Chapter 2

Sequences and Series

This chapter will deal with one of the main notion of calculus, the **limit of a sequence**. Although we are concerned with numerical sequence, almost all notions make sense in arbitrary metric spaces.
Given $a \in \mathbb{R}$ and $\varepsilon > 0$ we define

$$U_\varepsilon(a) := \{x \in \mathbb{R} \mid a - \varepsilon < x < a + \varepsilon\},$$

and call it the *ε-neighborhood of a.*

2.1 Convergent Sequences

A *sequence* is a mapping $x\colon \mathbb{N} \to \mathbb{R}$. To every $n \in \mathbb{N}$ we associate a real number x_n. We write this as $(x_n)_{n\in\mathbb{N}}$ or $(x_1, x_2, \dots)$.

Example 2.1 (a) $x_n = \frac{1}{n}$, $(1/n)$,$(1, 1/2, 1/3, \dots)$;
(b) $x_n = (-1)^n + 1$, $(0, 2, 0, 2, \dots)$;
(c) $x_n = a$ ($a \in \mathbb{R}$ fixed), $(a, a, \dots)$ (constant sequence),
(d) $x_n = a^n$ ($a \in \mathbb{R}_+$ fixed), $(a, a^2, a^3, \dots)$ (geometric sequence);

Definition 2.1 A sequence (x_n) is said to be *convergent to x* if

For every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies

$$|\, x_n - x \,| < \varepsilon.$$

x is called the *limit* of (x_n) and we write

$$x = \lim_{n\to\infty} x_n \quad \text{or simply} \quad x = \lim x_n \quad \text{or} \quad x_n \to x.$$

If there is no such x with the above property, the sequence (x_n) is said to be *divergent*.
In other words: (x_n) converges to x if any neighborhood $U_\varepsilon(x)$, $\varepsilon > 0$, contains "almost all" elements of the sequence (x_n). "Almost all" means "all but finitely many." Sometimes we say "for sufficiently large n" which means the same.

This is an equivalent formulation since $x_n \in U_\varepsilon(x)$ means $x - \varepsilon < x_n < x + \varepsilon$, hence $|\,x - x_n\,| < \varepsilon$. The n_0 in question need not to be the smallest possible.
We write

$$\lim_{n\to\infty} x_n = +\infty \tag{2.1}$$

if for all $E > 0$ there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $x_n \geq E$. Similarly, we write

$$\lim_{n\to\infty} x_n = -\infty \tag{2.2}$$

if for all $E > 0$ there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $x_n \leq -E$. In these cases we say that $+\infty$ and $-\infty$ are *improper limits* of (x_n). Note that in both cases (x_n) is divergent.

Example 2.2 (0) $\lim(-n)^3 = -\infty$, $\lim n = +\infty$. But $((-n)^n)$ and $(1, 2, 1, 3, 1, 4, 1, 5, \dots)$ both have no improper limit.
Let us have a look at the above Example 2.1.
(a) $\lim 1/n = 0$. Let $\varepsilon > 0$. We are seeking n_0 with $|\,1/n - 0\,| < \varepsilon$ for all $n \geq n_0$. This is equivalent to $1/\varepsilon < n$. Choose $n_0 > 1/\varepsilon$. Then for all $n \geq n_0 : n > 1/\varepsilon$; hence $1/n < \varepsilon$ and $|\,x_n - 0\,| < 0$. Therefore, (x_n) goes to 0.
(b) $x_n = (-1)^n + 1$ is divergent. Suppose to the contrary that x is the limit. To $\varepsilon = 1$ there is n_0 such that for $n \geq n_0$ we have $|\,x_n - x\,| < 1$. For even $n \geq n_0$ we find $|\,2 - x\,| < 1$ for odd $n \geq n_0$, $|\,0 - x\,| = |\,x\,| < 1$. The triangle inequality gives

$$2 = |\,(2 - x) + x\,| \leq |\,2 - x\,| + |\,x\,| < 1 + 1 = 2.$$

This is a contradiction. Hence, (x_n) is divergent.
(c) $x_n = a$. $\lim x_n = a$ since $|\,x_n - a\,| = |\,a - a\,| = 0 < \varepsilon$ for all $\varepsilon > 0$ and all $n \in \mathbb{N}$.
(d) $x_n = a^n$, $(a \geq 0)$.

$$\lim_{n\to\infty} a^n = \begin{cases} 1, & \text{if} \quad a = 1, \\ 0, & \text{if} \quad 0 \leq a < 1. \end{cases}$$

(a^n) is divergent for $a > 1$. Moreover, $\lim a^n = +\infty$. To prove this let $E > 0$ be given. By the Archimedean property of $\mathbb{R}$ and since $a - 1 > 0$ we find $m \in \mathbb{N}$ such that $m(a-1) > E$. Bernoulli's inequality gives

$$a^m \geq m(a - 1) + 1 > m(a - 1) > E.$$

By homework 4.1 (b), $n \geq m$ implies

$$a^n \geq a^m > E.$$

This proves the claim.
Clearly (a^n) is convergent in cases $a = 0$ and $a = 1$ since the sequence is constant then. Let $0 < a < 1$. Bernoulli's inequality gives

$$\left(\frac{1}{a}\right)^n \geq 1 + n\left(\frac{1}{a} - 1\right) > n\left(\frac{1}{a} - 1\right) \qquad |\cdot a^n \qquad |: (n(1/a - 1)) > 0$$

$$\frac{1}{n(1/a - 1)} > a^n > 0. \tag{2.3}$$

Let $\varepsilon > 0$. Choose $n_0 > \dfrac{1}{\varepsilon(1/a-1)}$. Then $\varepsilon > \dfrac{1}{n_0(1/a-1)}$ and $n \geq n_0$ implies

$$|\, a^n - 0 \,| = |\, a^n \,| = a^n \underset{(2.3)}{<} \frac{1}{n(1/a-1)} \leq \frac{1}{n_0(1/a-1)} < \varepsilon.$$

Hence, $a^n \to 0$.

Proposition 2.1 *The limit of a convergent sequence is uniquely determined.*

Proof. Suppose that $x = \lim x_n$ and $y = \lim x_n$ and $x \neq y$. Put $\varepsilon := |\, x - y \,|/2 > 0$. Then

$$\exists n_1 \in \mathbb{N}\ \forall n \geq n_1 : |\, x - x_n \,| < \varepsilon,$$
$$\exists n_2 \in \mathbb{N}\ \forall n \geq n_2 : |\, y - y_n \,| < \varepsilon.$$

Choose $m \geq \max\{n_1, n_2\}$. Then $|\, x - x_m \,| < \varepsilon$ and $|\, y - x_m \,| < \varepsilon$. Hence,

$$|\, x - y \,| \leq |\, x - x_m \,| + |\, y - x_m \,| < 2\varepsilon = |\, x - y \,|.$$

This contradiction establishes the statement. ∎

Proposition 2.1 holds in arbitrary metric spaces.

Definition 2.2 A sequence (x_n) is said to be *bounded* if the set of its elements is a bounded set; i. e. there is a $C \geq 0$ such that

$$|\, x_n \,| \leq C \quad \text{for all } n \in \mathbb{N}.$$

Similarly one defines that (x_n) is *bounded above* and *bounded below.*

Proposition 2.2 *If (x_n) is convergent, then (x_n) is bounded.*

Proof. Let $x = \lim x_n$. To $\varepsilon = 1$ there exists $n_0 \in \mathbb{N}$ such that $|\, x - x_n \,| < 1$ for all $n \geq n_0$. Then $|\, x_n \,| = |\, x_n - x + x \,| \leq |\, x_n - x \,| + |\, x \,| < |\, x \,| + 1$ for all $n \geq n_0$. Put

$$C := \max\{|\, x_1 \,|, \dots, |\, x_{n_0-1} \,|, |\, x \,| + 1\}.$$

Then $|\, x_n \,| \leq C$ for all $n \in \mathbb{N}$. ∎

The reversal statement is not true; there are bounded sequences which are not convergent, see Example 2.1 (b).

Corollary. If (x_n) has an improper limit, then (x_n) is divergent.

Proof. Suppose to the contrary that (x_n) is convergent; then it is bounded, say $|\, x_n \,| \leq C$ for all n. This contradicts $x_n > E$ as well as $x_n < -E$ for $E = C$ and sufficiently large n. Hence, (x_n) has no improper limits, a contradiction. ∎

2.1.1 Algebraic operations with sequences

The sum, difference, product, quotient and absolute value of sequences (x_n) and (y_n) are defined as follows

$$(x_n) \pm (y_n) := (x_n \pm y_n), \qquad (x_n) \cdot (y_n) := (x_n\, y_n),$$
$$\frac{(x_n)}{(y_n)} := \left(\frac{x_n}{y_n}\right), (y_n \neq 0) \qquad |\,(x_n)\,| := (|\,x_n\,|).$$

Proposition 2.3 *If (x_n) and (y_n) are convergent sequences and $c \in \mathbb{R}$, then their sum, difference, product, quotient (provided $y_n \neq 0$ and $\lim y_n \neq 0$), and their absolute values are also convergent:*
(a) $\lim(x_n \pm y_n) = \lim x_n \pm \lim y_n$;
(b) $\lim(cx_n) = c \lim x_n$, $\lim(x_n + c) = \lim x_n + c$.
(c) $\lim(x_n\, y_n) = \lim x_n \cdot \lim y_n$;
(d) $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}$ *if* $y_n \neq 0$ *for all* n *and* $\lim y_n \neq 0$;
(e) $\lim |\,x_n\,| = |\lim x_n\,|$.

