Chapter 11

The Hilbert Space

Functional analysis is a fruitful interplay between linear algebra and analysis. One defines function spaces with certain properties and certain topologies and considers linear operators between such spaces. The friendliest example of such spaces are Hilbert spaces. This chapter is devided into two parts—one describes the geometry of a Hilbert space, the second is concerned with linear operators on the Hilbert space.

11.1 The Geometry of the Hilbert Space

11.1.1 Unitary Spaces

Let E be a linear space over $\mathbb{K} = \mathbb{R}$ or over $\mathbb{K} = \mathbb{C}$.

Definition 11.1 An *inner product* on E is a function $\langle \cdot , \cdot \rangle : E \times E \to \mathbb{K}$ with

(a) $\langle \lambda_1 x_1 + \lambda_2 x_2, y \rangle = \lambda_1 \langle x_1, y \rangle + \lambda_2 \langle x_2, y \rangle$ (Linearity)
(b) $\langle x, y \rangle = \overline{\langle y, x \rangle}$. (Hermitian property)
(c) $\langle x, x \rangle \geq 0$ for all $x \in E$, and $\langle x, x \rangle = 0$ implies $x = 0$ (Positve definiteness)

A *unitary space* is a linear space together with an inner product.

Let us list some immediate consequences from these axioms: From (a) and (b) it follows that

$$\text{(d)} \quad \langle y, \lambda_1 x_1 + \lambda_2 x_2 \rangle = \overline{\lambda_1} \langle y, x_1 \rangle + \overline{\lambda_2} \langle y, x_2 \rangle .$$

A form on $E \times E$ satisfying (a) and (d) is called a *sesquilinear form*. (a) implies $\langle 0, y \rangle = 0$ for all $y \in E$. The mapping $x \mapsto \langle x, y \rangle$ is a linear mapping into $\mathbb{K}$ (a linear functional) for all $y \in E$.
By (c), we may define $\|x\|$, the *norm* of the vector $x \in E$ to be the square root of $\langle x, x \rangle$; thus

$$\|x\|^2 = \langle x, x \rangle . \tag{11.1}$$

Proposition 11.1 (Cauchy–Schwarz Inequality) *Let $(E, \langle\cdot,\cdot\rangle)$ be a unitary space. For $x, y \in E$ we have*

$$|\langle x,y\rangle| \le \|x\| \, \|y\| .$$

Proof. Choose $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $\alpha\langle y,x\rangle = |\langle x,y\rangle|$. For $\lambda \in \mathbb{R}$ we then have (since $\overline{\alpha}\langle x,y\rangle = \alpha\langle y,x\rangle = |\langle x,y\rangle|$)

$$\begin{aligned}\langle x-\alpha\lambda y, x-\alpha\lambda y\rangle &= \langle x,x\rangle - \alpha\lambda\langle y,x\rangle - \overline{\alpha}\lambda\langle x,y\rangle + \lambda^2\langle y,y\rangle\\ &= \langle x,x\rangle - 2\lambda\,|\langle x,y\rangle| + \lambda^2\langle y,y\rangle \ge 0.\end{aligned}$$

This is a quadratic polynomial $a\lambda^2+b\lambda+c$ in λ with real coefficients. Since this polynomial takes only non-negative values, its discriminant b^2-4ac must be non-positive:

$$4\,|\langle x,y\rangle|^2 - 4\,\|x\|^2\,\|y\|^2 - \le 0.$$

This imnplies $|\langle x,y\rangle| \le \|x\|\,\|y\|$. ∎

Corollary 11.2 $\|\cdot\|$ *defines a norm on E.*

Proof. It is clear that $\|x\| \ge 0$. From (c) it follows that $\|x\| = 0$ implies $x = 0$. Further, $\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{|\lambda|^2\langle x,x\rangle} = |\lambda|\,\|x\|$. We prove the triangle inequality. Since $2\,\mathrm{Re}\,(z) = z + \overline{z}$ we have by Proposition 1.24 and the Cauchy-Schwarz inequality

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\\ &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x,y\rangle\\ &\le \|x\|^2 + \|y\|^2 + 2\,|\langle x,y\rangle|\\ &\le \|x\|^2 + \|y\|^2 + 2\,\|x\|\,\|y\| = (\|x\| + \|y\|)^2;\end{aligned}$$

hence $\|x+y\| \le \|x\| + \|y\|$. ∎

By the corollary, any unitary space is a normed space with the norm $\|x\| = \sqrt{\langle x,x\rangle}$. Recall that any normed vector space is a metric space with the metric $d(x,y) = \|x-y\|$. Hence, the notions of open and closed sets, neighborhoods, converging sequences, Cauchy sequences, continuous mappings, and so on make sence in a unitary space. In particular $\lim\limits_{n\to\infty} x_n = x$ means that the sequence $(\|x_n - x\|)$ of non-negative real numbers tends to 0. Recall from Definition 6.8 that a metric space is said to be complete if every Cauchy sequence converges.

Definition 11.2 A complete unitary space is called a *Hilbert space.*

Example 11.1 Let $\mathbb{K} = \mathbb{C}$.

