Calculus – 2. Series

(turn in: October 31, 2003)

1. (a) Let
$$M = \left\{ \frac{1}{2^n} + \frac{1}{m} \;\middle|\; m, n \in \mathbb{N} \right\}.$$
Compute $\max M$, $\min M$, $\sup M$, and $\inf M$ if they exist.

   (b) Let $A \subset \mathbb{R}_+$ be a set with $\inf A > 0$. Prove that
$$\sup A^{-1} = \frac{1}{\inf A},$$
where $A^{-1} = \{1/a \mid a \in A\}$.

2. Solve for $x \in \mathbb{R}$
$$|\, 2x - 4 \,| < |\, x - 1 \,|.$$

3. Solve for $x \in \mathbb{R}$
$$\frac{1}{x-1} + \frac{1}{x+1} \geq 1.$$

4. Define a relation $\prec$ on $\mathbb{R}^2 = \{(x, y) \mid x,\, y \in \mathbb{R}\}$ by
$$(x, y) \prec (x', y') \quad \text{if} \quad (x < x' \quad \text{or} \quad (x = x' \quad \text{and} \quad y < y')).$$
Prove that $(\mathbb{R}^2, \prec)$ is an ordered set. Is $(\mathbb{R}^2, \prec)$ order complete?

5. Let n be a positive integer and $x_1, \ldots, x_n$ real numbers. Prove that
$$\left| \sum_{k=1}^{n} x_k \right| \leq \sum_{k=1}^{n} |\, x_k \,|, \tag{1}$$
$$\left| \prod_{k=1}^{n} x_k \right| = \prod_{k=1}^{n} |\, x_k \,|. \tag{2}$$