Proof. Let $x_n \to x$ and $y_n \to y$.
(a) Given $\varepsilon > 0$ then there exist integers n_1 and n_2 such that

$$n \geq n_1 \text{ implies } |\,x_n - x\,| < \varepsilon/2 \text{ and } n \geq n_2 \text{ implies } |\,y_n - y\,| < \varepsilon/2.$$

If $n_0 := \max\{n_1, n_2\}$, then $n \geq n_0$ implies

$$|\,(x_n + y_n) - (x + y)\,| \leq |\,x_n - x\,| + |\,y_n - y\,| \leq \varepsilon.$$

The proof for the difference is quite similar.
(b) follows from $|\,cx_n - cx\,| = |\,c\,|\,|\,x_n - x\,|$ and $|\,(x_n + c) - (x + c)\,| = |\,x_n - x\,|$.
(c) We use the identity

$$x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x). \tag{2.4}$$

Given $\varepsilon > 0$ there are integers n_1 and n_2 such that

$$n \geq n_1 \text{ implies } |\,x_n - x\,| < \sqrt{\varepsilon} \text{ and } n \geq n_2 \text{ implies } |\,y_n - y\,| < \sqrt{\varepsilon}.$$

If we take $n_0 = \max\{n_1, n_2\}$, $n \geq n_0$ implies

$$|\,(x_n - x)(y_n - y)\,| < \varepsilon,$$

so that

$$\lim_{n\to\infty} (x_n - x)(y_n - y) = 0.$$

Now we apply (a) and (b) to (2.4) and conclude that

$$\lim_{n\to\infty} (x_n y_n - xy) = 0.$$

(d) Choosing n_1 such that $|y_n - y| < |y|/2$ if $n \geq n_1$, we see that

$$|y| \leq |y - y_n| + |y_n| < |y|/2 + |y_n| \implies |y_n| > |y|/2.$$

Given $\varepsilon > 0$, there is an integer $n_2 > n_1$ such that $n \geq n_2$ implies

$$|y_n - y| < |y|^2 \varepsilon/2.$$

Hence, for $n \geq n_2$,

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \left| \frac{y_n - y}{y_n y} \right| < \frac{2}{|y|^2} |y_n - y| < \varepsilon$$

and we get $\lim(\frac{1}{y_n}) = \dfrac{1}{\lim y_n}$. The general case can be reduced to the above case using (c) and $(x_n/y_n) = (x_n \cdot 1/y_n)$.

(e) By Lemma 1.14 (e) we have $||x_n| - |x|| \leq |x_n - x|$. Given $\varepsilon > 0$, there is n_0 such that $n \geq n_0$ implies $|x_n - x| < \varepsilon$. By the above inequality, also $||x_n| - |x|| \leq \varepsilon$ and we are done. ∎

Example 2.3 (a) $z_n := \frac{n+1}{n}$. Set $x_n = 1$ and $y_n = 1/n$. Then $z_n = x_n + y_n$ and we already know that $\lim x_n = 1$ and $\lim y_n = 0$. Hence, $\lim \frac{n+1}{n} = \lim 1 + \lim \frac{1}{n} = 1 + 0 = 0$.

(b) $a_n = \frac{3n^2+13n}{n^2-2}$. We can write this as

$$a_n = \frac{3 + \frac{13}{n}}{1 - \frac{2}{n^2}}.$$

Since $\lim 1/n = 0$, by Proposition 2.3, we obtain $\lim 1/n^2 = 0$ and $\lim 13/n = 0$. Hence $\lim 2/n^2 = 0$ and $\lim (3 + 13/n) = 3$. Finally,

$$\lim_{n\to\infty} \frac{3n^2 + 13n}{n^2 - 2} = \frac{\lim_{n\to\infty} \left(3 + \frac{13}{n}\right)}{\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)} = \frac{3}{1} = 3.$$

(c) We introduce the notion of a polynomial and and a rational function.

Given $a_0, a_1, \dots, a_n \in \mathbb{R}$, $a_n \neq 0$. The function $p\colon \mathbb{R} \to \mathbb{R}$ given by $p(t) := a_r t^n + a_{r-1} t^{r-1} + \cdots + a_1 t + a_0$ is called a *polynomial*. The positive integer r is the *degree* of the polynomial $p(t)$, and $a_1, \dots a_r$ are called the *coefficients* of $p(t)$.

Given two polynomials p and q; put $D := \{t \in \mathbb{R} \mid q(t) \neq 0\}$. Then $r = p/q$ is a called a *rational function* where $r\colon D \to \mathbb{R}$ is defined by

$$r(t) := \frac{p(t)}{q(t)}.$$

Polynomials are special rational functions with $q(t) \equiv 1$.

If $x_n \to x$ and $p(t)$ the above polynomial. Then $\lim\limits_{n\to\infty} p(x_n) = p(x)$.We have $p(x_n) = \sum_{k=0}^m a_k x_n^k$. By Proposition 2.3

$$\lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} \sum_{k=0}^r a_k\, x_n^k = \sum_{k=0}^r \lim_{n\to\infty} a_k\, x_n^k = \sum_{k=0}^r a_k \lim_{n\to\infty} x_n^k = \sum_{k=0}^r a_k \left(\lim_{n\to\infty} x_n\right)^k$$

$$= \sum_{k=0}^r a_k\, x^k = p(x).$$

(d) The limit of a rational function of n. Let $p(t)$ as in the above example and $q(t) = \sum_{k=0}^{s} b_k t^k$ polynomials with coefficients $b_k \in \mathbb{R}$. Using Proposition 2.3 and (c) we have

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \lim_{n\to\infty} n^{r-s} \cdot \frac{\frac{a_0}{n^r} + \frac{a_1}{n^{r-1}} + \cdots + a_r}{\frac{b_0}{n^s} + \frac{b_1}{n^{s-1}} + \cdots + b_s}$$

$$= \lim_{n\to\infty} n^{r-s} \frac{a_r}{b_s} = \begin{cases} 0, & \text{for} \quad r < s \\ \frac{a_r}{b_r}, & \text{for} \quad r = s. \end{cases}$$

In case $r > s$ the sequence $\left(\dfrac{P(n)}{Q(n)}\right)$ is divergent. More precisely,

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} +\infty, & \text{if } a_r/b_s > 0, \\ -\infty, & \text{if } a_r/b_s < 0. \end{cases} \tag{2.5}$$

Using the fact that $P(n)/Q(n)$ is positive for all but finitely man n if and only if a_r/b_s is positive, the statement follows directly from the next Remark 2.1.

Remark 2.1 Let a_n be a sequence with $a_n \neq 0$ for every n and $\lim a_n = 0$. Then

$$\lim_{n\to\infty} \frac{1}{a_n} = +\infty, \quad \text{if } a_n > 0 \text{ for all but finitely many } n; \tag{2.6}$$

$$\lim_{n\to\infty} \frac{1}{a_n} = -\infty, \quad \text{if } a_n < 0 \text{ for all but finitely many } n. \tag{2.7}$$

Proof. We will prove (2.7). The proof of (2.6) is analogous. Let $\varepsilon > 0$. By assumption there is a positive integer n_0 such that $n \geq n_0$ implies $-\varepsilon < a_n < 0$. By Proposition 1.9 this implies $0 < -a_n < \varepsilon$ and further

$$\frac{1}{a_n} < -\frac{1}{\varepsilon} < 0. \tag{2.8}$$

Suppose $E > 0$ is given; choose $\varepsilon = 1/E$ and n_0 as above. Then by (2.8), $n \geq n_0$ implies

$$\frac{1}{a_n} < -\frac{1}{\varepsilon} < -E.$$

This shows (2.7). ■

In the German literature the next proposition is known as the 'Theorem of the two policemen'.

Proposition 2.4 (Sandwich Theorem) *Let a_n, b_n and x_n be real sequences with $a_n \leq x_n \leq b_n$ for all but finitely many $n \in \mathbb{N}$. Further let $\lim a_n = \lim b_n = x$. Then x_n is also convergent to x.*

Proof. Let $\varepsilon > 0$. There exist n_1, n_2, and $n_3 \in \mathbb{N}$ such $n \geq n_1$ implies $a_n \in U_\varepsilon(x)$, $n \geq n_2$ implies $b_n \in U_\varepsilon(x)$, and $n \geq n_3$ implies $a_n \leq x_n \leq b_n$. Choosing $n_0 = \max\{n_1, n_2, n_3\}$, $n \geq n_0$ implies $x_n \in U_\varepsilon(x)$. Hence, $x_n \to x$. ■

Example 2.4 $x_n = \sqrt{1+1/n}$. Using Bernoulli's inequality, $1 < \sqrt{1+1/n} < 1+1/(2n)$. Since both the left hand side (1) and the right hand side $(1+1/(2n))$ are convergent to 1, $\lim x_n = 1$.