(a) $E = \mathbb{C}^n$, $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, $y = (y_1, \dots, y_n) \in \mathbb{C}^n$. Then $\langle x,y\rangle = \sum\limits_{k=1}^{n} x_k\overline{y_k}$ defines

an inner product, with the euclidean norm $\|x\| = (\sum_{k=1}^n |\, x_k \,|)^{\frac{1}{2}}$. $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$ is a Hilbert space.
(b) $E = \mathrm{L}^2(X,\mu)$ is a Hilbert space with the inner product $\langle f, g\rangle = \int_X f\overline{g}\,\mathrm{d}\mu$.
Using Young's inequality (Proposition 1.30) and the monotony of the Lebesgue integral, one easily proves Hölder's inequality

$$\int_X |\, fg \,|\, \mathrm{d}\mu \le \left(\int_X |\, f \,|^p\, \mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X |\, g \,|^q\, \mathrm{d}\mu\right)^{\frac{1}{q}}, \tag{11.2}$$

where p and q are conjugate exponents, $1/p + 1/q = 1$. For more details, consult [14, 3.5 Theorem, p. 62]. In case $p = q = 2$ we obtain the Cauchy–Schwarz inequality $|\int_X f\,\overline{g}\,\mathrm{d}\mu\,|^2 \le \int_X |\, f\,|^2\,\mathrm{d}\mu \int_X |\, g\,|^2\,\mathrm{d}\mu$. This shows two things, first, $f\overline{g}$ is integrable and the inner product is well-defined on $\mathrm{L}^2(X,\mu)$, secondly, $\mathrm{L}^2(X,\mu)$ is a linear space—the triangle inequality $\|f+g\| \le \|f\| + \|g\|$ shows that $f, g \in \mathrm{L}^2$ implies $f+g \in \mathrm{L}^2$.
Note that the inner product is positive definite since $\int_X |\, f\,|^2\,\mathrm{d}\mu = 0$ implies (by Proposition 10.17) that $|\, f\,| = 0$ a. e. However, in $\mathrm{L}^2(X,\mu)$ it follows $f = 0$ since we do not distinguish between function which are equal almost everywhere. Open: the completeness of $\mathrm{L}^2(X,\mu)$.
(c) $E = \ell_2$, $\ell_2 = \mathrm{L}^2(\mathbb{N},\mu)$ with the counting measure μ on $\mathbb{N}$, i. e.

$$\ell_2 = \{(x_n) \mid x_n \in \mathbb{C}, n \in \mathbb{N}, \sum_{n=1}^{\infty} |\, x_n\,|^2 < \infty\}.$$

Note that the Cauchy–Schwarz's inequality (Corollary 1.34) implies

$$\left|\sum_{n=1}^{k} x_n\overline{y_n}\right|^2 \le \sum_{n=1}^{k} |\, x_n\,|^2 \sum_{n=1}^{k} |\, y_n\,|^2 \le \sum_{n=1}^{\infty} |\, x_n\,|^2 \sum_{n=1}^{\infty} |\, y_n\,|^2.$$

Taking the supremum over all $k \in \mathbb{N}$ on the right, we have

$$\left|\sum_{n=1}^{\infty} x_n\overline{y_n}\right|^2 \le \sum_{n=1}^{\infty} |\, x_n\,|^2 \sum_{n=1}^{\infty} |\, y_n\,|^2;$$

hence

$$\langle (x_n), (y_n)\rangle = \sum_{n=1}^{\infty} x_n\,\overline{y_n}$$

is an absolutely converging series such that the inner product is well-defined on ℓ_2.

Lemma 11.3 *Let E be a unitary space. For any fixed $y \in E$ the mappings*

$$x \mapsto \langle x, y\rangle\,, \quad \text{and} \quad x \mapsto \langle y, x\rangle$$

are continuous functions on E.

Proof. The Cauchy–Schwarz inequality implies that for $x_1, x_2 \in E$

$$|\langle x_1, y\rangle - \langle x_2, y\rangle| = |\langle x_1 - x_2, y\rangle| \leq \|x_1 - x_2\| \, \|y\|,$$

which proves that the map $x \mapsto \langle x, y\rangle$ is in fact uniformly continuous (Given $\varepsilon > 0$ choose $\delta = \varepsilon / \|y\|$. Then $\|x_1 - x_2\| < \delta$ implies $|\langle x_1, y\rangle - \langle x_2, y\rangle| < \varepsilon$). The same is true for $x \mapsto \langle y, x\rangle$. ■

Definition 11.3 Let H be a unitary space. We call x and y *orthogonal* to each other, and write $x \perp y$, if $\langle x, y\rangle = 0$. Two subsets $M, N \subset H$ are called *orthogonal* to each other if $x \perp y$ for all $x \in M$ and $y \in N$.
For a subset $M \subset H$ define the *orthogonal complement* $M^\perp$ of M to be the set

$$M^\perp = \{x \in H \mid \langle x, m\rangle = 0, \quad \text{for all } m \in M\}.$$

For example, $E = \mathbb{R}^n$ with the standard inner product and $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, $v \neq 0$ yields

$$\{v\}^\perp = \{x \in \mathbb{R}^n \mid \sum_{k=1}^n x_k \, v_k = 0\}.$$

This is a hyperplane in $\mathbb{R}^n$ which is orthogonal to v.

Lemma 11.4 *Let H be a unitary space and $M \subset H$ be an arbitrary subset. Then, $M^\perp$ is a closed linear subspace of H.*

Proof. (a) Suppose that $x, y \in M^\perp$. Then for $m \in M$ we have

$$\langle \lambda_1 x + \lambda_2 y, m\rangle = \lambda_1 \langle x, m\rangle + \lambda_2 \langle y, m\rangle = 0;$$

hence $\lambda_1 x + \lambda_2 y \in M^\perp$. This shows that $M^\perp$ is a linear subspace.
(b) We show that any converging sequence (x_n) of elements of $M^\perp$ has its limit in $M^\perp$. Suppose $\lim_{n\to\infty} x_n = x$, $x_n \in M^\perp$, $x \in H$. Then for all $m \in M$, $\langle x_n, m\rangle = 0$. Since the inner product is continuous in the first argument (see Lemma 11.3) we obtain

$$0 = \lim_{n\to\infty} \langle x_n, m\rangle = \langle x, m\rangle.$$

This shows $x \in M^\perp$; hence $M^\perp$ is closed. ■

11.1.2 Norm and Inner product

Let $(E, \langle\cdot,\cdot\rangle)$ be a unitary space; $\|x\| = \sqrt{\langle x, x\rangle}$. then we have

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right), \quad \text{if} \quad \mathbb{K} = \mathbb{R}. \tag{11.3}$$

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\,\|x+\mathrm{i}y\|^2 - \mathrm{i}\,\|x-\mathrm{i}y\|^2\right), \quad \text{if} \quad \mathbb{K} = \mathbb{C}. \tag{11.4}$$

These equations are called *polarization identities.* They simply follow by evaluating the right sides.
Problem. Given a normed linear space $(E, \|\cdot\|)$. Does there exist an inner product $\langle\cdot,\cdot\rangle$ on E such that $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in E$? In this case we call $\|\cdot\|$ an inner product norm.

