Chapter 6

Basic Topology

In the study of functions of several variables we need some topological notions like neighborhood, open set, closed set, and compactness.

6.1 Finite, Countable, and Uncountable Sets

Definition 6.1 If there exists a 1-1 mapping of the set A onto the the B (a bijection), we say that A and B have the same *cardinal number* (the same *cardinality* or A and B are *equivalent*) and we write $A \sim B$.

This relation clearly has the following properties:

(R) It is *reflexive*: $A \sim A$.
(S) It is *symmetric*: If $A \sim B$ then $B \sim A$.
(R) It is *transitive*: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any such relation with these three properties is called an *equivalence relation*.

Definition 6.2 For any nonnegative integer $n \in \mathbb{N}_0$ let N_n be the set $\{1, 2, \dots, n\}$. For any set A we say that:

(a) A is *finite* if $A \sim N_n$ for some n. The empty set $\varnothing$ is also considered to be finite.
(b) A is *infinite* if A is not finite.
(c) A is *countable* if $A \sim \mathbb{N}$.
(d) A is *uncountable* if A is neither finite nor countable.
(e) A is *at most countable* if A is finite or countable.

For finite sets A and B we evidently have $A \sim B$ if A and B have the same number of elements. For infinite sets, however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

Example 6.1 $\mathbb{Z}$ is countable. Indeed, the arrangement

$$0, 1, -1, 2, -2, 3, -3, \dots$$

gives a bijection between $\mathbb{N}$ and $\mathbb{Z}$. Explicitly, the bijection $f\colon \mathbb{N} \to \mathbb{Z}$ is given by $f(2n) = n$, $f(2n-1) = -n+1$, $n \in \mathbb{N}$.
We see that an infinite set can be equivalent to one of its proper subsets. The set X is infinite if and only if X is equivalent to one of its proper subsets. Any countable set can be arranged in a sequence.

Proposition 6.1 *Every infinite subset of a countable set A is countable.*

Proof. Suppose $E \subset A$ is an infinite subset. Arrange the elements of A in a sequence (x_n) of distinct elements. Construct a subsequence (x_{n_k}) as follows. Let n_1 be the smallest positive integer with $x_{n_1} \in E$. Having chosen $x_{n_1}, x_{n_2}, \dots, x_{n_{k-1}}$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$. Putting $f(k) = x_{n_k}$ we obtain a bijection between $\mathbb{N}$ and E. ■

Roughly speaking, the theorem shows that countable sets represent the "smallest" infinity: No uncountable set can be a subset of a countable set.

Theorem 6.2 (Cantor's First Diagonal Process) *Let (E_n) $n \in \mathbb{N}$, be a sequence of countable sets, and put*

$$S = \bigcup_{n\in\mathbb{N}} E_n.$$

Then S is countable.

Proof. Let every set E_n be arranged in a sequence (x_{nk}), $k = 1, 2, \dots$, and consider the infinite array

$$\begin{array}{ccccccccc}
x_{11} & \rightarrow & x_{12} & & x_{13} & \rightarrow & x_{14} & & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
x_{21} & & x_{22} & & x_{23} & & x_{24} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
x_{31} & & x_{32} & & x_{33} & & x_{34} & & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & & \\
x_{41} & & x_{42} & & x_{43} & & x_{44} & & \cdots \\
\downarrow & \nearrow & & & & & & & \\
x_{51} & & \cdots & & & & & &
\end{array}$$

in which the elements of E_n form the nth row. The array contains all elements of S. As indicated by arrows, these elements can be arranged in a sequence

$$x_{11}, x_{12}, x_{21}, x_{31}, x_{22}, x_{13}, x_{14}, \dots \tag{6.1}$$

If any two of the sets E_n have elements in common, these will appear more than once in (6.1). Hence there is a subset T of $\mathbb{N}$ such that $T \sim S$ which shows that S is at most countable. Since $E_1 \subset S$ and E_1 is infinite, S is infinite and thus countable. ■

Proposition 6.3 *Let A be a countable set and let B_n be the set off all n-tuples $(a_1, \dots, a_n)$ where $a_k \in A$ and the elements $a_1, \dots, a_n$ need not to be distinct. Then B_n is countable.*

Proof. We use induction on n. That B_1 is countable is evident since $B_1 = A$. The elements of B_n are of the form

$$(b, a), \quad b \in B_{n-1},\ a \in A.$$

For every fixed b, the set of pairs (b, a) is a set equivalent to A, and hence countable. Thus B_n is the union of a countable set of countable set. By Theorem 6.2, B_n is countable. ■

Corollary 6.4 *The set $\mathbb{Q}$ of all rational numbers is countable.*

Proof. We apply the above theorem with $n = 2$, noting that every rational number r is of the form $r = p/q$ where p and q are integers. The set of pairs (p, q) and therefore the set of fractions p/q is countable. ■

Theorem 6.5 (Cantor's Second Diagonal Process) *Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.*

Proof. Suppose to the contrary that A is countable and arrange the elements of A in a sequence $(s_n)_{n\in\mathbb{N}}$ of distinct elements of A. We construct a sequence s as follows. If the nth element in s_n is 1 we let the nth digit of s be 0, and vice versa. Then the sequence s differs from every member $s_1, s_2, \dots$ at least in one place; hence $s \not\in A$—a contradiction since s is indeed an element of A. This proves, A is uncountable. ■

To illustrate the proof let $s_n = (a_{n1}, a_{n2}, \dots)$ be the nth sequence of digits $a_{nk} \in \{0, 1\}$. Suppose the sequence $(a_{11}, a_{22}, a_{33}, a_{44}, \dots)$ of diagonal elements of the array a_{nk} is $(0, 1, 1, 0, \dots)$, then we choose s to be the "complementary" sequence $s = (1, 0, 0, 1, \dots)$. This sequence s can't be s_1 since the sequence s differs in the first element from s_1, also, s differs in the second element from s_2 and so on. Thus, s is not contained in the sequence of elements of A.