Q 14. Prove the following statement. Let (x_n) be a real sequence and $x \in \mathbb{R}$. Suppose there is a sequence $d_n \to 0$ such that for all but finitely many $n \in \mathbb{N}$

$$| x_n - x | \leq | d_n | .$$

Then $\lim x_n = x$.

2.1.2 Some special sequences

Proposition 2.5 (a) *If $p > 0$, then* $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.

(b) *If $p > 0$, then* $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

(d) *If $a > 1$ and $\alpha \in \mathbb{R}$, then* $\lim\limits_{n\to\infty} \dfrac{n^\alpha}{a^n} = 0$.

Proof. (a) Let $\varepsilon > 0$. Take $n_0 > (1/\varepsilon)^{1/p}$ (Note that the Archimedean Property of the real numbers is used here). Then $n \geq n_0$ implies $1/n^p < \varepsilon$.

(b) If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then, $x_n > 0$ and by Bernoulli's inequality we have

$$0 < x_n \leq \frac{1}{n}(p-1)$$

By Proposition 2.4, $x_n \to 0$. If $p = 1$, (b) is trivial, and if $0 < p < 1$ the result is obtained by taking reciprocals.

(c) Put $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$, and, by the binomial theorem,

$$n = (1+x_n)^n \geq \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}, \quad (n \geq 2).$$

(d) Put $p = a - 1$, then $p > 0$. Let k be an integer such that $k > \alpha$, $k > 0$. For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence,

$$0 < \frac{n^\alpha}{a^n} = \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k} \quad (n > 2k).$$

Since $\alpha - k < 0$, $n^{\alpha-k} \to 0$ by (a). ∎

Q 15. Let (x_n) be a convergent sequence, $x_n \to x$. Then the sequence of arithmetic means $s_n := \frac{1}{n}\sum_{k=1}^{n} x_k$ also converges to x.

Q 16. Let $(x_n) > 0$ a convergent sequence of positive numbers with and $\lim x_n = x > 0$. Then $\sqrt[n]{x_1 x_2 \cdots x_n} \to x$. *Hint:* Consider $y_n = \log x_n$.

2.1.3 Monotonic Sequences

Definition 2.3 A sequence (x_n) of is said to be

(a) *monotonically increasing* if $x_n \leq x_{n+1}$ for all n;
(b) *monotonically decreasing* if $x_n \geq x_{n+1}$ for all n.

The class of *monotonic sequences* consists of the increasing and decreasing sequences.

A sequence is said to be *strictly monotonically increasing or decreasing* if $x_n < x_{n+1}$ or $x_n > x_{n+1}$ for all n, respectively. We write $x_n \nearrow$ and $x_n \searrow$.

Proposition 2.6 *A monotonic and bounded sequence is convergent. More precisely, if* (x_n) *is increasing and bounded above, then* $\lim x_n = \sup\{x_n\}$*. If* (x_n) *is decreasing and bounded below, then* $\lim x_n = \inf\{x_n\}$*.*

Proof. Suppose $x_n \leq x_{n+1}$ for all n (the proof is analogous in the other case). Let $E := \{x_n \mid n \in \mathbb{N}\}$ and $x = \sup E$. Then $x_n \leq x$, $n \in \mathbb{N}$. For every $\varepsilon > 0$ there is an integer $n_0 \in \mathbb{N}$ such that

$$x - \varepsilon < x_{n_0} < x,$$

for otherwise $x - \varepsilon$ would be an upper bound of E. Since x_n increases, $n \geq n_0$ implies

$$x - \varepsilon < x_n < x,$$

which shows that (x_n) converges to x. ∎

2.1.4 Subsequences

Definition 2.4 Let (x_n) be a sequence and $(n_k)_{k\in\mathbb{N}}$ a strictly increasing sequence of positive integers $n_k \in \mathbb{N}$. We call $(x_{n_k})_{k\in\mathbb{N}}$ a *subsequence* of $(x_n)_{n\in\mathbb{N}}$. If (x_{n_k}) converges, its limit is called a *subsequential limit* of (x_n).

Example 2.5 (a) $x_n = 1/n$, $n_k = 2^k$. then $(x_{n_k}) = (1/2, 1/4, 1/8, \dots)$.
(b) $(x_n) = (1, -1, 1, -1, \dots)$. $(x_{2k}) = (-1, -1, \dots)$ has the subsequential limit -1; $(x_{2k+1}) = (1, 1, 1, \dots)$ has the subsequential limit 1.

Proposition 2.7 *Subsequences of convergent sequences are convergent with the same limit.*

Proof. Let $\lim x_n = x$ and x_{n_k} be a subsequence. To $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $n \geq m_0$ implies $|\, x_n - x \,| < \varepsilon$. Since $n_m \geq m$ for all m, $m \geq m_0$ implies $|\, x_{n_m} - x \,| < \varepsilon$; hence $\lim x_{n_m} = x$. ■

Definition 2.5 Let (x_n) be a sequence. We call $x \in \mathbb{R}$ a *limit point* of (x_n) if every neighborhood of x contains infinitely many elements of (x_n).

Proposition 2.8 *The point x is limit point of the sequence (x_n) if and only if x is a subsequential limit.*

Proof. If $\lim\limits_{k\to\infty} x_{n_k} = x$ then every neighborhood $U_\varepsilon(x)$ contains all but finitely many x_{n_k}; in particular, it contains infinitely many elements x_n. That is, x is a limit point of (x_n). Suppose x is a limit point of (x_n). To $\varepsilon = 1$ there exists $x_{n_1} \in U_1(x)$. To $\varepsilon = 1/k$ there exists n_k with $x_{n_k} \in U_{1/k}(x)$ and $n_k > n_{k-1}$. We have constructed a subsequence (x_{n_k}) of (x_n) with

$$|\, x - x_{n_k} \,| < \frac{1}{k};$$

Hence, (x_{n_k}) converges to x. ■

Question: Which sequences do have limit points? The answer is: Every *bounded* sequence has limit points.

Proposition 2.9 (Principle of nested intervals) *Let $I_n := [a_n, b_n]$ a sequence of closed nested intervals $I_{n+1} \subseteq I_n$ such that their lengths $b_n - a_n$ tend to 0:*

Given $\varepsilon > 0$ there exists n_0 such that $0 \leq b_n - a_n < \varepsilon$ for all $n \geq n_0$.

For any such interval sequence $\{I_n\}$ there exists a unique real number $x \in \mathbb{R}$ which is a member of all intervals, i. e. $\{x\} = \bigcap_{n\in\mathbb{N}} I_n$.

Proof. Since the intervals are nested, $(a_n) \nearrow$ is an increasing sequence bounded above by each of the b_k, and $(b_n) \searrow$ is decreasing sequence bounded below by each of the a_k. Consequently, by Proposition 2.6 we have

$$\exists x = \lim_{n\to\infty} a_n = \sup\{a_n\} \leq b_m, \quad \text{for all } m, \text{ and } \quad \exists y = \lim_{m\to\infty} b_m = \inf\{b_m\} \geq x.$$

Since $a_n \leq x \leq y \leq b_n$ for all n,

$$\varnothing \neq [x, y] \subseteq \bigcap_{n\in\mathbb{N}} I_n.$$

Since $x = \sup a_n$ and $y = \inf b_n$, $[x, y] = \bigcap_{n\in\mathbb{N}} I_n$. Now we will use the second assumption. Given $\varepsilon > 0$ we find n such that $y - x \leq b_n - a_n \leq \varepsilon$. Hence $y - x \leq 0$; therefore $x = y$, and the intersection contains a unique point x. ■

Proposition 2.10 (Bolzano–Weierstraß) *A bounded real sequence has a limit point.*

Proof. We use the principle of nested intervals. Let (x_n) be bounded, say $|\,x_n\,| \leq C$. Hence, the interval $[-C, C]$ contains infinitely many x_k. Consider the intervals $[-C, 0]$ and $[0, C]$. At least one of them contains infinitely many x_k, say $I_1 := [a_1, b_1]$. Suppose, we have already constructed $I_n = [a_n, b_n]$ of length $b_n - a_n = C/2^{n-1}$ which contains infinitely many x_k. Consider the two intervals $[a_n, (a_n + b_n)/2]$ and $[(a_n + b_n)/2, b_n]$ of length $C/2^n$. At least one of them still contains infinitely many x_k, say $I_{n+1} := [a_{n+1}, b_{n+1}]$. In this way we have constructed a nested sequence of intervals which length go to 0. By Proposition 2.9, there exists a unique $x \in \bigcap_{n\in\mathbb{N}} I_n$. We will show that x is a subsequential limit of (x_n) (and hence a limit point). For, choose $x_{n_k} \in I_k$; this is possible since I_k contains infinitely many x_m. Then, $a_k \leq x_{n_k} \leq b_k$ for all $k \in \mathbb{N}$. Proposition 2.4 gives $x = \lim a_k \leq \lim x_{n_k} \leq \lim b_k = x$; hence $\lim x_{n_k} = x$. ∎

Remark 2.2 The principle of nested intevals is *equivalent* to the order completeness of $\mathbb{R}$. Using (2.9) we can prove that any subset of $\mathbb{R}$ which is bounded above has a least upper bound. The method is quite similar to that used in the proof of Proposition 2.10. Suppose $E \subset \mathbb{R}$ is bounded above by b_1 and non-empty, say $e \in E$. Set $a_1 = e - 1$; then a_1 is not an upper bound for E. Suppose a_n and b_n are already constructed, where a_n is not an upper bound of E whereas b_n is. Consider $m = (a_n + b_n)/2$; if m is an upper bound of E, set $b_{n+1} = m$, $a_{n+1} = a_n$, if not, set $a_{n+1} = m$ and $b_{n+1} = b_n$. Then a_{n+1} is still not an upper bound and b_{n+1} is still an upper bound of E. Obviously, the lengths of the intervals $[a_n, b_n]$ tend to 0. By Proposition 2.9 there exists a unique number x which belongs to all intervals. Using $x = \inf\{b_m \mid m \in \mathbb{N}\}$ it is not difficult to see that x is an upper bound of E. Using $x = \sup\{a_m \mid m \in \mathbb{N}\}$ one can see that $x - \varepsilon$ is not an upper bound for every $\varepsilon > 0$. Hence, $x = \sup E$.