Proposition 11.5 *A norm $\|\cdot\|$ on a linear space E over $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ is an inner product norm if and only if the* parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad x, y \in E. \tag{11.5}$$

is satisfied. If (11.5) *is satisfied, the inner product $\langle\cdot,\cdot\rangle$ is given by* (11.3) *in the real case $\mathbb{K} = \mathbb{R}$ and by* (11.4) *in the complex case $\mathbb{K} = \mathbb{C}$.*

It is easy to see that (11.5) is a necessary condition for E to be a unitary space.

11.1.3 Two Theorems of F. Riesz

(born: January 22, 1880 in Austria-Hungary, died: February 28, 1956, founder of functional analysis)

Definition 11.4 Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ be Hilbert spaces. Let $H = \{(x_1, x_2) \mid x_1 \in H_1,\ x_2 \in H_2\}$ be the direct sum of the Hilbert spaces H_1 and H_2. Then

$$\langle (x_1, x_2), (y_1, y_2)\rangle = \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2$$

defines an inner product on H. With this inner product H becomes a Hilbert space. $H = H_1 \oplus H_2$ is called the (direct) *orthogonal sum* of H_1 and H_2.

Definition 11.5 Two Hilbert spaces H_1 and H_2 are called *isomorphic* if there exists a bijective linear mapping $\varphi\colon H_1 \to H_2$ such that

$$\langle \varphi(x), \varphi(y)\rangle_2 = \langle x, y\rangle_1, \quad x, y \in H_1.$$

φ is called *isometric isomorphism* or a *unitary map.*

Back to the orthogonal sum $H = H_1 \oplus H_2$. Let $\tilde{H}_1 = \{(x_1, 0) \mid x_1 \in H_1\}$ and $\tilde{H}_2 = \{(0, x_2) \mid x_2 \in H_2\}$. Then $x_1 \mapsto (x_1, 0)$ and $x_2 \mapsto (0, x_2)$ are isometric isomorphisms from $H_i \to \tilde{H}_i$, $i = 1, 2$. We have $\tilde{H}_1 \perp \tilde{H}_2$ and $\tilde{H}_i$, $i = 1, 2$ are closed linear subspaces of H. In this situation we say that H is the *inner* othogonal sum of the two closed subspaces $\tilde{H}_1$ and $\tilde{H}_2$.

(a) Riesz's First Theorem

Problem. Let H_1 be a closed linear subspace of H. Does there exist another closed linear subspace H_2 such that $H = H_1 \oplus H_2$?
Answer: YES.

Lemma 11.6 *Let C be a convex and closed subset of the Hilbert space H. For $x \in H$ let*

$$\varrho(x) = \inf\{\|x - y\| \mid y \in C\}.$$

Then there exists a unique element $c \in K$ such that

$$\varrho(x) = \|x - c\|.$$

Proof. Existence. Since $\varrho(x)$ is an infimum, there exists a sequence (y_n), $y_n \in C$, which approximates the infimum, $\lim_{n\to\infty} \|x - y_n\| = \varrho(x)$. We will show, that (y_n) is a Cauchy sequence. By the parallelogram law (see Proposition 11.5) we have

$$\begin{aligned}\|y_n - y_m\|^2 &= \|y_n - x + x - y_m\|^2 \\ &= 2\|y_n - x\|^2 + 2\|x - y_m\|^2 - \|2x - y_n - y_m\|^2 \\ &= 2\|y_n - x\|^2 + 2\|x - y_m\|^2 - 4\left\|x - \frac{y_n + y_m}{2}\right\|^2.\end{aligned}$$

Since C is convex, $(y_n + y_m)/2 \in C$ and therefore $\left\|x - \frac{y_n+y_m}{2}\right\| \geq \varrho(x)$. Hence

$$\leq 2\|y_n - x\|^2 + 2\|x - y_m\|^2 - 4\varrho(x)^2.$$

By the choice of (y_n), the first two sequences tend to $\varrho(x)^2$ as $m, n \to \infty$. Thus,

$$\lim_{m,n\to\infty} \|y_n - y_m\|^2 = 2(\varrho^2(x) + \varrho(x)^2) - 4\varrho(x)^2 = 0,$$

hence (y_n) is a Cauchy sequence. Since H is complete, there exists an element $c \in H$ such that $\lim_{n\to\infty} y_n = c$. Since $y_n \in C$ and C is closed, $c \in C$. By construction, we have $\|y_n - x\| \longrightarrow \varrho(x)$. On the other hand, since $y_n \longrightarrow c$ and the norm is continuous (see homework 37.1), we have

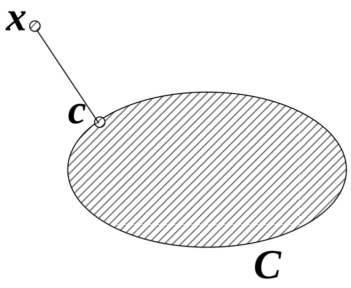
$$\|y_n - x\| \longrightarrow \|c - x\|.$$

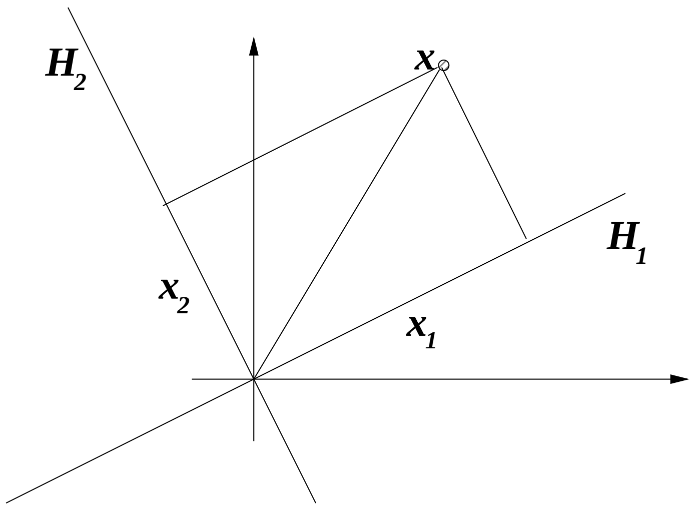
This implies $\varrho(x) = \|c - x\|$.