Corollary 6.6 $\mathbb{R}$ *is uncountable.*

Proof. Using the binary expansion of real numbers of the interval $I = [0, 1)$ we see that every element of $s \in A$, $s = (a_1, a_2, \dots)$, corresponds to a real number $x \in I$, $x = \sum_{n=1}^{\infty} a_n/2^n$. There are only countably many pairs of sequences (having zeros or ones in the end) corresponding the same x. ■

6.2 Metric Spaces and Normed Spaces

Definition 6.3 A set X is said to be a *metric space* if for any two points $x, y \in X$ there is associated a real number $d(x, y)$, called the *distance* of x and y such that

(a) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in X$ (triangle inequality).

Any function d with these three properties is called a *distance function* or *metric* on X.

Example 6.2 (a) $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and any subsets of these sets are metric spaces with $d(x, y) := |\, y - x \,|$.
(b) The *real plane* $\mathbb{R}^2$ is a metric space with respect to

$$d_2((x_1, x_2), (y_1, y_2)) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$
$$d_1((x_1, x_2), (y_1, y_2)) := |\, x_1 - x_2 \,| + |\, y_1 - y_2 \,| .$$

d_2 is called the *euclidean metric.*
(c) Let X be a set. Define

$$d(x, y) := \begin{cases} 1, & \text{if} \quad x \neq y, \\ 0, & \text{if} \quad x = y. \end{cases}$$

Then (X, d) becomes a metric space.

Definition 6.4 Let E be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). Suppose on E there is given a real-valued function which associates to each $x \in E$ a real number $\|x\|$ such that the following three conditions are satisfied:

(i) $\|x\| \geq 0$ for every $x \in E$, and $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|\lambda x\| = |\, \lambda \,| \, \|x\|$ for all $\lambda \in \mathbb{C}$ (in $\mathbb{R}$, resp.)
(iii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in E$.

Then E is called a *normed (vector) space* and $\|x\|$ is the *norm* of x.

Clearly, every normed vector space E is a metric space if we put

$$d(x, y) = \|x - y\| .$$

Example 6.3 (a) $E = \mathbb{R}^k$. We define an *inner product* (or *scalar product*) of two vectors $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$, $x, y \in \mathbb{R}^k$ by

$$\langle x, y \rangle = x \cdot y = \sum_{i=1}^{k} x_i y_i$$

and the (euclidean) *norm* of x by

$$\|x\| = \sqrt{x \cdot x} = \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} . \tag{6.2}$$

The vector space $\mathbb{R}^k$ together with the above inner product and norm is called *euclidean* k-space. Minkowski's inequality (Proposition 1.35) with $p = 2$ shows that (iii) is satisfied.
We define an *inner product* (or *scalar product*) of two vectors $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$, $x, y \in \mathbb{C}^k$ by

$$\langle x, y\rangle = x \cdot y = \sum_{i=1}^{k} \overline{x_i}\, y_i$$

and a norm of x by

$$\| x \| = \sqrt{x \cdot x} = \left(\sum_{i=1}^{k} |\, x_i \,|^2 \right)^{\frac{1}{2}}. \tag{6.3}$$

Sometimes we write $\| x \|_2$ in place of $\| x \|$ to emphasize that we are dealing with the inner product norm. Sometimes $\mathbb{C}^k$ with this inner product and this norm is called *unitary* k-*space*. Note that the inner product on $\mathbb{C}^k$ is linear in the *second* argument and *anti-linear* in the first argument.
There are other possibilities to define a norm on E. Let $p \geq 1$

$$\| x \|_\infty = \sup_{i=1,\dots,k} |\, x_i \,|, \quad \text{supremum norm}$$

$$\| x \|_p = \left(\sum_{i=1}^{k} |\, x_i \,|^p \right)^{\frac{1}{p}}.$$

We show that $\| x \|_p$ indeed defines a norm on $\mathbb{R}^k$ or $\mathbb{C}^k$. It is obvious that $\| x \|_p \geq 0$. Let $\| x \|_p = 0$, then $\sum_{i=1}^{k} |\, x_i \,|^p = 0$ which implies $x_i = 0$ for every $i = 1, \dots, k$; hence $x = 0$.
Moreover

$$\| \lambda x \|_p = \left(\sum_{i=1}^{k} |\, \lambda x_i \,|^p \right)^{1/p} = \left(\sum_{i=1}^{k} |\, \lambda \,|^p \,|\, x_i \,|^p \right)^{1/p} = |\, \lambda \,| \left(\sum_{i=1}^{k} |\, x_i \,|^p \right)^{1/p} = |\, \lambda \,| \, \| x \|_p .$$

Finally, Minkowski's inequality (see Proposition 1.35 and Corollary 1.36) gives $\| x + y \|_p \leq \| x \|_p + \| y \|_p$.
(b) $E = \mathrm{C}([a, b])$. Let $p \geq 1$. Then

$$\| f \|_\infty = \sup_{x \in [a,b]} |\, f(x) \,|,$$

$$\| f \|_p = \left(\int_a^b |\, f(t) \,|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

define norms on E. Note that $\| f \|_p \leq \sqrt[p]{b - a}\; \| f \|_\infty$.
(c) $E = \ell_2 = \{(x_n) \mid \sum_{n=1}^{\infty} |\, x_n \,|^2 < \infty\}$. Then

$$\| x \|_2 = \left(\sum_{n=1}^{\infty} |\, x_n \,|^2 \right)^{\frac{1}{2}}$$

defines a norm on ℓ_2.

6.3 Basic Notions

Definition 6.5 Let X be a metric space with metric d. All points and subsets mentioned below are understood to be elements and subsets of X.

(a) The set $U_\varepsilon(x) = \{y \mid d(x, y) < \varepsilon\}$ with some $\varepsilon > 0$ is called the *ε-neighborhood* of x. The number ε is called the *radius* of the neighborhood $U_\varepsilon(x)$.

(b) A point p is an *interior* or *inner* point of E if there is a neighborhood $U_\varepsilon(p)$ completely contained in E. E is *open* if every point of E is an interior point.