Example 2.6 (a) $x_n = (-1)^{n-1} + 1/n$; set of limit points: $\{-1, 1\}$.
(b) $x_n = n - 5\left[\dfrac{n}{5}\right]$, where $[x]$ denotes the least integer less or equal to x ($[\pi] = [3] = 3$, $[-2.8] = -3$, $[1/2] = 0$).
$(x_n) = (1, 2, 3, 4, 0, 1, 2, 3, 4, 0, \dots)$; set of limit points: $\{0, 1, 2, 3, 4\}$
(c) One can enumerate the rational numbers in $(0, 1)$ in the following way.

$$\begin{array}{llllll} \frac{1}{2}, & & & x_1, & & \\ \frac{1}{3}, & \frac{2}{3}, & & x_2, & x_3 & \\ \frac{1}{4}, & \frac{2}{4}, & \frac{3}{4}, & x_4, & x_5, & x_6, \\ & \cdots & & & \cdots & \end{array}$$

The set of limit points is the whole interval $[0, 1]$.

(d) $x_n = n$ has no limit point. Since it is not bounded, Bolzano-Weierstraß fails to apply.

Let (x_n) be a bounded sequence, say $|\,x_n\,| \leq C$, with A the set of limit points of (x_n). Then, $A \subseteq [-C, C]$ is bounded. Put $\overline{x} := \sup A$ and $\underline{x} := \inf A$.

Definition 2.6 Let (x_n) be a bounded sequence and A its set of limit points. Then $\overline{x}$ is called the *upper limit* of (x_n) and $\underline{x}$ is called the *lower limit* of (x_n). We write

$$\overline{x} = \varlimsup_{n\to\infty} x_n, \qquad \underline{x} = \varliminf_{n\to\infty} x_n.$$

If (x_n) is not bounded above, we write $\varlimsup x_n = +\infty$. It was shown in Homework 5.5 that in this case (x_n) has a subsequence with improper limit $+\infty$. If moreover $+\infty$ is the only limit point, $\lim x_n = +\infty$, and we can also write $\varliminf x_n = +\infty$. If (x_n) is not bounded below, $\varliminf x_n = -\infty$. Then (x_n) has a subsequence diverging to $-\infty$. If $-\infty$ is the only limit point, $\lim x_n = -\infty$ and we can also write $\varlimsup x_n = +\infty$.

Proposition 2.11 *Let (x_n) be a bounded sequence and A the set of limit points of (x_n). Then $\overline{x} := \sup A$ and $\underline{x} := \inf A$ are also limit points of A.*

Proof. Let $\varepsilon > 0$. By the definition of sup there exists $x' \in A$ with $x' > \overline{x} - \varepsilon/2$. Since x' is a limit point, $U_{\varepsilon/2}(x')$ contains infinitely many elements x_k. By construction, $U_{\varepsilon/2}(x') \subseteq U_\varepsilon(\overline{x})$. Hence, $\overline{x}$ is a limit point, too. The proof for $\underline{x}$ is similar. ∎

Proposition 2.12 *Let $b \in \mathbb{R}$ be fixed. Suppose (x_n) is a sequence which is bounded above, then*

$$\begin{aligned} x_n \le b \quad &\textit{for all but finitely many } n \textit{ implies} \\ \varlimsup_{n\to\infty} x_n \le b.& \end{aligned} \tag{2.9}$$

Similarly, if (x_n) is bounded below, then

$$\begin{aligned} x_n \ge b \quad &\textit{for all but finitely many } n \textit{ implies} \\ \varliminf_{n\to\infty} x_n \ge b.& \end{aligned} \tag{2.10}$$

Proof. We prove only the first part. Proving statement for $\varliminf x_n$ is similar.
Suppose to the contrary that $t := \varlimsup x_n > b$. Set $\varepsilon = (t-b)/2$, then $U_\varepsilon(t)$ contains infinitely many x_n (t is a limit point) which are *all greater than* b; this contradicts $x_n \le b$ for all but finitely many n. Hence $\varlimsup x_n \le b$. ∎

Remarks 2.3 (a) Note, that for sup and inf we have

$$\begin{aligned} x_n \le b \quad &\text{for all } n \text{ implies} \\ \sup\{x_n\} \le b.& \end{aligned}$$

Similarly, if (x_n) is bounded below, then

$$\begin{aligned} x_n \ge b \quad &\text{for all } n \text{ implies} \\ \inf\{x_n\} \ge b.& \end{aligned}$$

This in particular implies $\overline{\lim}\, x_n \leq \sup\{x_n\}$ and $\underline{\lim}\, x_n \geq \inf\{x_n\}$.
Let (x_n) be bounded.
(b) $\underline{\lim}\, x_n$ and $\overline{\lim}\, x_n$ are the smallest and the greatest subsequential limits of (x_n) by Proposition 8.
(c) $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$ since the set A of limit points of (x_n) is non-empty (by Proposition 2.10) and therefore $\inf A \leq \sup A$.

The next proposition is a converse statement to Proposition 2.7.

Proposition 2.13 *Let (x_n) be a bounded sequence with a unique limit point x. Then (x_n) converges to x.*

Proof. Suppose to the contrary that (x_n) diverges; that is, there exists some $\varepsilon > 0$ such that infinitely many x_n are outside $U_\varepsilon(x)$. We view these elements as a subsequence $(y_k) := (x_{n_k})$ of (x_n). Since (x_n) is bounded, so is (y_k). By Proposition 2.10 there exists a limit point y of (y_k) which is in turn also a limit point of (x_n). Since $y \notin U_\varepsilon(x)$, $y \neq x$ is a second limit point; a contradiction! We conclude that (x_n) converges to x. ∎

Note that $\overline{\lim}\, x_n + \varepsilon$ is an upper bound for all but finitely many x_n and $\underline{\lim}\, x_n - \varepsilon$ is a lower bound for all but finitely many x_n, see homework 6.3. Let us consider the above examples.

Example 2.7 (a) $x_n = (-1)^{n-1} + 1/n$; $\underline{\lim}\, x_n = -1$, $\overline{\lim}\, x_n = 1$.
(b) $x_n = n - 5\left[\dfrac{n}{5}\right]$, $\underline{\lim}\, x_n = 0$, $\overline{\lim}\, x_n = 4$.
(c) (x_n) is the sequence of rational numbers of $(0, 1)$; $\underline{\lim}\, x_n = 0$, $\overline{\lim}\, x_n = 1$.
(d) $x_n = n$; $\underline{\lim}\, x_n = \overline{\lim}\, x_n = +\infty$.

Proposition 2.14 *If $s_n \leq t_n$ for all but finitely many n, then*

$$\overline{\lim_{n\to\infty}}\, s_n \leq \overline{\lim_{n\to\infty}}\, t_n, \qquad \underline{\lim_{n\to\infty}}\, s_n \leq \underline{\lim_{n\to\infty}}\, t_n.$$

Proof. We show the first inequality. The proof of the second is analogous. Suppose (t_n) is bounded above. Then (s_n) is also bounded above. Put $t^* = \overline{\lim}\, t_n$ and $s^* = \overline{\lim}\, s_n$. Since s^* is a limit point of (s_n), there is a subsequence (s_{n_k}) converging to s^*. Since $s_{n_k} \leq t_{n_k}$, for all but finitely many k, (t_{n_k}) is bounded below by $s^* - \varepsilon$ for any $\varepsilon > 0$ and bounded above since (t_n) is bounded above. Hence, by Proposition 2.10, (t_{n_k}) has a convergent subsequence, say $t_{n_{k_m}} \to t^{**}$. Then

$$s^* - \varepsilon \leq t^{**} \leq t^*.$$

Since ε was arbitrary, $s^* \leq t^*$.