Uniqueness. Let c, c' two such elements. Then, by the parallelogram law,

$$\begin{aligned}0 \leq \|c - c'\|^2 &= \|c - x + x - c'\|^2 \\ &= 2\|c - x\|^2 + 2\|x - c'\|^2 - 4\left\|x - \frac{c + c'}{2}\right\|^2 \\ &\leq 2(\varrho(x)^2 + \varrho(x)^2) - 4\varrho(x)^2 = 0.\end{aligned}$$

This implies $c = c'$; the point $c \in C$ which realizes the infimum is unique. ■





Theorem 11.7 (Riesz's first theorem) *Let H_1 be a closed linear subspace of the Hilbert space H. Then we have*

$$H = H_1 \oplus H_1^\perp,$$

that is, any $x \in H$ has a unique representation $x = x_1 + x_2$ with $x_1 \in H_1$ and $x_2 \in H_1^\perp$.

Proof. Let $x \in H$. Apply Lemma 11.6 to the convex, closed set H_1. There exists a unique $x_1 \in H_1$ such that

$$\varrho(x) = \inf\{\|x - x_1\| \mid x_1 \in H_1\} \le \|x - x_1 - ty_1\|$$

for all $t \in \mathbb{K}$ and $y_1 \in H_1$. Homework 37.3 (c) now implies $x_2 = x - x_1 \perp y_1$ for all $y_1 \in H_1$. Hence $x_2 \in H_1^\perp$. Therefore, $x = x_1 + x_2$, and the existence of such a representation is shown.

We show the uniqueness. Suppose that $x = x_1 + x_2 = x_1' + x_2'$ are two possibilities to write x as a sum of elements of $x_1, x_1' \in H_1$ and $x_2, x_2' \in H_1^\perp$. Then

$$x_1 - x_1' = x_2' - x_2 = u$$

belongs to both H_1 and $H_1^\perp$ (by linearity of H_1 and H_2). Hence $\langle u, u\rangle = 0$ which implies $u = 0$. That is, $x_1 = x_1'$ and $x_2 = x_2'$. ■

Let $x = x_1 + x_2$ as above. Then the mappings $P_1(x) = x_1$ and $P_2(x) = x_2$ are well-defined on H. They are called *orthogonal projections* of H onto H_1 and H_2, respectively. We will consider projections in more detail later.

(b) Riesz's Representation Theorem

Recall from Section 9.7 that a *linear functional* on the vector space E is a mapping $F\colon E \to \mathbb{K}$ such that $F(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 F(x_1) + \lambda_2 F(x_2)$ for all $x_1, x_2 \in E$ and $\lambda_1, \lambda_2 \in \mathbb{K}$.

Definition 11.6 Let $(E, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$. A linear functional $F\colon E \to \mathbb{K}$ is called *continuous* if $x_n \longrightarrow x$ in E implies $F(x_n) \longrightarrow F(x)$.
The set of all continuous linear functionals F on E form a linear space E' with the same linear operations as in E^*.

Now let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. By Lemma 11.3, $F_y\colon H \to \mathbb{K}$, $F_y(x) = \langle x, y\rangle$ defines a continuous linear functional on H. Riesz's representation theorem states that *any* continuous linear functional on H is of this form.

Theorem 11.8 (Riesz's Representation Theorem) *Let F be a continuous linear functional on the Hilbert space H.*
Then there exists a unique element $y \in H$ such that $F(x) = F_y(x) = \langle x, y\rangle$ for all $x \in H$.

Proof. Existence. Let $H_1 = \ker F$ be the null-space of the linear functional F. H_1 is a linear subspace (since F is linear). H_1 is closed since $H_1 = F^{-1}(\{0\})$ is the preimage of the closed set $\{0\}$ under the continuous map F. By Riesz's first theorem, $H = H_1 \oplus H_1^\perp$.
Case 1. $H_1^\perp = \{0\}$. Then $H = H_1$ and $F(x) = 0$ for all x. We can choose $y = 0$; $F(x) = \langle x, 0\rangle$.
Case 2. $H_1^\perp \neq \{0\}$. Suppose $u \in H_1^\perp$, $u \neq 0$. Then $F(u) \neq 0$ (otherwise, $u \in H_1^\perp \cap H_1$ such that $\langle u, u\rangle = 0$ which implies $u = 0$). We have

$$F\left(x - \frac{F(x)}{F(u)}\,u\right) = F(x) - \frac{F(x)}{F(u)}\,F(u) = 0.$$

Hence $x - \dfrac{F(x)}{F(u)}\,u \in H_1$. Since $u \in H_1^\perp$ we have

$$0 = \left\langle x - \frac{F(x)}{F(u)}\,u, u\right\rangle = \langle x, u\rangle - \frac{F(x)}{F(u)}\,\langle u, u\rangle$$
$$F(x) = \frac{F(u)}{\langle u, u\rangle}\,\langle x, u\rangle = \left\langle x, \frac{\overline{F(u)}}{\|u\|^2}\,u\right\rangle = F_y(x),$$

where $y = \dfrac{\overline{F(u)}}{\|u\|^2}\,u$.
Uniqueness. Suppose that both $y_1, y_2 \in H$ give the same functional F, i. e. $F(x) = \langle x, y_1\rangle = \langle x, y_2\rangle$ for all x. This implies

$$\langle y_1 - y_2, x\rangle = 0, \quad x \in H.$$

In particular, choose $x = y_1 - y_2$. This gives $\|y_1 - y_2\|^2 = 0$; hence $y_1 = y_2$. ∎

(c) Application

Any continuous linear functionals on $\mathrm{L}^2(X, \mu)$ are of the form $F(f) = \int_X f\,\overline{g}\,\mathrm{d}\mu$ with some $g \in \mathrm{L}^2(X, \mu)$.

