Calculus – 13. Series

(turn in: January 30, 2004)

1. Let $f(x) = x^x$ be defined for positive real numbers $x > 0$. Compute $\lim\limits_{x\to 0+0} f(x)$. Determine the local extrema of f. On which parts of its domain is f convex; on which parts is f concave?

2. Prove that for every $x > 0$
$$\frac{x}{1+x} \leq \log(1+x) \leq x.$$
Hint. Apply the mean value theorem to $f(x) = \log(1+x)$.

3. Compute the limits where $a > 0$ and $b > 0$ denote fixed positive real numbers.
   (a) $\displaystyle\lim_{x\to 0} \frac{\log\cosh(ax)}{\log\cos(bx)}$
   (b) $\displaystyle\lim_{x\to\pi/2} \frac{1-\sin^{a+b} x}{\sqrt{(1-\sin^a x)(1-\sin^b x)}}$
   (c) $\displaystyle\lim_{x\to+\infty} \arccos(\sqrt{x^2+x} - x)$
   (d) $\displaystyle\lim_{x\to+\infty} x\left(\frac{\pi}{4} - \arctan\frac{x}{x+1}\right)$
   (e) $\displaystyle\lim_{x\to 0} \frac{\cos\frac{1}{x}\sin^2 x}{x}$

4. Compute the Taylor polynomial of degree 3 of the function
$$f(x) = \mathrm{e}^{-x}\cos x$$
at $x_0 = 0$. Give an estimate for the remainder if $|\,x\,| \leq \frac{1}{2}$.

5. (a) Compute the Taylor series T_f and T_g of $f(x) = \cos x$ and $g(x) = \log(1+x)$ at $x_0 = 0$, respectively. Compute their radii of convergence.
(b) Show that $T_f(x)$ converges to $f(x)$ for all $x \in \mathbb{R}$. Show that $T_g(x)$ converges to $g(x)$ for all $x \in (0,1)$.