Calculus – 10. Series

(turn in: January 9, 2004)

1. Determine all values of $c \in \mathbb{R}$ such that

$$f(x) = \begin{cases} (x-2)^2 & \text{if} \quad x \leq 4, \\ cx^2 - 8 & \text{if} \quad x > 4 \end{cases}$$

   is continuous on $\mathbb{R}$.

2. (a) Prove the following fixed point theorem. Let $D = [a, b]$ be a finite closed interval. Every continuous function $f\colon D \to D$ has a fixed point, i. e. there exists $c \in [a, b]$ such that $f(c) = c$.
   Give examples of functions $f\colon D \to D$ such that the fixed point theorem fails if
   (b) D is a closed infinite interval.
   (c) $D = [a, b)$.
   (d) $D = [0, 1] \cup [2, 3]$.
   (e) f is not continuous.
   Hint. Use the intermediate value theorem for (a).

3. Let a and b be real numbers with $1 < a < b$. Prove that the equation

$$\frac{x^7 + 1}{x - a} + \frac{x^3 - 1}{x - b} = 0$$

   has a solution $x \in (a, b)$.
   Hint. Define an appropriate function f and apply the intermediate value theorem.

4. Prove. If $f\colon [a, b] \to \mathbb{R}$ is continuous at $x_0 \in (a, b)$ and $f(x_0) = A > 0$ then there exists a real number $\delta > 0$ such that for every $x \in [a, b]$ the inequality $|\, x - x_0 \,| < \delta$ implies $f(x) > A/2$.
   (In other words: If a continuous function is nonzero at a point x_0, then f is nonzero on a whole neighborhood $U_\delta(x_0)$)

5. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$, whereas $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}_+$.

Merry Christmas and a Happy New Year!