Any continuous linear functional on ℓ_2 is given by

$$F((x_n)) = \sum_{n=1}^{\infty} x_n \overline{y_n}, \quad \text{with} \quad (y_n) \in \ell_2.$$

The mapping $H \to H'$, $y \mapsto F_y$ is an anti-linear isomorphism of linear spaces (that is, $F_{\lambda y} = \overline{\lambda} F_y$, $F_{y_1+y_2} = F_{y_1} + F_{y_2}$).

11.1.4 Orthogonal Sets and Fourier Expansion

Motivation. Let $E = \mathbb{R}^n$ be the euclidean space with the standard inner product and standard basis $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$. Then we have with $x_i = \langle x, \mathrm{e}_i \rangle$

$$x = \sum_{k=1}^{n} x_k \, \mathrm{e}_k, \quad \|x\|^2 = \sum_{k=1}^{n} |\, x_k \,|^2, \quad \langle x, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}.$$

We want to generalize these formulas to arbitrary Hilbert spaces.

(a) Orthonormal Sets

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

Definition 11.7 Let $\{x_i \mid i \in I\}$ be a family of elements of H.
$\{x_i\}$ is called an *orthogonal set* or *OS* if $\langle x_i, x_j \rangle = 0$ for $i \neq j$.
$\{x_i\}$ is called an *orthonormal set* or *NOS* if $\langle x_i, x_j \rangle = \delta_{ij}$ for all $i, j \in I$.

Example 11.2 (a) $H = \ell_2$, $\mathrm{e}_n = (0, 0, \dots, 0, 1, 0, \dots)$ with the 1 at the nth component. Then $\{\mathrm{e}_n \mid n \in \mathbb{N}\}$ is an OS in H.
(b) $H = \mathrm{L}^2((0, 2\pi))$ with the Lebesgue measure, $\langle f, g \rangle = \int_0^{2\pi} f \overline{g} \, \mathrm{d}\lambda$. Then $\{1, \sin(nx), \cos(nx) \mid n \in \mathbb{N}\}$ is an OS.
Note that $\cos(m+n)x + \cos(m-n)x = 2\cos(mx)\cos(nx)$ yields

$$\langle \cos(mx), \cos(nx) \rangle = \int_0^{2\pi} \cos(mx)\cos(nx) \, \mathrm{d}x = \frac{1}{2} \int_0^{2\pi} (\cos(m+n)x + \cos(m-n)x) \, \mathrm{d}x = 0,$$

if $n \neq m$ and π if $n = m$. Similarly computations hold for $\sin x$ and 1. Finally,

$$\|1\| = 2\pi, \quad \|\sin(nx)\| = \pi, \quad \|\cos(nx)\| = \pi,$$

and we obtain

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}} \mid n \in \mathbb{N} \right\}, \quad \left\{ \frac{\mathrm{e}^{\mathrm{i}nx}}{\sqrt{2\pi}} \mid n \in \mathbb{N} \right\}$$

to be orthonormal sets of H.

Lemma 11.9 (The Pythagorean Theorem) *Let $\{x_1, \dots, x_k\}$ be an OS in H, then*

$$\|x_1 + \dots + x_k\|^2 = \|x_1\|^2 + \dots + \|x_k\|^2.$$

The easy proof is left to the reader.

Lemma 11.10 *Let $\{x_n\}$ be an OS in H. Then $\sum_{k=1}^{\infty} x_k$ converges if and only if $\sum_{k=1}^{\infty} \|x_k\|^2$ converges.*

Note that the convergence of a series $\sum_{i=1}^{\infty} x_i$ of elements x_i of a Hilbert space H is defined to be the limit of the partial sums $\lim_{n\to\infty} \sum_{i=1}^{n} x_i$. In particular, the Cauchy criterion applies since H is complete:

> The series $\sum y_i$ converges if and only if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $m, n \geq n_0$ imply $\left\| \sum_{i=m}^{n} y_i \right\| < \varepsilon$.

Proof. By the above discussion, $\sum_{k=1}^{\infty} x_k$ converges if and only if $\|\sum_{k=m}^{n} x_k\|^2$ becomes small for sufficiently large $m, n \in \mathbb{N}$. By the Pythagorean theorem this term equals

$$\sum_{k=m}^{n} \|x_k\|^2 ;$$

hence the series $\sum x_k$ converges, if and only if the series $\sum \|x_k\|^2$ converges. ■

(b) Fourierexpansion

Throughout this parapgraph let $\{x_n \mid n \in \mathbb{N}\}$ an NOS in the Hilbert space H.

Definition 11.8 The numbers $\langle x, x_n \rangle$, $n \in \mathbb{N}$, are called *Fourier coefficients* of $x \in H$ with respect to the NOS $\{x_n\}$.

Example 11.3 Consider the NOS $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}} \mid n \in \mathbb{N} \right\}$ from the previuous example on $H = \mathrm{L}^2((0, 2\pi))$. Let $f \in H$. Then

$$\begin{aligned} \left\langle f, \frac{\sin(nx)}{\sqrt{\pi}} \right\rangle &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(t) \sin(nt)\, \mathrm{d}t, \\ \left\langle f, \frac{\cos(nx)}{\sqrt{\pi}} \right\rangle &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(t) \cos(nt)\, \mathrm{d}t, \\ \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t)\, \mathrm{d}t, \end{aligned}$$

These are the usual Fourier coefficients—up to a factor. Note that we have another normalization than in Definition 7.4 since the inner product there has the factor $1/(2\pi)$.