(c) A point p is called an *accumulation* or *limit* point of E if every neighborhood of p has a point $q \neq p$ such that $q \in E$.

(d) E is said to be *closed* if every accumulation point of E is a point of E. The *closure* of E (denoted by $\overline{E}$) is E together with all accumulation points of E. In other words $p \in \overline{E}$, if and only if every neighborhood of x has a non-empty intersection with E.

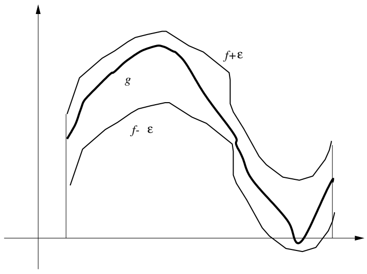
(e) The *complement* of E (denoted by E^{c}) is the set of all points $p \in X$ such that $p \notin E$.

(f) E is *bounded* if there exists a real number M such that $d(x, y) < M$ for all $x, y \in E$.

(g) E is *dense* in X if every point of X is an accumulation point of E or a point of E (or both), i. e. $\overline{E} = X$.

Example 6.4 (a) $(a, b) \subset \mathbb{R}$ is an open set. Indeed, for every $x \in (a, b)$ we have $U_\varepsilon(x) \subset (a, b)$ if ε is small enough, say $\varepsilon \leq \min\{|\,x - a\,|, |\,x - b\,|\}$. Hence, x is an inner point of (a, b). Since x was arbitrary, (a, b) is open.
$[a, b)$ is not open since a is not an inner point of $[a, b)$. Indeed, $U_\varepsilon(a) \subsetneq [a, b)$ for every $\varepsilon > 0$.
a is an accumulation point of both (a, b) and $[a, b)$. This is true since every neighborhood $U_\varepsilon(a)$, $\varepsilon < b - a$, has $a + \varepsilon/2 \in (a, b)$ (resp. in $[a, b)$) which is different from a. The closure $\overline{E}$ of both sets (a, b) and $[a, b)$ is $[a, b]$. By the description of the closure $\overline{E}$ of E in (g) it is clear that $\overline{E} \supseteq [a, b]$. For any point $x \notin [a, b]$ we find a neighborhood $U_\varepsilon(x)$ with $U_\varepsilon(x) \cap [a, b) = \varnothing$; hence $x \notin \overline{E}$.
The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$. Indeed, every neighborhood $U_\varepsilon(r)$ of every real number r contains a rational number, see Proposition 1.11 (b).
For the real line one can prove: Every open set is the at most countable union of disjoint open intervals. A similar description for closed subsets of $\mathbb{R}$ is false. There is no similar description of open subsets of $\mathbb{R}^k$, $k \geq 2$.
(b) For every metric space X, both the whole space X and the empty set $\varnothing$ are open as well as closed.
(c) Let $B = \{x \in \mathbb{R}^k \mid \|x\|_2 < 1\}$ be the *open unit ball* in $\mathbb{R}^k$. B is open (see Lemma 6.7

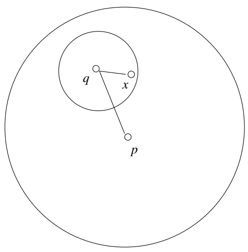
below); B is not closed. For example, $x_0 = (1, 0, \dots, 0)$ is an accumulation point of B since $x_n = (1 - 1/n, 0, \dots, 0)$ is a sequence of elements of B converging to x_0, however, $x_0 \notin B$. The accumulation points of B are $\overline{B} = \{x \in \mathbb{R}^k \mid \|x\|_2 \le 1\}$. This is also the closure of B in $\mathbb{R}^k$.



(d) Consider $E = \mathrm{C}([a, b])$ with the supremum norm. Then $g \in E$ is in the ε-neighborhood of a function $f \in E$ if and only if

$$|\, f(t) - g(t)\,| < \varepsilon, \quad \text{for all} \quad x \in [a, b].$$

Lemma 6.7 *Every neighborhood $U_r(p)$, $r > 0$, of a point p is an open set.*



Proof. Let $q \in U_r(p)$. Then there exists $\varepsilon > 0$ such that $d(q, p) = r - \varepsilon$. We will show that $U_\varepsilon(q) \subset U_r(p)$. For, let $x \in U_\varepsilon(q)$. Then by the triangle inequality we have

$$d(x, p) \le d(x, q) + d(q, p) < \varepsilon + (r - \varepsilon) = r.$$

Hence $x \in U_r(p)$ and q is an interior point of $U_r(p)$. Since q was arbitrary, $U_r(p)$ is open. ∎

Remarks 6.1 (a) If p is an accumulation point of a set E, then every neighborhood of p contains infinitely many points of E.
(b) A finite set has no accumulation points.

Example 6.5 (a) The open complex unit disc, $\{z \in \mathbb{C} \mid |\,z\,| < 1\}$.
(b) The closed unit disc, $\{z \in \mathbb{C} \mid |\,z\,| \le 1\}$.
(c) A finite set.
(d) The set $\mathbb{Z}$ of all integers.
(e) $\{1/n \mid n \in \mathbb{N}\}$.
(f) The set $\mathbb{C}$ of all complex numbers.
(g) The interval (a, b).