Second proof. (a) We keep the notations s^* and t^* for the upper limits of (s_n) and (t_n), respectively. Set $\underline{s} = \underline{\lim}\, s_n$ and $\underline{t} = \underline{\lim}\, t_n$. Let $\varepsilon > 0$. By homework 6.3 (a)

$$\begin{aligned}
& \underline{s} - \varepsilon \leq s_n \quad \text{for all but finitely many } n \\
\underset{\text{by assumption}}{\Longrightarrow} \quad & \underline{s} - \varepsilon \leq s_n \leq t_n \quad \text{for all but finitely many } n \\
\underset{\text{by Prp. 2.12}}{\Longrightarrow} \quad & \underline{s} - \varepsilon \leq \underline{\lim}\, t_n \\
\underset{\text{by Remark 2.3 (a)}}{\Longrightarrow} \quad & \sup\{\underline{s} - \varepsilon \mid \varepsilon > 0\} \leq \underline{t} \\
& \underline{s} \leq \underline{t}.
\end{aligned}$$

(b) Now we proof the second part using homework 1.4 (a), $-\sup E = \inf(-E)$. By assumption $-t_n \leq -s_n$ for all but finitely many n. Part (a) of this proof gives

$$\begin{aligned}
& \underline{\lim}\,(-t_n) \leq \underline{\lim}\,(-s_n) \\
\underset{\text{hw 1.4 (a)}}{\Longrightarrow} \quad & -\overline{\lim}\, t_n \leq -\overline{\lim}\, s_n \\
& -t^* \leq -s^* \\
& s^* \leq t^*.
\end{aligned}$$

∎

2.2 Cauchy Sequences

The aim of this section is to characterize convergent sequences without knowing their limits.

Definition 2.7 A sequence (x_n) is said to be a *Cauchy sequence* if:

> For every $\varepsilon > 0$ there exists a positive integer n_0 such that $|\, x_n - x_m \,| < \varepsilon$ for all $m, n \geq n_0$.

The definition makes sense in arbitrary metric spaces. The definition is equivalent to

$$\forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0\, \forall k \in \mathbb{N} : \, |\, x_{n+k} - x_n \,| < \varepsilon.$$

Lemma 2.15 *Every convergent sequence is a Cauchy sequence.*

Proof. Let $x_n \to x$. To $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $x_n \in U_{\varepsilon/2}(x)$. By triangle inequality, $m, n \geq n_0$ implies

$$|\, x_n - x_m \,| \leq |\, x_n - x \,| + |\, x_m - x \,| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, (x_n) is a Cauchy sequence. ∎

Proposition 2.16 (Cauchy convergence criterion) *A real sequence is convergent if and only if it is a Cauchy sequence.*

Proof. One direction is Lemma 2.15. We prove the other direction. Let (x_n) be a Cauchy sequence. First we show that (x_n) is bounded. To $\varepsilon = 1$ there is a positive integer n_0 such that $m, n \geq n_0$ implies $|\, x_m - x_n \,| < 1$. In particular $|\, x_n - x_{n_0} \,| < 1$ for all $n \geq n_0$; hence $|\, x_n \,| < 1 + |\, x_{n_0} \,|$. Setting

$$C = \max\{|\, x_1 \,|, |\, x_2 \,|, \dots, |\, x_{n_0-1} \,|, |\, x_{n_0} \,| + 1\},$$

$|\, x_n \,| < C$ for all n.

By Proposition 2.10 there exists a limit point x of (x_n); and by Proposition 2.8 a subsequence (x_{n_k}) converging to x. We will show that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Since $x_{n_k} \to x$ we find $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies $|\, x_{n_k} - x \,| < \varepsilon/2$. Since (x_n) is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $m, n \geq n_0$ implies $|\, x_n - x_m \,| < \varepsilon/2$. Put $n_1 := \max\{n_0, n_{k_0}\}$ and choose k_1 with $n_{k_1} \geq n_1 \geq n_{k_0}$. Then $n \geq n_1$ implies

$$|\, x - x_n \,| \leq \left|\, x - x_{n_{k_1}} \right| + \left|\, x_{n_{k_1}} - x_n \right| < 2 \cdot \varepsilon/2 = \varepsilon.$$

∎

Example 2.8 (a) $x_n = \sum_{k=1}^{n} 1/k = 1 + 1/2 + 1/3 + \dots + 1/n$. We show that (x_n) is not a Cauchy sequence. For, consider

$$x_{2^{m+1}} - x_{2^m} = \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k} \geq \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{2^{m+1}} = \frac{2^m}{2^{m+1}} = \frac{1}{2}.$$

Hence, there is no n_0 such that $p, n \geq n_0$ implies $|\, x_p - x_n \,| < \frac{1}{2}$.

(b) $x_n = \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = 1 - 1/2 + 1/3 - + \dots + (-1)^{n+1} 1/n$. Consider

$$\begin{aligned} x_{n+k} - x_n &= (-1)^n \left[\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \dots + (-1)^{k+1} \frac{1}{n+k} \right] \\ &= (-1)^n \left[\left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \left(\frac{1}{n+3} - \frac{1}{n+4} \right) + \cdots \right. \\ &\qquad\qquad + \begin{cases} \left(\frac{1}{n+k-1} - \frac{1}{n+k} \right), & k \text{ even} \\ \frac{1}{n+k}, & k \text{ odd} \end{cases} \end{aligned}$$

Since all summands in parentheses are positive, we conclude

$$\begin{aligned} |\, x_{n+k} - x_n \,| &= \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \left(\frac{1}{n+3} - \frac{1}{n+4} \right) + \dots + \begin{cases} \left(\frac{1}{n+k-1} - \frac{1}{n+k} \right), & k \text{ even} \\ \frac{1}{n+k}, & k \text{ odd} \end{cases} \\ &= \frac{1}{n+1} - \left[\left(\frac{1}{n+2} - \frac{1}{n+3} \right) + \dots + \begin{cases} \frac{1}{n+k}, & k \text{ even} \\ \left(\frac{1}{n+k-1} - \frac{1}{n+k} \right), & k \text{ even} \end{cases} \right. \end{aligned}$$

$$|\, x_{n+k} - x_n \,| < \frac{1}{n+1},$$

since all summands in parentheses are positive. Hence, (x_n) is a Cauchy sequence and converges.

2.3 Series

Definition 2.8 Given a sequence (a_n), we associate with (a_n) a sequence (s_n), where

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

For (s_n) we also use the symbol

$$\sum_{k=1}^{\infty} a_k, \tag{2.11}$$

and we call it an *infinite series* or just a *series*. The numbers s_n are called the *partial sums* of the series. If (s_n) converges to s, we say that the series *converges*, and write

$$\sum_{k=1}^{\infty} a_k = s.$$

The number s is called the *sum* of the series.

Remarks 2.4 (a) The sum of a series should be clearly understood as *the limit of the sequence of partial sums*; it is not simply obtained by addition.
(b) If (s_n) diverges, the series is said to be *divergent*.
(c) The symbol $\sum_{k=1}^{\infty} a_k$ means both, the sequence of partial sums as well as the limit of this sequence (if it exists). Sometimes we use series of the form $\sum_{k=k_0}^{\infty} a_k$, $k_0 \in \mathbb{N}$. We simply write $\sum a_k$ if there is no ambiguity.

Example 2.9 (1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. This is the *harmonic series*.
(2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent. It is an example of an *alternating series* (the summands are changing their signs, and the absolute value of the summands form a decreasing to 0 sequence).
(3) $\displaystyle\sum_{n=0}^{\infty} q^n$ is called the *geometric series*. It is convergent for $|\,q\,| < 1$ with $\sum_0^{\infty} q^n = \frac{1}{1-q}$. This is easily seen from $\sum_{k=0}^{n} q^n = (1 - q^{n+1})/(1-q)$. The series is divergent for $|\,q\,| \geq 1$. The general formula in case $|\,q\,| < 1$ is

$$\sum_{n=n_0}^{\infty} cq^n = \frac{cq^{n_0}}{1-q}. \tag{2.12}$$

2.3.1 Properties of Convergent Series

Lemma 2.17 (1) *If* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\sum_{k=m}^{\infty} a_k$ *is convergent for any* $m \in \mathbb{N}$.
(2) *If* $\sum a_n$ *is convergent, then the sequence* $r_n := \sum_{k=n}^{\infty} a_k$ *tends to* 0.
(3) *If* (a_n) *is a sequence of nonnegative real numbers, then* $\sum a_n$ *converges if and only if the partial sums are bounded.*

Proof. (1) is obvious. We prove (2). Suppose $\sum_1^{\infty} a_n$ is convergent. By (1) $r_n = \sum_{k=n+1}^{\infty} a_k$ is also a convergent series for all n. We have

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k\\ &\Longrightarrow s = s_n + r_n\\ &\Longrightarrow r_n = s - s_n\\ &\Longrightarrow \lim_{n\to\infty} r_n = s - s = 0.\end{aligned}$$

(3) Suppose $a_n \geq 0$, then $s_{n+1} = s_n + a_{n+1} \geq s_n$. Hence, (s_n) is an increasing sequence. By Proposition 2.6, (s_n) converges.
The other direction is trivial since every convergent sequence is bounded. ■

Proposition 2.18 (Cauchy criterion) $\sum a_n$ *converges if and only if for every* $\varepsilon > 0$ *there is an integer* $n_0 \in \mathbb{N}$ *such that*

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon \tag{2.13}$$

if $m, n \geq n_0$.