Proposition 11.11 (Bessel's Inequality) *For $x \in H$ we have*

$$\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \leq \|x\|^2 . \tag{11.6}$$

Proof. Let $n \in \mathbb{N}$ be a positive integer and $y_n = x - \sum_{k=1}^{n} \langle x, x_k \rangle \; x_k$. Then

$$\langle y_n, x_m \rangle = \langle x, x_m \rangle - \sum_{k=1}^{n} \langle x, x_k \rangle \; \langle x_k, x_m \rangle = \langle x, x_m \rangle - \sum_{k=1}^{n} \langle x, x_k \rangle \; \delta_{km} = 0$$

for $m = 1, \dots, n$. Hence, $\{y_n, \langle x, x_1 \rangle \, x_1, \dots, \langle x, x_n \rangle \, x_n\}$ is an OS. By Lemma 11.9

$$\|x\|^2 = \left\| y_n + \sum_{k=1}^{n} \langle x, x_k \rangle \; x_k \right\| = \|y_n\|^2 + \sum_{k=1}^{n} | \langle x, x_k \rangle |^2 \, \|x_k\|^2 \geq \sum_{k=1}^{n} | \langle x, x_k \rangle |^2 ,$$

since $\|x_k\|^2 = 1$ for all k. Taking the supremum over all n on the right, the assertion follows. ∎

Corollary 11.12 *For any $x \in H$ the series $\sum_{k=1}^{\infty} \langle x, x_k \rangle \; x_k$ converges in H.*

Proof. Since $\{\langle x, x_k \rangle \, x_k\}$ is an OS, by Lemma 11.10 the series converges if and only if the series $\sum_{k=1}^{\infty} \|\langle x, x_k \rangle \; x_k\|^2 = \sum_{k=1}^{\infty} | \langle x, x_k \rangle |^2$ converges. By Bessel's inequality, this series converges. ∎

We call $\sum_{k=1}^{\infty} \langle x, x_k \rangle \; x_k$ the *Fourier series* of x with respect to the NOS $\{x_k\}$.

Remarks 11.1 (a) In general, the Fourierseries of x does *not* converge to x.
(b) The NOS $\{\frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}\}$ gives the ordinary Fourier series of a function f which is integrable over $(0, 2\pi)$.

Theorem 11.13 *Let $\{x_k \mid k \in \mathbb{N}\}$ be an NOS in H. The following are equivalent:*

(a) $x = \sum_{k=1}^{\infty} \langle x, x_k \rangle \; x_k$ *for all $x \in H$, i.e. the Fourier series of x converges to x.*

(b) $\langle z, x_k \rangle = 0$ *for all $k \in \mathbb{N}$ implies $z = 0$, i. e. the NOS is* maximal.

(c) *For every $x \in H$ we have* $\|x\|^2 = \sum_{k=1}^{\infty} | \langle x, x_k \rangle |^2$.

(d) *If $x \in H$ and $y \in H$, then* $\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, x_k \rangle \; \langle x_k, y \rangle$.

Formula (d) *is called* Parseval's identity.

Definition 11.9 An orthonormal set $\{x_i \mid i \in \mathbb{N}\}$ which satisfies the above (equivalent) properties is called a *complete orthonormal system*, CNOS for short, or *orthonormal basis.*

Proof. (a) → (d): Since the inner product is continuous in both components we have

$$\langle x, y\rangle = \left\langle \sum_{k=1}^{\infty} \langle x, x_k\rangle\ x_k, \sum_{n=1}^{\infty} \langle y, x_n\rangle\ x_n \right\rangle = \sum_{k,n=1}^{\infty} \langle x, x_k\rangle\, \overline{\langle y, x_n\rangle}\, \underbrace{\langle x_k, x_n\rangle}_{\delta_{kn}}$$
$$= \sum_{k=1}^{\infty} \langle x, x_k\rangle \langle x_k, y\rangle\,.$$

(d) → (c): Put $y = x$.
(c) → (b): Suppose $\langle z, x_k\rangle = 0$ for all k. By (c) we then have

$$\|z\|^2 = \sum_{k=1}^{\infty} |\,\langle z, x_k\rangle\,|^2 = 0; \quad \text{hence} \quad z = 0.$$

(b) → (a): Fix $x \in H$ and put $y = \sum_{k=1}^{\infty} \langle x, x_k\rangle\ x_k$ which converges according to Corollary 11.12. With $z = x - y$ we have for all positive integers $n \in \mathbb{N}$

$$\langle z, x_n\rangle = \langle x - y, x_n\rangle = \left\langle x - \sum_{k=1}^{\infty} \langle x, x_k\rangle\ x_k, x_n \right\rangle$$
$$\langle z, x_n\rangle = \langle x, x_n\rangle - \sum_{k=1}^{\infty} \langle x, x_k\rangle\ \langle x_k, x_n\rangle = \langle x, x_n\rangle - \langle x, x_n\rangle = 0.$$

This shows $z = 0$ and therefore $x = y$, i. e. the Fourier series of x converges to x. ■

Example 11.4 (a) $H = \ell_2$, $\{\mathrm{e}_n \mid n \in \mathbb{N}\}$ is an NOS. We show that this NOS is complete. For, let $x = (x_n)$ be orthogonal to every e_n, $n \in \mathbb{N}$; that is, $0 = \langle x, \mathrm{e}_n\rangle = x_n$. Hence, $x = (0, 0, \dots) = 0$. By (b), $\{\mathrm{e}_n\}$ is a CNOS. How does the Fourier series of x look like? The Fourier coefficients of x are $\langle x, \mathrm{e}_n\rangle = x_n$ such that

$$x = \sum_{n=1}^{\infty} x_n\, \mathrm{e}_n$$

is the Fourier series of x . The NOS $\{\mathrm{e}_n \mid n \geq 2\}$ is not complete.
(b) $H = \mathrm{L}^2\,((0, 2\pi))$,

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}} \mid n \in \mathbb{N} \right\}, \quad \text{and} \quad \left\{ \frac{\mathrm{e}^{\mathrm{i}nx}}{\sqrt{2\pi}} \mid n \in \mathbb{N} \right\}$$

are both CNOSs in H. This was shown in Theorem7.19

(c) Existence of CNOS in a Separabel Hilbert Space

Definition 11.10 A metric space E is called *separabel* if there exists a countable dense subset of E.

