left(\frac{a}{b}\right) bd \underset{\text{by Def.}}{=} \left(a\frac{1}{b}\right) bd \underset{(M\,3)}{=} a\left(\frac{1}{b}b\right) d \underset{(M\,2)}{=} a\left(b\frac{1}{b}\right) d$$
$$\underset{(M\,5)}{=} a \cdot 1 \cdot d \underset{(M\,4)}{=} a \cdot d.$$

Similarly, $\frac{c}{d}\, bd = bc$. This proves the first direction of (a).

Suppose now $ad = bc$. Multiplication of this equation by $\frac{1}{b}\frac{1}{d}$ gives

$$\begin{aligned} ad\,\frac{1}{b}\frac{1}{d} &= bc\frac{1}{b}\,\frac{1}{d} \\ \underset{(M\,2)}{\Longrightarrow} ad\frac{1}{d}\frac{1}{b} &= cb\frac{1}{b}\frac{1}{d} \\ \underset{(M\,5)}{\Longrightarrow} a \cdot 1 \cdot \frac{1}{b} &= c \cdot 1 \cdot \frac{1}{d} \\ \underset{(M\,4)}{\Longrightarrow} a \cdot \frac{1}{b} &= c \cdot \frac{1}{d} \\ \underset{\text{Def.}}{\Longrightarrow} \frac{a}{b} &= \frac{c}{d}. \end{aligned}$$

This proves the second part of (a).
(b) Multiplying $\frac{a}{b} + \frac{c}{d}$ by bd we obtain

$$\begin{aligned} \left(\frac{a}{b} + \frac{c}{d}\right) bd &\underset{(D)}{=} \left(a\frac{1}{b}\right) bd + \left(c\frac{1}{d}\right) bd \underset{(M2),\,(M3)}{=} a\left(\frac{1}{b}b\right) d + c\left(\frac{1}{d}d\right) b \\ &\underset{(M5)}{=} a \cdot 1 \cdot d + c \cdot 1 \cdot b \underset{(M4),\,(M2)}{=} ad + bc. \end{aligned} \tag{3}$$

On the other hand,

$$\frac{ad + bc}{bd}\,bd \underset{(M4),\,(M5)}{=} ad + bc. \tag{4}$$

Comparing (3) and (4) we have

$$\left(\frac{a}{b} + \frac{c}{d}\right) bd = \frac{ad + bc}{bd}\,bd.$$

Using cancellation law, Proposition 6 (a), we get

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

which completes the proof of (b). ■