Here (d), (e), and (g) are regarded as subsets of $\mathbb{R}$. Some properties of these sets are tabulated below:

	Closed	Open	Bounded
(a)	No	Yes	Yes
(b)	Yes	No	Yes
(c)	Yes	No	Yes
(d)	Yes	No	No
(e)	No	No	Yes
(f)	Yes	Yes	No
(g)	No	Yes	Yes

Proposition 6.8 *A subset $E \subset X$ of a metric space X is open if and only if its complement E^c is closed.*

Proof. First, suppose E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not an accumulation point of E^c. Hence there exists a neighborhood U of x such that $U \cap E^c$ is empty, that is $U \subset E$. Thus x is an interior point of E and E is open.
Next, suppose that E is open. Let x be an accumulation point of E^c. Then every neighborhood of x contains a point of E^c, so that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed. ∎

Lemma 6.9 *Let X be a set and $\{E_\alpha\}$ a family of subsets of X. Then*

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c \quad \textit{(de Morgan's rule).} \tag{6.4}$$

Using $(A^c)^c = A$ for all subsets $A \subset X$ we obtain de Morgan's second rule by taking the complement of (6.4):

$$\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha^c.$$

Proof. Let A and B be the left and right members of (6.4), respectively. If $x \in A$, then $x \notin \bigcup_\alpha E_\alpha$, hence $x \notin E_\alpha$ for every α. Thus, $x \in E_\alpha^c$ for any α, so that $x \in B$.
Conversely, if $x \in B$ then $x \in E_\alpha^c$ for every α, hence $x \notin E_\alpha$ for any α, hence $x \notin \bigcup_\alpha E_\alpha$, so that $x \in A$; thus $B \subset A$. It follows that $A = B$. ∎

Proposition 6.10
(a) *For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.*
(b) *For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.*
(c) *For any finite collection $G_1, \dots, G_n$ of open sets, $\displaystyle\bigcap_{i=1}^n G_i$ is open.*
(d) *For any finite collection $F_1, \dots, F_n$ of closed sets, $\displaystyle\bigcup_{i=1}^n F_i$ is closed.*

Proof. Put $G = \bigcup_\alpha G_\alpha$. If $x \in G$, then $x \in G_\alpha$ for some α. Since x is an interior point of G_α, x is also an interior point of G, and G is open. This proves (a).
By (6.4) and since F_α^c is open, (a) implies that $(\bigcap_\alpha F_\alpha)^c = \bigcup_\alpha F_\alpha^c$ is open. Hence by Proposition 6.8, $\bigcap_\alpha F_\alpha$ is closed.
Next, put $H = \bigcap_{i=1}^n G_i$. For any $x \in H$ there exist neighborhoods $U_{r_i}(x)$ such that $U_{r_i}(x) \subset G_i$, $i = 1, \dots, n$. Put $r := \min\{r_1, \dots, r_n\}$. Then $U_r(x) \subset G_i$ for $i = 1, \dots, n$, so that $U_r(x) \subset H$, and H is open.
By taking complements, (d) follows from (c): $\displaystyle\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n F_i^c.$ ∎

Example 6.6 In parts (c) and (d) of the preceding proposition, the finiteness of the collection is essential. For let $G_n = (-1/n, 1/n)$, $n \in \mathbb{N}$. Then G_n is an open subset of $\mathbb{R}$. Put $G = \bigcap_{n\in\mathbb{N}} G_n$. Then $G = \{0\}$ consists of a single point and is therefore not an open subset of $\mathbb{R}$. Similarly, the union of an infinite collection of closed sets need not to be closed.

6.3.1 Topological Spaces

Forgetting about the metric d and taking the open sets as the basic notion we arrive at the notion of a *topological space.*

Definition 6.6 Let X be a set. A family $\mathcal{T}$ of subsets of X is called a *topology* on X if

(a) $\varnothing$, $X \in \mathcal{T}$.
(b) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.
(c) If $\mathcal{F} \subset \mathcal{T}$ then $\bigcup \mathcal{F} \in \mathcal{T}$.

Recall that $\bigcup \mathcal{F} = \{x \mid \exists F \in \mathcal{F} \colon x \in F\}$ is the union of the family $\mathcal{F}$. The elements of $\mathcal{T}$ are called *open* sets of X. A subset E of X is said to be *closed* if $E^{\mathsf{c}} \in \mathcal{T}$. A subset U of X is called a *neighborhood* of a point $x \in X$ if there is an open set G with $x \in G \subset U$. Every metric space is a topological space.

Proposition 6.11 *Let $E_1 = (E, \|\cdot\|_i)$, $i = 1, 2$, be normed vector spaces such that there exist positive numbers $c_1, c_2 > 0$ with*

$$c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1\,, \quad \textit{for all} \quad x \in E. \tag{6.5}$$

Then E_1 and E_2 have the same topology, i. e. G is open in E_1 if and only if G is open in E_2.

Proof. Condition (6.5) is obviously symmetric with respect to E_1 and E_2 since $\|x\|_2/c_2 \leq \|x\|_1 \leq \|x\|_2/c_1$. It is sufficient to show the following: If $\|x\|_2 \leq c\|x\|_1$ then every open set $G \subset E_2$ is also open in E_1 (we say that the topology of E_1 is *stronger* or *finer* than the topology of E_2 since E_1 has more open sets).
For, let $p \in G \subset E_2$, then p is an inner point and there is a neighborhood $U^2_\varepsilon(p) = \{x \mid \|x-p\|_2 < \varepsilon\}$ contained in G. We will show that $U^1_{\varepsilon/c}(p) \subset U^2_\varepsilon(p)$.

$$x \in U^1_{\varepsilon/c}(p) \Longrightarrow \|x-p\|_1 < \varepsilon/c \Longrightarrow \|x-p\|_2 \leq c\|x-p\|_1 < c \cdot \frac{\varepsilon}{c} = \varepsilon.$$

We conclude $U^1_{\varepsilon/c}(p) \subset U^2_\varepsilon(p) \subset G$, and p is an inner point of $G \subset E_1$. ∎

Example 6.7 (a) The normed vector spaces $(E, \|\cdot\|_p)$, $p \in [1, \infty]$, $E = \mathbb{R}^k$ or $E = \mathbb{C}^k$, are equivalent as topological vector spaces. Since

$$\|x\|_\infty^p \leq \sum_{i=1}^k |\,x_i\,|^p \leq \sum_{i=1}^k \|x\|_\infty^p = k\,\|x\|_\infty^p\,,$$
$$\|x\|_\infty \leq \|x\|_p \leq \sqrt[p]{k}\;\|x\|_\infty\,. \tag{6.6}$$

This shows that $(E, \|\cdot\|_p)$ and $(E, \|\cdot\|_\infty)$ have the same open sets.
(b) The inequality

$$\|f\|_p \leq \sqrt[p]{b-a}\ \|f\|_\infty$$

in Example 6.3 (b) shows that the supremum norm on $C([a,b])$ defines a finer topology than the L^p-norm. The topologies are *not* equivalent.