Example 11.5 (a) $\mathbb{R}^n$ is separabel. $M = \{(r_1, \dots, r_n) \mid r_1, \dots, r_n \in \mathbb{Q}\}$ is a countable dense set in $\mathbb{R}^n$.
(b) $\mathbb{C}^n$ is separabel. $M = \{(r_1+\mathrm{i}s_1, \dots, r_n+\mathrm{i}s_n) \mid r_1, \dots, r_n, s_1, \dots, s_n \in \mathbb{Q}\}$ is a countable dense subset of $\mathbb{C}^n$.
(c) $\mathrm{L}^2([a,b])$ is separabel. First note that uniform convergence $f_n \rightrightarrows f$ of continuous functions implies convergence in $\mathrm{L}^2([a,b])$ since by Lebesgue's theorem

$$\lim_{n\to\infty} \int_a^b | f_n(x) - f(x) |^2 \, \mathrm{d}x = \int_a^b \lim_{n\to\infty} | f_n(x) - f(x) |^2 \, \mathrm{d}x = 0.$$

By the theorem of Weierstraß (Theorem 7.16) any continuous function on $[a,b]$ can be uniformly approximated by polynomials. By the above argument, any continuous function is the L^2-limit of polynomial. Since the continuous functions are dense in $\mathrm{L}^2([a,b])$ (see homework 16.5 where the same was proved for the Riemann integral), we conclude that any L^2-function can be approximated by polynomials in the L^2-norm. Hence,

$$M = \{p(x) \mid p(x) = \sum_{k=0}^{n} (r_k + \mathrm{i}s_k)x^k, \;\; n \in \mathbb{N}, r_k, s_k \in \mathbb{Q}\}$$

is a countable dense subset of $\mathrm{L}^2([a,b])$.

Proposition 11.14 (Schmidt's Orthogonalization Process) *Let $\{y_k\}$ be an at most countable linearly independent subset of the Hilbert space H. Then there exists an NOS $\{x_k\}$ such that for every n*

$$\mathrm{lin}\,\{y_1, \dots, y_n\} = \mathrm{lin}\,\{x_1, \dots, x_n\}.$$

Proof. Since $\{y_k\}$ is linearly independent, $y_1 \neq 0$. Put $x_1 = y_1 / \|y_1\|$.
Suppose we have already constructed the NOS $\{x_1, \dots, x_n\}$ with the above property. However $\mathrm{lin}\,\{x_1, \dots, x_n\} \subsetneqq \mathrm{lin}\,\{y_k\}$. That is, there exists, say y_{n+1}, which is not in the linear span of $x_1, \dots, x_n$. We try the ansatz

$$\tilde{x}_{n+1} = \sum_{k=1}^{n} \lambda_j x_j + y_{n+1}.$$

The orthogonality property $\langle \tilde{x}_{n+1}, x_k \rangle = 0$ then yields

$$0 = \langle \tilde{x}_{n+1}, x_k \rangle = \left\langle \sum_{j=1}^{n} \lambda_j x_j + y_{n+1}, x_k \right\rangle = \lambda_k + \langle y_{n+1}, x_k \rangle .$$

$$\lambda_k = - \langle y_{n+1}, x_k \rangle , \quad k = 1, \dots, n.$$

Then

$$\tilde{x}_{n+1} = - \sum_{k=1}^{n} \langle y_{n+1}, x_k \rangle \, x_k + y_{n+1}$$

is orthogonal and linearly independent to $\{x_1, \dots, x_n\}$. Putting

$$x_{n+1} = \frac{\tilde{x}_{n+1}}{\|\tilde{x}_{n+1}\|}$$

$\{x_1, \dots, x_{n+1}\}$ is an NOS which linear span coincides with the linear span of $\{y_1, \dots, y_{n+1}\}$. ■

Corollary 11.15 *Let $\{\mathrm{e}_k \mid k \in N\}$ be an NOS where $N = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ or $N = \mathbb{N}$. Suppose that $H_1 = \mathrm{lin}\,\{\mathrm{e}_k \mid k \in N\}$ is the linear span of the NOS. Then $x_1 = \sum_{k\in N} \langle x, \mathrm{e}_k\rangle\, \mathrm{e}_k$ is the orthogonal projection of $x \in H$ onto H_1.*

Proof. $x_1 \in H_1$ is obvious since $\mathrm{e}_k \in H_1$ for all $k \in \mathbb{N}$. For any $\mathrm{e}_j \in H_1$ we have by continuity of $\langle\cdot,\cdot\rangle$

$$\begin{aligned}\langle x - x_1, \mathrm{e}_j\rangle &= \langle x, \mathrm{e}_j\rangle - \left\langle \sum_{k\in N} \langle x, \mathrm{e}_k\rangle\, \mathrm{e}_k, \mathrm{e}_j \right\rangle = \langle x, \mathrm{e}_j\rangle - \sum_{k\in N} \langle x, \mathrm{e}_k\rangle \langle \mathrm{e}_k, \mathrm{e}_j\rangle \\ &= \langle x, \mathrm{e}_j\rangle - \sum_{k\in N} \langle x, \mathrm{e}_k\rangle\, \delta_{kj} = \langle x, \mathrm{e}_j\rangle - \langle x, \mathrm{e}_j\rangle = 0.\end{aligned}$$

Hence, $x - x_1 \perp H_1$ such that $x = x_1 + (x - x_1)$ is the unique representation of x with $x_1 \in H_1$ and $x - x_1 \in H_1^\perp$. ■