Proof. Clear from Proposition 2.16. ■

Corollary 2.19 *If* $\sum a_n$ *converges, then* (a_n) *converges to* 0.

Proof. Take $m = n$ in (2.13); this becomes $|\, a_n \,| < \varepsilon$. Hence (a_n) tends to 0. ■

Proposition 2.20 (Comparison test) (a) *If* $|\, a_n \,| \leq C b_n$ *for some* $C > 0$ *and for almost all* $n \in \mathbb{N}$, *and if* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
(b) *If* $a_n \geq C d_n \geq 0$ *for some* $C > 0$ *and for almost all* n, *and if* $\sum d_n$ *diverges, then* $\sum a_n$ *diverges.*

Proof. (a) Suppose $n \geq n_1$ implies $|\, a_n \,| \leq C b_n$. Given $\varepsilon > 0$, there exists $n_0 \geq n_1$ such that $m, n \geq n_0$ implies

$$\sum_{k=m}^{n} b_k < \varepsilon / C$$

by the Cauchy criterion. Hence

$$\left| \sum_{k=m}^{n} a_k \right| \le \sum_{k=m}^{n} |\, a_k \,| \le \sum_{k=m}^{n} C b_k < \varepsilon,$$

and (a) follows by the Cauchy criterion.
(b) follows from (a), for if $\sum a_n$ converges, so must $\sum d_n$. ∎

2.3.2 Operations with Convergent Series

Definition 2.9 If $\sum a_n$ and $\sum b_n$ are series, we define sums and differences as follows $\sum a_n \pm \sum b_n := \sum (a_n \pm b_n)$ and $c \sum a_n := \sum c a_n$, $c \in \mathbb{R}$.
Let $c_n := \sum_{k=1}^{n} a_k b_{n-k+1}$, then $\sum c_n$ is called the *Cauchy product* of $\sum a_n$ and $\sum b_n$.

If $\sum a_n$ and $\sum b_n$ are convergent, it is easy to see that $\sum_1^\infty (a_n + b_n) = \sum_1^\infty a_n + \sum_1^\infty b_n$ and $\sum_1^n c a_n = c \sum_1^n a_n$.
Caution, the product series $\sum c_n$ need not to be convergent.
Q 17. Let $a_n := b_n := (-1)^n / \sqrt{n}$. Show that $\sum a_n$ and $\sum b_n$ are convergent but $\sum c_n$ is not convergent, when $c_n = \sum_{k=1}^{n} a_k b_{n-k+1}$.

2.3.3 Series of Nonnegative Numbers

Proposition 2.21 (Compression Theorem) *Suppose $a_1 \ge a_2 \ge \cdots \ge 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots \tag{2.14}$$

converges.

Proof. By Lemma 2.17 (3) it suffices to consider boundedness of the partial sums. Let

$$\begin{aligned} s_n &= a_1 + \cdots + a_n, \\ t_k &= a_1 + 2a_2 + \cdots + 2^k a_{2^k}. \end{aligned}$$

For $n < 2^k$

$$\begin{aligned} s_n &\le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ s_n &\le a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t_k. \end{aligned} \tag{2.15}$$

On the other hand, if $n > 2^k$,

$$\begin{aligned} s_n &\ge a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ s_n &\ge \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\ s_n &\ge \frac{1}{2} t_k. \end{aligned} \tag{2.16}$$

By (2.15) and (2.16), the sequences s_n and t_k are either both bounded or both unbounded. This completes the proof. ∎

Example 2.10 (a) $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
If $p \leq 0$, divergence follows from Corollary 2.19. If $p > 0$ Proposition 2.21 is applicable, and we are led to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

Now, $2^{1-p} < 1$ if and only if $p > 1$, and the result follows by comparison with the geometric series.
(b) If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \tag{2.17}$$

converges; if $p \leq 1$, the series diverges. "$\log n$" denotes the logarithm to the base e.
If $p < 0$, $\frac{1}{n(\log n)^p} > \frac{1}{n}$ and divergence follows by comparison with the harmonic series. Now let $p > 0$. By Lemma 1.37 (b), $\log n < \log(n+1)$. Hence $(n(\log n)^p)$ increases and $1/(n(\log n))^p$ decreases; we can apply Proposition 2.21 to (2.17). This leads us to the series

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k(\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p},$$

and the assertion follows from example (a).
This procedure can evidently be continued. For instance $\sum_{n=3}^{\infty} 1/(n \log n \log\log n)$ diverges, whereas $\sum_{n=3}^{\infty} 1/(n \log n (\log\log n)^2)$ converges.

2.3.4 The Number e

We define

$$\mathrm{e} := \sum_{n=0}^{\infty} \frac{1}{n!}, \tag{2.18}$$

where $0! = 1! = 1$ by definition. Since

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdots n} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3, \end{aligned}$$

the series converges and the definition makes sense. In fact, the series converges very rapidly and allows us to compute e with great accuracy. It is of interest to note that e can also defined by means of another limit process.

Proposition 2.22

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \tag{2.19}$$

Proof. Let

$$s_n = \sum_{k=0}^{n}\frac{1}{k!}, \quad t_n = \left(1+\frac{1}{n}\right)^n.$$

By the binomial theorem,

$$\begin{aligned} t_n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots 1}{n!}\cdot\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \\ &\qquad + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \end{aligned}$$

Hence, $t_n \le s_n$, so that

$$\varlimsup_{n\to\infty} t_n \le \mathrm{e}, \tag{2.20}$$

by Proposition 2.14. Next if $n \ge m$,

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right).$$

Let $n\to\infty$, keeping m fixed. We get

$$\varliminf_{n\to\infty} t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = s_m.$$

Letting $m\to\infty$, we finally get

$$\mathrm{e} \le \varliminf_{n\to\infty} t_n. \tag{2.21}$$

The proposition follows from (2.20) and (2.21). ∎

The rapidity with which the series $\sum 1/n!$ converges can be estimated as follows.

$$\begin{aligned} \mathrm{e} - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots \\ &< \frac{1}{(n+1)!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right] = \frac{1}{(n+1)!}\cdot\frac{1}{1-\frac{1}{n+1}} = \frac{1}{n!\,n} \end{aligned}$$

so that

$$0 < \mathrm{e} - s_n < \frac{1}{n!\,n}. \tag{2.22}$$

We use the preceding inequality to compute e. For $n = 9$ we find

$$s_9 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40,320} + \frac{1}{362,880} = 2.718281526... \tag{2.23}$$

By (2.22)

$$\mathrm{e} - s_9 < \frac{3.1}{10^7}$$

such that the first six digits of e in (2.23) are correct.

Proposition 2.23 e *is irrational.*

Proof. Suppose e is rational, say $\mathrm{e} = p/q$ with positive integers p and q. By (2.22)

$$0 < q!(\mathrm{e} - s_q) < \frac{1}{q}. \tag{2.24}$$

By our assumption, $q!\mathrm{e}$ is an integer. Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$$

is also an integer, we see that $q!(\mathrm{e} - s_q)$ is an integer. Since $q \geq 1$, (2.24) implies the existence of an integer between 0 and 1 which is absurd. ■

2.3.5 The Root and the Ratio Tests

Theorem 2.24 (Root Test) *Given* $\sum a_n$, *put* $\alpha = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}$.
Then

(a) *if* $\alpha < 1$, $\sum a_n$ *converges;*
(b) *if* $\alpha > 1$, $\sum a_n$ *diverges;*
(c) *if* $\alpha = 1$, *the test gives no information.*

In other words: (a) $\sum a_n$ converges, if there exists $q < 1$ such that $\sqrt[n]{|a_n|} \leq q$ for almost all n. (b) $\sum a_n$ diverges if $\sqrt[n]{|a_n|} \geq 1$ for infinitely many n.
Proof. If $\alpha < 1$ choose β such that $\alpha < \beta < 1$, and an integer n_0 such that

$$\sqrt[n]{|a_n|} < \beta$$

for $n \geq n_0$ (such n_0 exists since α is the supremum of the limit set of $(\sqrt[n]{|a_n|})$). That is, $n \geq n_0$ implies

$$|a_n| < \beta^n.$$

Since $0 < \beta < 1$, $\sum \beta^n$ converges. Convergence of $\sum a_n$ now follows from the comparison test.
If $\alpha > 1$ there is a subsequence (a_{n_k}) such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Hence $|a_n| > 1$ for

infinitely many n, so that the necessary condition for convergence, $a_n \to 0$, fails.
To prove (c) consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. For each of the series $\alpha = 1$, but the first diverges, the second converges. ∎

Theorem 2.25 (Ratio Test) *The series* $\sum a_n$

(a) *converges if* $\overline{\lim\limits_{n\to\infty}} \left| \frac{a_{n+1}}{a_n} \right| < 1$,

(b) *diverges if* $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ *for all but finitely many* n.

In other words, $\sum a_n$ converges if there is a $q < 1$ such that $\left| \frac{a_{n+1}}{a_n} \right| \leq q$ for almost all n.

Corollary 2.26 *The series* $\sum a_n$

(a) *converges if* $\lim \left| \frac{a_{n+1}}{a_n} \right| < 1$,

(b) *diverges if* $\lim \left| \frac{a_{n+1}}{a_n} \right| > 1$.