6.4 Limits and Continuity

In this section we generalize the notions of convergent sequences and continuous functions to arbitrary metric spaces.

Definition 6.7 Let X be a metric space and (x_n) a sequence of elements of X. We say that (x_n) *converges to* $x \in X$ if for every neighborhood U_ε, $\varepsilon > 0$, of x there exists an $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $x_n \in U_\varepsilon$. We write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \underset{n\to\infty}{\longrightarrow} x$; formally

$$\forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N}\ \forall\, n \geq n_0 \colon x_n \in U_\varepsilon(x).$$

We have $\lim\limits_{n\to\infty} x_n = x$ if and only if, the sequence of real numbers $(d(x_n, x))$ converges to 0 as $n \to \infty$, $\lim\limits_{n\to\infty} d(x_n, x) = 0$.
Note that a subset F of a metric space X is closed if and only if F contains all limits of convergent sequences (x_n), $x_n \in F$. That is, our first definition of a closed set $F \subset \mathbb{R}$ given in Definition 3.6 works in arbitrary metric spaces.
The following Proposition is quite analogous to Proposition 2.31 with $k = 2$.

Proposition 6.12 *Let (x_n) be a sequence of vectors of the euclidean space $\mathbb{R}^k$,*

$$x_n = (x_{n1}, \dots, x_{nk}).$$

Then (x_n) converges to $a = (a_1, \dots, a_k) \in \mathbb{R}^k$ if and only if

$$\lim_{n\to\infty} x_{ni} = a_i, \quad i = 1, \dots, k.$$

Proof. Suppose that $\lim\limits_{n\to\infty} x_n = a$. Given $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\|x_n - a\|_2 < \varepsilon$. Thus, for $i = 1, \dots, k$ we have

$$|\, x_{ni} - a_i \,| \leq \|x_n - a\|_2 < \varepsilon;$$

hence $\lim\limits_{n\to\infty} x_{ni} = a_i$.
Conversely, suppose that $\lim\limits_{n\to\infty} x_{ni} = a_i$ for $i = 1, \dots, k$. Given $\varepsilon > 0$ there are $n_{0i} \in \mathbb{N}$ such that $n \geq n_{0i}$ implies

$$|\, x_{ni} - a_i \,| < \frac{\varepsilon}{\sqrt{k}}.$$

For $n \geq \max\{n_{01}, \dots, n_{0k}\}$ we have (see (6.6))

$$\|x_n - a\|_2 \leq \sqrt{k}\ \|x_n - a\|_\infty < \varepsilon.$$

hence $\lim\limits_{n\to\infty} x_n = a$. ∎

Corollary 6.13 *Let $B \subset \mathbb{R}^k$ be a bounded subset and (x_n) a sequence of elements of B. Then (x_n) has a converging subsequence.*

Proof. Since B is bounded all coordinates of B are bounded; hence there is a subsequence $(x_n^{(1)})$ of (x_n) such that the first coordinate converges. Further, there is a subsequence $(x_n^{(2)})$ of $(x_n^{(1)})$ such that the second coordinate converges. Finally there is a subsequence $(x_n^{(k)})$ of $(x_n^{(k-1)})$ such that all coordinates converge. By the above proposition the subsequence $(x_n^{(k)})$ converges in $\mathbb{R}^k$. ■

The same statement is true for subsets $B \subset \mathbb{C}^k$.

Definition 6.8 Let (X, d) be a metric space. A sequence (x_n) of elements of X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{for all} \quad m, n \geq n_0.$$

A metric space is said to be *complete* if every Cauchy sequence converges.
A complete normed vector space is called a *Banach* space.

Proposition 6.14 *The euclidean k-space $\mathbb{R}^k$ is complete.*

Proof. Let (x_n), $x_n = (x_{n1}, \dots, x_{nk})$, be a Cauchy sequence in $\mathbb{R}^k$. Since

$$|\, x_{ni} - x_{mi} \,| \leq \|x_n - x_m\|$$

for every $i = 1, \dots, k$, the sequences $(x_{ni})_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is complete (Proposition 2.16), (x_{ni}) converges and so does (x_n) by Proposition 6.12. ■

Note that the unitary k-space $\mathbb{C}^k$ is also complete.
Q 20. Consider the normed vector spaces $E_j = (\mathbb{R}^k, \|\cdot\|_j)$, $j = 1, 2, \infty$.
Prove that a sequence (x_n), $x_n \in \mathbb{R}^k$, converges in E_1 if and only if it converges in E_2 if and only if it converges in E_∞.

Definition 6.9 (ϵ-δ definition) A mapping $f\colon X \to Y$ from the metric space X into the metric space Y is said to be *continuous* at $a \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$

$$d(x, a) < \delta \quad \text{implies} \quad d(f(x), f(a)) < \varepsilon. \tag{6.7}$$

The mapping f is said to be *continuous* on X if f is continuous at every point a of X.

As in the case of real functions we have an equivalent characterization of continuous functions using sequences. The proof is completely the same as the proof of Proposition 3.1 3.1; we omit it.

Proposition 6.15 *A mapping $f\colon X \to Y$ from the metric space X into the metric space Y is continuous at a point $a \in X$ if and only if*

$$\lim_{x\to a} f(x) = f(a), \tag{6.8}$$

i. e. if for any sequence (x_n), $x_n \in X$, $x_n \neq a$, with $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} f(x_n) = f(a)$.

Proposition 6.16 *The composition of two continuous mappings is continuous.*

The proof is completely the same as in the real case (see Proposition 3.4) and we omit it.

Proposition 6.17 (a) *The projection mapping* $\mathrm{p}_i\colon \mathbb{R}^k \to \mathbb{R}$, $i = 1, \dots, k$, *given by* $\mathrm{p}_i(x_1, \dots, x_k) = x_i$ *is continuous.*
(b) *Let* $U \subseteq \mathbb{R}^k$ *be open and* $f, g\colon U \to R$ *continuous functions on* U. *Then* $f + g$, fg, $|\,f\,|$, *and,* f/g $(g \neq 0)$ *are continuous functions on* U.
(c) *Let* X *be a metric space. A mapping*

$$f = (f_1, \dots, f_k)\colon X \to \mathbb{R}^k$$

is continuous if and only if all components $f_i\colon X \to \mathbb{R}$, $i = 1, \dots, k$, *are continuous.*

Proof. (a) Let (x_n) be a sequence converging to $a = (a_1, \dots, a_k) \in \mathbb{R}^k$. Then the sequence $(\mathrm{p}_i(x_n))$ converges to $a_i = \mathrm{p}_i(a)$ by Proposition 6.12. This shows continuity of p_i at a.
(b) The proofs are quite similar to the proofs in the real case, see Proposition 2.3. As a sample we carry out the proof in case fg. Let $a \in U$ and put $M = \max\{|\,f(a)\,|, |\,f(b)\,|\}$. Let $\varepsilon > 0$, $\varepsilon < 3M^2$, be given. Since f and g are continuous at a, there exists $\delta > 0$ such that

$$\begin{aligned} \|x - a\| < \delta \quad &\text{implies} \quad |\,f(x) - f(a)\,| < \frac{\varepsilon}{3M}, \\ \|x - a\| < \delta \quad &\text{implies} \quad |\,g(x) - g(a)\,| < \frac{\varepsilon}{3M}. \end{aligned} \tag{6.9}$$

Note that

$$fg(x) - fg(a) = (f(x) - f(a))(g(x) - g(a)) + f(a)(g(x) - g(a)) + g(a)(f(x) - f(a)).$$

Taking the absolute value of the above identity, using the triangle inequality as well as (6.9) we have that $\|x - a\| < \delta$ implies

$$\begin{aligned} |\,fg(x) - fg(a)\,| &\le |\,f(x) - f(a)\,|\,|\,g(x) - g(a)\,| + |\,f(a)\,|\,|\,g(x) - g(a)\,| + |\,g(a)\,|\,|\,f(x) - f(a)\,| \\ &\le \frac{\varepsilon^2}{9M^2} + M\frac{\varepsilon}{3M} + M\frac{\varepsilon}{3M} \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This proves continuity of fg at a.
(c) Suppose first that f is continuous at $a \in X$. Since $f_i = \mathrm{p}_i \circ f$, f_i is continuous by the result of (a) and Proposition 6.16.
Suppose now that all the f_i, $i = 1, \dots, k$, are continuous at a. Let (x_n), $x_n \neq a$, be a sequence in X with $\lim_{n\to\infty} x_n = a$ in X. Since f_i is continuous, the sequences $(f_i(x_n))$ of numbers converge to $f_i(a)$. By Proposition 6.12, the sequence of vectors $f(x_n)$ converges to $f(a)$; hence f is continuous at a. ∎

Example 6.8 Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$f(x, y, z) = \begin{pmatrix} \sin \frac{x^2 + \mathrm{e}^z}{\sqrt{x^2+y^2+z^2+1}} \\ \log |\,x^2 + y^2 + z^2 + 1\,| \end{pmatrix}.$$

Then f is continuous on U. Indeed, since product, sum, and composition of continuous functions are continuous, $\sqrt{x^2+y^2+z^2+1}$ is a continuous function on $\mathbb{R}^3$. We also made use of Proposition 6.17 (a); the coordinate functions x, y, and z are continuous. Since the denominator is nonzero, $f_1(x,y,z) = \sin \frac{x^2+e^z}{\sqrt{x^2+y^2+z^2+1}}$ is continuous. Since $|\,x^2+y^2+z^2+1\,| > 0$, $f_2(x,y,z) = \log|\,x^2+y^2+z^2+1\,|$ is continuous. By Proposition 6.17 (c) f is continuous.

We give the topological description of continuous functions.

Proposition 6.18 *Let X and Y be metric spaces. A mapping $f\colon X \to Y$ is continuous if and only if the preimage of any open set in Y is open in X.*

Proof. Suppose that f is continuous and $G \subset Y$ is open. If $f^{-1}(G) = \varnothing$, there is nothing to prove; the empty set is open. Otherwise there exists $x_0 \in f^{-1}(G)$, and therefore $f(x_0) \in G$. Since G is open, there is $\varepsilon > 0$ such that $U_\varepsilon(f(x_0)) \subset G$. Since f is continuous in x_0, to ε there exists $\delta > 0$ such that $x \in U_\delta(x_0)$ implies $f(x) \in U_\varepsilon(f(x_0)) \subset G$. That is, $U_\delta(x_0) \subset f^{-1}(G)$, and x_0 is an inner point of $f^{-1}(G)$; hence $f^{-1}(G)$ is open.
Suppose now that the condition of the proposition is fulfilled. We will show that f is continuous. Fix $x_0 \in X$ and $\varepsilon > 0$. Since $G = U_\varepsilon(f(x_0))$ is open by Lemma 6.7, $f^{-1}(G)$ is open by assumption. In particular, $x_0 \in f^{-1}(G)$ is an inner point. Hence, there exists $\delta > 0$ such that $U_\delta(x_0) \subset f^{-1}(G)$. It follows that $f(U_\delta(x_0)) \subset U_\varepsilon(x_0)$; this means that f is continuous at x_0. Since x_0 was arbitrary, f is continuous on X. ∎

Remark 6.2 Since the complement of an open set is a closed set, it is obvious that the proposition holds if we replace "open set" by "closed set."
In general, the image of an open set under a continuous function need not to be open; consider for example $f(x) = \sin x$ and $G = (0, 2\pi)$ which is open; however, $f((0,2\pi)) = [-1,1]$ is not open.