Proof of the corollary. If (a_{n+1}/a_n) converges, the limit coincides with the upper limit and (a) follows from the theorem. If (a_{n+1}/a_n) converges to some $a > 1$, then $U_\varepsilon(a)$, $\varepsilon = (a-1)/2$, contains almost all element of the quotient sequence. In particular, almost all quotients are greater than 1 and (b) follows from the theorem. ∎

Proof of Theorem 2.25. If condition (a) holds, we can find $\beta < 1$ and an integer m such that $n \geq m$ implies

$$\left| \frac{a_{n+1}}{a_n} \right| < \beta.$$

In particular,

$$\begin{aligned} |\, a_{m+1} \,| &< \beta \,|\, a_m \,|, \\ |\, a_{m+2} \,| &< \beta \,|\, a_{m+1} \,| < \beta^2 \,|\, a_m \,|, \\ &\dots \\ |\, a_{m+p} \,| &< \beta^p \,|\, a_m \,|. \end{aligned}$$

That is,

$$|\, a_n \,| < |\, a_m \,| \, \beta^{-m} \cdot \beta^n$$

for $n \geq m$, and (a) follows from the comparison test, since $\sum \beta^n$ converges.
If $|\, a_{n+1} \,| \geq |\, a_n \,|$ for $n \geq n_0$, it is easily seen that the condition $a_n \to 0$ does not hold, and (b) follows. ∎

Remark 2.5 Homework 7.5 shows that in (b) "all but finitely many" cannot be replaced by the weaker assumption "infinitely many."

Example 2.11 (a) The series $\sum_{n=0}^{\infty} n^2/2^n$ converges since, if $n \geq 3$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2 2^n}{2^{n+1} n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right)^2 \leq \frac{1}{2}\left(1+\frac{1}{3}\right)^2 = \frac{8}{9} < 1.$$

(b) Consider the series

$$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots,$$

where $\underline{\lim} \dfrac{a_{n+1}}{a_n} = \dfrac{1}{8}$, $\overline{\lim} \dfrac{a_{n+1}}{a_n} = 2$, but $\lim \sqrt[n]{a_n} = \frac{1}{2}$. The root test indicates convergence; the ratio test does not apply.

Q 18. Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots.$$

Compute $\underline{\lim} \dfrac{a_{n+1}}{a_n}$, $\underline{\lim} \sqrt[n]{a_n}$, $\overline{\lim} \sqrt[n]{a_n}$, and $\overline{\lim} \dfrac{a_{n+1}}{a_n}$. Apply the ratio and the root tests. The ratio test is frequently easier to apply than the root test. However, the root test has wider scope.

Remark 2.6 For any sequence (c_n) of positive real numbers,

$$\underline{\lim}_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \underline{\lim}_{n\to\infty} \sqrt[n]{c_n} \leq \overline{\lim}_{n\to\infty} \sqrt[n]{c_n} \leq \overline{\lim}_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

For the proof, see [7, 3.37 Theorem]. In particular, if $\lim \dfrac{c_{n+1}}{c_n}$ exists, then $\lim \sqrt[n]{c_n}$ also exists and both limits coincide.

Proposition 2.27 (Leibniz criterion) *Let $\sum b_n$ be an alternating sum, i. e. $\sum b_n = \sum (-1)^{n+1} a_n$ with a decreasing sequence of positive numbers $a_1 \geq a_2 \geq \cdots \geq 0$. If $\lim a_n = 0$ then $\sum b_n$ converges.*

Proof. The proof is quite the same as in Example 2.8 (b). We find for the partial sums s_n of $\sum b_n$

$$|\, s_n - s_m \,| \leq a_{m+1}$$

if $n \geq m$. Since (a_n) tends to 0, the Cauchy criterion applies to (s_n). Hence, $\sum b_n$ is convergent. ∎

2.3.6 Absolute Convergence

The series $\sum a_n$ is said to *converge absolutely* if the series $\sum |a_n|$ converges.

Proposition 2.28 *If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.*

Proof. The assertion follows from the inequality

$$\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k|$$

plus the Cauchy criterion. ∎

Remarks 2.7 For series with positive terms, absolute convergence is the same as convergence. If $\sum a_n$ converges but $\sum |a_n|$ diverges, we say that $\sum a_n$ *converges nonabsolutely.* For instance $\sum(-1)^{n+1}/n$ converges nonabsolutely. The comparison test as well as the root and the ratio tests, is really a test for absolute convergence and cannot give any information about nonabsolutely convergent series.
We shall see that we may operate with absolutely convergent series very much as with finite sums. We may multiply them, we may change the order in which the additions are carried out without effecting the sum of the series. But for nonabsolutely convergent sequences this is no longer true and more care has to be taken when dealing with them.

Proposition 2.29 *If $\sum a_n$ converges absolutely with $\sum_{n=0}^{\infty} a_n = A$, $\sum b_n$ converges, $\sum_{n=0}^{\infty} b_n = B$, $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n \in \mathbb{Z}_+$.*
Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

Proof. Put

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n := \sum_{k=0}^{n} c_k, \quad \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0. \end{aligned}$$

Put

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0.$$

We wish to show that $C_n \to AB$. Since $A_nB \to AB$, it suffices to show that $\gamma_n \to 0$. Since $\sum a_n$ converges absolutely,

$$\alpha := \sum_{n=0}^{\infty} |\, a_n \,|$$

exists. Let $\varepsilon > 0$ be given. Since $\sum b_n = B$, $\beta_n \to 0$. Hence, we can choose n_0 such that $|\, \beta_n \,| \le \varepsilon$ for $n \ge n_0$, in which case

$$\begin{aligned} |\, \gamma_n \,| &\le |\, \beta_0 a_n + \cdots + \beta_{n_0} a_{n-n_0} \,| + |\, \beta_{n_0+1} a_{n-n_0+1} + \cdots + \beta_n a_0 \,| \\ &\le |\, \beta_0 a_n + \cdots + \beta_{n_0} a_{n-n_0} \,| + \varepsilon\alpha. \end{aligned}$$

Keeping n_0 fixed, and letting $n \to \infty$. By Proposition 2.14 and by $\overline{\lim}\, a_{n-k} = 0$ for fixed $k = 0, \dots, n_0$, we get

$$\overline{\lim_{n\to\infty}}\ |\, \gamma_n \,| \le \alpha\varepsilon.$$

Since ε is arbitrary, $\overline{\lim}\ |\, \gamma_n \,| = 0$, and $\lim \gamma_n = 0$ follows. ■

2.3.7 Decimal Expansion of Real Numbers

Proposition 2.30 (a) *Let α be a real number with $0 \le \alpha < 1$. Then there exists a sequence (a_n), $a_n \in \{0, 1, 2, \dots, 9\}$ such that*

$$\alpha = \sum_{n=1}^{\infty} a_n\, 10^{-n}. \tag{2.25}$$

The sequence (a_n) is called a decimal expansion *of α.*
(b) Given a sequence (a_k), $a_k \in \{0, 1, \dots, 9\}$, then there exists a real number $\alpha \in [0, 1]$ such that

$$\alpha = \sum_{n=1}^{\infty} a_n\, 10^{-n}.$$

Proof. (b) Comparison with the geometric series yields

$$\sum_{n=1}^{\infty} a_n 10^{-n} \le 9 \sum_{n=1}^{\infty} 10^{-n} = \frac{9}{10} \cdot \frac{1}{1 - 1/10} = 1.$$

Hence the series $\sum_{n=1}^{\infty} a_n 10^{-n}$ converges to some $\alpha \in [0, 1]$.
(a) Given $\alpha \in [0, 1)$ we use induction to construct a sequence (a_n) with (2.25) and

$$s_n \le \alpha < s_n + 10^{-n}, \quad \text{where} \quad s_n = \sum_{k=1}^{n} a_k\, 10^{-k}.$$

First, cut $[0, 1]$ into 10 pieces $I_j := [j/10, (j+1)/10)$, $j = 0, \dots, 9$, of equal length. If $\alpha \in I_j$, put $a_1 := j$. Then,

$$s_1 = \frac{a_1}{10} \le \alpha < s_1 + \frac{1}{10}.$$

Suppose $a_1, \dots, a_n$ are already constructed and

$$s_n \leq \alpha < s_n + 10^{-n}.$$

Consider the intervals $I_j := [s_n + j/10^{n+1}, s_n + (j+1)/10^{n+1})$, $j = 0, \dots, 9$. There is exactly one j such that $\alpha \in I_j$. Put $a_{n+1} := j$, then

$$s_n + \frac{a_{n+1}}{10^{n+1}} \leq \alpha < s_n + \frac{a_{n+1}+1}{10^{n+1}}$$
$$s_{n+1} \leq \alpha < s_{n+1} + 10^{-n-1}.$$

The induction step is complete. By construction $|\,\alpha - s_n\,| < 10^{-n}$, that is, $\lim s_n = \alpha$. ∎

Remarks 2.8 (a) The proof shows that any real number $\alpha \in [0, 1)$ can be approximated by rational numbers.
(b) The construction avoids decimal expansion of the form $\alpha = \dots a9999\dots$, $a < 9$, and gives instead $\alpha = \dots (a + 1)000\dots$. It gives a bijective correspondence between the real numbers of the interval $[0, 1)$ and the sequences (a_n), $a_n \in \{0, 1, \dots, 9\}$, not ending with nines.
(c) It is not difficult to see that $\alpha \in [0, 1)$ is rational if and only if there exist positive integers n_0 and p such that $n \geq n_0$ implies $a_n = a_{n+p}$—the decimal expansion is *periodic* from n_0 on.