Definition 6.10 Two metric spaces X and Y are said to be *homeomorphic* if there exists a bijective continuous mapping $f\colon X \to Y$ such that its inverse mapping is also continuous.

Example 6.9 (a) The mapping $f\colon \mathbb{R} \to (-1,1)$, $f(x) = \frac{2}{\pi}\arctan x$ is continuous and bijective and its inverse $g\colon (-1,1) \to \mathbb{R}$, $x = g(y) = \tan\left(\frac{\pi y}{2}\right)$ is also continuous. Hence, $(-1,1)$ and $\mathbb{R}$ are homeomorphic.
(b) The two intervals $[0,1)$ and $[0,1]$ are not homeomorphic. There is no continuous surjective mapping $f\colon [0,1] \to [0,1)$ since the image of a compact set under a continuous mapping is compact, see Proposition 6.22 below.

6.5 Compact Sets

By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition 6.11 (Covering definition) A subset K of a metric space X is said to be *compact* if every open cover contains a finite subcover. More explicitly, if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

The notion of compactness is of great importance in analysis, especially in connection with continuity. In Proposition 6.19 below we will see that the above definition of compactness is equivalent to the one already given in Definition 3.7 for subsets of $\mathbb{R}$ and $\mathbb{C}$. Note that the definition does not state that a set is compact if there exists a finite open cover—the whole space X is open and a cover consisting of only one member. Instead, *every* open cover has a finite subcover.

Example 6.10 (a) It is clear that every finite set is compact.
(b) Let (x_n) be a converging to x sequence in a metric space X. Then

$$A = \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$$

is compact.
Proof. Let $\{G_\alpha\}$ be any open cover of A. In particular, the limit point x is covered by, say, G_0. Then there is an $n_0 \in \mathbb{N}$ such that $x_n \in G_0$ for every $n \geq n_0$. Finally, x_k is covered by some G_k, $k = 1, \dots, n_0 - 1$. Hence the collection

$$\{G_k \mid k = 0, 1, \dots, n_0 - 1\}$$

is a finite subcover of A; therefore A is compact. ■

Proposition 6.19 *A subset K of a metric space X is compact if and only if every sequence in K contains a convergent in K subsequence.*

Proof. (a) Let K be compact and suppose to the contrary that (x_n) is a sequence in K without any convergent to some point of K subsequence. Then every $x \in K$ has a neighborhood U_x containing only finitely many elements of the sequence (x_n). (Otherwise x would be a limit point of (x_n) and there were a converging to x subsequence.) By construction,

$$K \subset \bigcup_{x \in X} U_x.$$

Since K is compact, there are finitely many points $y_1, \dots, y_m \in K$ with

$$K \subset U_{y_1} \cup \dots \cup U_{y_m}.$$

Since every U_{y_i} contains only finitely many elements of (x_n), there are only finitely many elements of (x_n) in K—a contradiction.
(b) The proof is an the appendix to this chapter. ■

Corollary 6.20 (Heine–Borel) *A subset K of $\mathbb{R}^k$ or $\mathbb{C}^k$ is compact if and only if K is bounded and closed.*

Proof. Suppose K is closed and bounded. Let (x_n) be a sequence in K. By Corollary 6.13 (x_n) has a convergent subsequence. Since K is closed, the limit is in K. By the above proposition K is compact.
The proof of the converse direction is completely the same as in Proposition 3.7. ■

The following Proposition is also an easy consequence of Proposition 6.19. But we give an independent proof based on the covering definition of compactness in the appendix.

Proposition 6.21 (a) *A compact subset of a metric space is closed and bounded.*
(b) *A closed subsets of a compact set is compact.*

As in the real case (see Proposition 3.9 and Theorem 3.10) we have the analogous results for metric spaces.

Proposition 6.22 *Let X be a compact metric space.*
(a) *Let $f\colon X \to Y$ be a continuous mapping into the metric space Y. Then $f(X)$ is compact.*
(b) *Let $f\colon X \to \mathbb{R}$ a continuous mapping. Then f is bounded and attains its maximum and minimum, that is there are points p and q in X such that*

$$f(p) = \sup_{x\in X} f(x), \quad f(q) = \inf_{x\in X} f(x).$$

Proof. (a) Let $\{G_\alpha\}$ be an open covering of $f(X)$. By Proposition 6.18 $f^{-1}(G_\alpha)$ is open for every α. Hence, $\{f^{-1}(G_\alpha)\}$ is an open cover of X. Since X is compact there is an open subcover of X, say $\{f^{-1}(G_{\alpha_1}),\dots,f^{-1}(G_{\alpha_n})\}$. Then $\{G_{\alpha_1},\dots,G_{\alpha_n}\}$ is a finite subcover of $\{G_\alpha\}$ covering $f(X)$. Hence, $f(X)$ is compact. We skip (b). ■

Similarly as for real function we have the following proposition about uniform continuity. The proof is in the appendix.

Proposition 6.23 *Let $f\colon K \to \mathbb{R}$ be a continuous function on a compact set $K \subset \mathbb{R}$. Then f is uniformly continuous on K.*

Q 21. For $x, y \in \mathbb{R}$ define

$$\begin{aligned}
d_1(x,y) &= (x-y)^2,\\
d_2(x,y) &= \sqrt{|\,x-y\,|},\\
d_3(x,y) &= \left|\,x^2-y^2\,\right|\\
d_4(x,y) &= |\,x-2y\,|,\\
d_5(x,y) &= \frac{|\,x-y\,|}{1+|\,x-y\,|},\\
d_6(x,y) &= \arctan(x-y).
\end{aligned}$$

Determine for each of these, whether it is a metric or not.