2.3.8 Complex Sequences and Series

Almost all notions and theorems carry over from real sequences to complex sequences. For example

> A sequence (z_n) of complex numbers *converges to* z if for every (real) $\varepsilon > 0$ there exists a positive integer $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies
>
> $$|\,z - z_n\,| < \varepsilon.$$

The following proposition shows that convergence a complex sequence can be reduced to the convergence of two real sequences.

Proposition 2.31 *The complex sequence (z_n) converges to some complex number z if and only if the real sequences $(\operatorname{Re} z_n)$ converges to $\operatorname{Re} z$ and the real sequence $(\operatorname{Im} z_n)$ converges to $\operatorname{Im} z$.*

Proof. Using the (complex) limit law $\lim(z_n + c) = c + \lim z_n$ it is easy to see that we can restrict ourselves to the case $z = 0$. Suppose first $z_n \to 0$. Proposition 1.24 (d) gives $|\operatorname{Re} z_n| \leq |\,z_n\,|$. Hence $\operatorname{Re} z_n$ tends to 0 as $n \to \infty$. Similarly, $|\operatorname{Im} z_n| \leq |\,z_n\,|$ and therefore $\operatorname{Im} z_n \to 0$.

Suppose now $x_n := \operatorname{Re} z_n \to 0$ and $y_n := \operatorname{Im} z_n \to 0$ as n goes to infinity. Since $|z_n|^2 = x_n^2 + y_n^2$, $|z_n|^2 \to 0$ as $n \to \infty$; this implies $z_n \to 0$. ∎

Since the complex field $\mathbb{C}$ is not an ordered field, all notions and propositions where the *order* is involved do not make sense for complex series or they need modifications. The sandwich theorem does not hold; there is no notion of monotonic sequences, upper and lower limits. But still there are bounded sequences ($|z_n| \le C$), limit points, subsequences, Cauchy sequences, series, and absolute convergence. The following theorems are true for complex sequences, too:

Proposition 1, 2, 3, 7, 8, 10, 13, 15, 16.

The Bolzano–Weierstraß Theorem for bounded complex sequences (z_n) can be proved by considering the real and the imaginary sequences $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ separately.
The comparison test for series now reads:

(a) If $|a_n| \le C|b_n|$ for some $C > 0$ and for almost all $n \in \mathbb{N}$, and if $\sum |b_n|$ converges, then $\sum a_n$ converges.
(b) If $|a_n| \ge C|d_n|$ for some $C > 0$ and for almost all n, and if $\sum |d_n|$ diverges, then $\sum a_n$ diverges.

The Cauchy criterion, the root, and the ratio tests are true for complex series as well. Proposition 28, 29 are true for complex series.

2.3.9 Power Series

Definition 2.10 Given a sequence (c_n) of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n \tag{2.26}$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

In general, the series will converge or diverge, depending on the choice of z. More specifically, with every power series there is associated a circle with center 0, the circle of convergence, such that (2.26) converges if z is in the interior of the circle and diverges if z is in the exterior.

Theorem 2.32 *Given a power series $\sum c_n z^n$, put*

$$\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}. \tag{2.27}$$

If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$. Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

The behavior on the circle of convergence cannot be described so simple.

Proof. Put $a_n = c_n z^n$ and apply the root test:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\,a_n\,|} = |\,z\,| \overline{\lim_{n\to\infty}} \sqrt[n]{|\,c_n\,|} = \frac{|\,z\,|}{R}.$$

This gives convergence if $|\,z\,| < R$ and divergence if $|\,z\,| > R$. ■

The nonnegative number R is called the *radius of convergence*.

Example 2.12 (a) The series $\sum n^n z^n$ has $R = 0$.

(b) The series $\sum \frac{z^n}{n!}$ has $R = +\infty$. (In this case the ratio test is easier to apply than the root test. Indeed,

$$\alpha = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

and therefore $R = +\infty$.)

(c) The series $\sum z^n$ has $R = 1$. If $|\,z\,| = 1$ diverges since (z^n) does not tend to 0. This generalizes the geometric series; formula (2.12) still holds if $|\,q\,| < 1$:

$$\sum_{n=2}^{\infty} 2\left(\frac{\mathrm{i}}{3}\right)^n = \frac{2(\mathrm{i}/3)^2}{1-\mathrm{i}/3} = -\frac{3+\mathrm{i}}{15}.$$

(d) The series $\sum z^n/n$ has $R = 1$. It diverges if $z = 1$. It converges for all other z with $|\,z\,| = 1$ (without proof).

(e) The series $\sum z^n/n^2$ has $R = 1$. It converges for all z with $|\,z\,| = 1$ by the comparison test, since $|\,z^n/n^2\,| = 1/n^2$.

2.3.10 Rearrangements

Definition 2.11 Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be a bijective mapping, that is in the sequence $(\sigma(1), \sigma(2), \dots)$ every positive integer appears once and only once. Putting

$$a'_n = a_{\sigma(n)}, \quad (n = 1, 2, \dots),$$

we say that $\sum a'_n$ is a *rearrangement* of $\sum a_n$.

If (s_n) and (s'_n) are the partial sums of $\sum a_n$ and a rearrangement $\sum a'_n$ of $\sum a_n$, it is easily seen that, in general, these two sequences consist of entirely different numbers. We are led to the problem of determining under what conditions all rearrangements of a convergent series will converge and whether the sums are necessarily the same.

Example 2.13 (a) Consider the convergent series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - + \cdots \tag{2.28}$$

and one of its rearrangements

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots. \tag{2.29}$$

If s is the sum of (2.28) then $s>0$ since

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots>0.$$

We will show that (2.29) converges to $s'=s/2$. Namely

$$\begin{aligned}
s'=\sum a_n' &= \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots\\
&=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots\\
&=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\cdots\right)=\frac{1}{2}s
\end{aligned}$$

Since $s\neq 0$, $s'\neq s$. Hence, there exist rearrangements which converge; however to a different limit.
(b) Consider the following rearrangement of the series (2.28)

$$\begin{aligned}
\sum a_n' = 1 &-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\\
&+\left(\frac{1}{5}+\frac{1}{7}\right)-\frac{1}{6}+\\
&+\left(\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}\right)-\frac{1}{8}\\
&+\cdots\\
&+\left(\frac{1}{2^n+1}+\frac{1}{2^n+3}+\cdots+\frac{1}{2^{n+1}-1}\right)-\frac{1}{2n+2}+\cdots
\end{aligned}$$

Since for every positive integer $n\geq 10$

$$\left(\frac{1}{2^n+1}+\frac{1}{2^n+3}+\cdots+\frac{1}{2^{n+1}-1}\right)-\frac{1}{2n+2}>2^{n-1}\cdot\frac{1}{2^{n+1}}-\frac{1}{2n+2}>\frac{1}{4}-\frac{1}{2n+2}>\frac{1}{5}$$

the rearranged series diverges to $+\infty$.

Without proof (see [7, 3.54 Theorem]) we remark the following surprising theorem. It shows (together with the Proposition 2.34) that the absolute convergence of a series is necessary and sufficient for every rearrangement to be convergent (to the same limit).

Proposition 2.33 *Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose $-\infty\leq\alpha\leq\beta\leq+\infty$. Then there exists a rearrangement $\sum a_n'$ with partial sums s_n' such that*

$$\varliminf_{n\to\infty} s_n'=\alpha,\quad \varlimsup_{n\to\infty} s_n'=\beta.$$

Proposition 2.34 *If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.*

Proof. Let $\sum a'_n$ be a rearrangement with partial sums s'_n. Given $\varepsilon > 0$, by the Cauchy criterion for the series $\sum |a_n|$ there exists $n_0 \in \mathbb{N}$ such that $n \geq m \geq n_0$ implies

$$\sum_{k=m}^{n} |a_k| < \varepsilon. \tag{2.30}$$

Now choose p such that the integers $1, 2, \ldots, n_0$ are all contained in the set $\sigma(1), \sigma(2), \ldots, \sigma(p)$.

$$\{1, 2, \ldots, n_0\} \subseteq \{\sigma(1), \sigma(2), \ldots, \sigma(p)\}.$$

Then, if $n \geq p$, the numbers $a_1, a_2, \ldots, a_{n_0}$ will cancel in the difference $s_n - s'_n$, so that

$$|s_n - s'_n| = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_{\sigma(k)} \right| \leq \left| \sum_{k=n_0+1}^{n} \pm a_k \right| \leq \sum_{k=n_0+1}^{n} |a_k| < \varepsilon,$$

by (2.30). Hence (s'_n) converges to the same sum as (s_n).
The same argument shows that $\sum a'_n$ also absolutely converges. ∎