6.6 Appendix E

Proof of Proposition 6.21. (a) Let K be a compact subset of a metric space X. We shall prove that the complement of K is an open subset of X.
Suppose that $p \in X$, $p \notin K$. If $q \in K$, let V^q and $U(q)$ be neighborhoods of p and q, respectively, of radius less than $d(p,q)/2$. Since K is compact, there are finitely many points $q_1, \dots, q_n$ in K such that

$$K \subset U_{q_1} \cup \dots \cup U_{q_n} =: U.$$

If $V = V^{q_1} \cap \dots \cap V^{q_n}$, then V is a neighborhood of p which does not intersect U. Hence $U \subset K^{\mathsf{c}}$, so that p is an interior point of K^{c}, and K is closed. We show that K is bounded. Let $\varepsilon > 0$ be given. Since K is compact the open cover $\{U_\varepsilon(x) \mid x \in K\}$ of K has a finite subcover, say $\{U_\varepsilon(x_1), \dots, U_\varepsilon(x_n)\}$. Let $U = \bigcup_{i=1}^n U_\varepsilon(x_i)$, then the maximal distance of two points x and y in U is bounded by

$$2\varepsilon + \sum_{1 \le i < j \le n} d(x_i, x_j).$$

This completes the proof of (a).
(b) Suppose $F \subset K \subset X$, F is closed in X, and K is compact. Let $\{U^\alpha\}$ be an open cover of F. Since F^{c} is open, $\{U^\alpha, F^{\mathsf{c}}\}$ is an open cover Ω of K. Since K is compact, there is a finite subcover Φ of Ω, which covers K. If F^{c} is a member of Φ, we may remove it from Φ and still retain an open cover of F. Thus we have shown that a finite subcollection of $\{U^\alpha\}$ covers F. ∎

Proof of Proposition 6.19 (b). This direction is hard to proof. It does not work in arbitrary topological spaces and essentially uses that X is a metric space. The prove is roughly along the lines of Exercises 22 to 26 in [8]. We give the proof of Bredon (see [1, 9.4 Theorem])
Suppose that every sequence in K contains a converging in K subsequence.
1) K contains a countable dense set. For, we show that for every $\varepsilon > 0$, K can be covered by a finite number of ε-balls (ε is fixed). Suppose, this is not true, i. e. K can't be covered by any finite number of ε-balls. Then we construct a sequence (x_n) as follows. Take an arbitrary x_1. Suppose $x_1, \dots, x_n$ are already found; since K is not covered by a finite number of ε-balls, we find x_{n+1} which distance to every preceding element of the sequence is greater than or equal to ε. Consider a limit point x of this sequence and an $\varepsilon/2$-neighborhood U of x. Almost all elements of a suitable subsequence of (x_n) belong to U, say x_r and x_s with $s > r$. Since both are in U their distance is less than ε. But this contradicts the construction of the sequence.
Now take the union of all those finite sets corresponding to $\varepsilon = 1/n$, $n \in \mathbb{N}$. This is a countable dense set of K.
2) Any open cover $\{U_\alpha\}$ of K has a countable subcover. Let $x \in K$ be given. Since $\{U_\alpha\}_{\alpha \in I}$ is an open cover of K we find $\beta \in I$ and $n \in \mathbb{N}$ such that $U_{2/n}(x) \subset U_\alpha$. Further,

since $\{x_i\}_{i\in\mathbb{N}}$ is dense in K, we find $i, n \in \mathbb{N}$ such that $d(x, x_i) < 1/n$. By the triangle inequality

$$x \in U_{1/n}(x_i) \subset U_{2/n}(x) \subset U_\beta.$$

To each of the countably many $U_{1/n}(x_i)$ choose one $U_\beta \supset U_{1/n}(x_i)$. This is a countable subcover of $\{U_\alpha\}$.
3) Rename the countable open subcover by $\{V_n\}_{n\in\mathbb{N}}$ and consider the decreasing sequence C_n of closed sets

$$C_n = K \setminus \bigcup_{k=1}^{n} V_k, \quad C_1 \supset C_2 \supset \cdots .$$

If $C_k = \varnothing$ we have found a finite subcover, namely $V_1, V_2, \dots, V_k$. Suppose that all the C_n are nonempty, say $x_n \in C_n$. Further, let x be the limit of the subsequence (x_{n_i}). Since $x_{n_i} \in C_m$ for all $n_i \geq m$ and C_m is closed, $x \in C_m$ for all m. Hence $x \in \bigcap_{m\in\mathbb{N}} C_m$. However,

$$\bigcap_{m\in\mathbb{N}} C_m = K \setminus \bigcup_{m\in\mathbb{N}} V_m = \varnothing.$$

This contradiction completes the proof. ∎

Proof of Proposition 6.23. Let $\varepsilon > 0$ be given. Since f is continuous, we can associate to each point $p \in K$ a positive number $\delta(p)$ such that $q \in K \cap U_{\delta(p)}(p)$ implies $|\, f(q) - f(p)\,| < \varepsilon/2$. Let $J(p) = \{q \in K \mid |\, p - q\,| < \delta(p)/2\}$.
Since $p \in J(p)$, the collection $\{J(p) \mid p \in K\}$ is an open cover of K; and since K is compact, there is a finite set of points $p_1, \dots, p_n$ in K such that

$$K \subset J(p_1) \cup \cdots \cup J(p_n). \tag{6.10}$$

We put $\delta := \frac{1}{2}\min\{\delta(p_1), \dots, \delta(p_n)\}$. Then $\delta > 0$. Now let p and q be points of K with $|\, x - y\,| < \delta$. By (6.10), there is an integer m, $1 \leq m \leq n$, such that $p \in J(p_m)$; hence

$$|\, p - p_m\,| < \frac{1}{2}\delta(p_m),$$

and we also have

$$|\, q - p_m\,| \leq |\, p - q\,| + |\, p - p_m\,| < \delta + \frac{1}{2}\delta(p_m) \leq \delta(p_m).$$

Finally, continuity at p_m gives

$$|\, f(p) - f(q)\,| \leq |\, f(p) - f(p_m)\,| + |\, f(p_m) - f(q)\,| < \varepsilon.$$

This completes the proof. ∎