Proposition 11.16 *A Hilbert space H has an at most countable complete orthonormal system (CNOS) if and only if H is separabel.*

Proof. Suppose $\{x_k \mid k \in N\}$ is an CNOS, where $N = \mathbb{N}$ or $N = \{1, \dots, n\}$. By Theorem 11.13, $x = \sum_k \langle x, x_k\rangle\, x_k$ for all $x \in H$. Then

$$M = \left\{ \sum_{k\in N} (r_k + \mathrm{i} s_k)\, x_k \mid r_k, s_k \in \mathbb{Q} \right\}$$

is a countable dense subset of H

Other direction. Suppose that H has a countable dense subset $M = \{\tilde{y}_1, \dots, \tilde{y}_n, \dots\}$. Cancel all elements $\tilde{y}_n$ in the sequence which are linearly dependent from its predecessors. We obtain a linearly independent set $\{y_1, \dots y_n, \dots\}$. Using Schmidt's orthogonalization process we obtain an NOS $\{x_1, \dots\}$, with the property $\mathrm{lin}\,\{x_1, \dots, x_r\} = \mathrm{lin}\,\{y_1, \dots, y_r\}$ for all $r \in \mathbb{N}$. Hence $\langle z, x_k\rangle = 0$ for all k implies $\langle z, m\rangle = 0$ for all $m \in M$ implies $\langle z, x\rangle = 0$ for all $x \in \overline{M} = H$. This implies $z = 0$. By Theorem 11.13 (b), $\{x_n\}$ is a CNOS. ■

Corollary 11.17 *Let H be a separabel Hilbert space. Then H is either isomorphic to $\mathbb{K}^n$ for some $n \in \mathbb{N}$ or to ℓ_2.*

Proof. Let $\dim H = \infty$. By Proposition 11.16, there exists a countable CNOS $\{x_k \mid k \in \mathbb{N}\}$. Every $x \in H$ can be written as $x = \sum_{k=1}^\infty \langle x, x_k\rangle\, x_k$. Define an isomorphism $T\colon H \to \ell_2$ by $T(x) = (\langle x, x_k\rangle_{k\in\mathbb{N}})$. By Bessel's inequality, the sequence of Fourier coefficients is indeed in ℓ_2. T is linear since $x \mapsto \langle x, x_k \rangle$ is linear for all k. We have to show T preserves the inner product. Let $x, y \in H$; by Parseval's identity

$$\langle x, y \rangle_H = \sum_{k=1}^{\infty} \langle x, x_k \rangle \, \overline{\langle y, x_k \rangle} = \langle T(x), T(y) \rangle_{\ell_2} .$$

In particular, $\|T(x)\| = \|x\|$ for all $x \in H$. This implies T to be injective since $T(x) = 0$ yields $x = 0$. Moreover T is surjective since any $(a_n) \in \ell_2$ has a preimage $x = \sum_{n=1}^{\infty} a_n \, x_n$ with $T(x) = (a_n)$. This proves that T is an isometric isomrphism from H onto ℓ_2. ■

Remarks 11.2 (a) Let $G \subset \mathbb{R}^n$ be a region. Then $\mathrm{L}^2(G)$ is an infinite dimensional separabel Hilbert space hence isomorphic to ℓ_2.
(b) Any Hilbert space is isomorphic to some $\mathrm{L}^2(X, \mu)$ where μ is the counting measure on X; $X = \mathbb{N}$ gives ℓ_2. X uncountable gives a non-separabel Hilbet space.

11.2 Linear Operators in Hilbert Spaces

11.2.1 Bounded Linear Operators

Definition 11.11 (a) Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be normed linear space. A linear map $T\colon E_1 \to E_2$ is called *continuous* if $x_n \longrightarrow x$ in E_1 implies $T(x_n) \longrightarrow T(x)$ in E_2.
(b) A linear map $T\colon E_1 \to E_2$ is called *bounded* if there exist a positive real number $C > 0$ such that

$$\|T(x)\|_2 \leq C \, \|x\|_1 , \quad \text{for all} \quad x \in E_1. \tag{11.7}$$

Definition 11.12 Suppose that $T\colon E_1 \to E_2$ is a bounded linear map. We define its *operator norm* by

$$\begin{aligned}
&\text{(a)} \quad \|T\| = \sup \left\{ \frac{\|T(x)\|_2}{\|x\|_1} \mid x \in E_1, \ x \neq 0 \right\}, \\
&\text{(b)} \quad \|T\| = \sup \{ \|T(x)\|_2 \mid \|x\|_1 \leq 1 \} \\
&\text{(c)} \quad \|T\| = \sup \{ \|T(x)\|_2 \mid \|x\|_1 = 1 \} \\
&\text{(d)} \quad \|T\| = \inf \{ C > 0 \mid \|T(x)\|_2 \leq C \, \|x\|_1 \} .
\end{aligned}$$

We can restrict ourselves to unit vectors since

$$\frac{\|T(\alpha x)\|_2}{\|\alpha x\|_1} = \frac{|\,\alpha\,| \, \|T(x)\|_2}{|\,\alpha\,| \, \|x\|_1} = \frac{\|T(x)\|_2}{\|x\|_1}.$$

This shows the equivalence of (a) and (c). Since $\|T(\alpha x)\|_2 = |\,\alpha\,| \, \|T(x)\|_2$, the suprema (b) and (c) are equal. From The last equality follows from the fact that the least upper bound is the infimum over all upper bounds. From (a) and (d) it follows,

$$\|T(x)\|_2 \leq \|T\| \, \|x\|_1 .$$

Proposition 11.18 *For a linear map $T\colon E_1 \to E_2$ of a normed space E_1 into a normed space E_2 the following are equivalent:*

(a) *T is bounded.*
(b) *T is continuous.*
(c) *T is continuous at one point of E_1.*

Proof. (a) $\to$ (b). This follows from the fact

$$\|T(x_1) - T(x_2)\| = \|T(x_1 - x_2)\| \leq \|T\| \, \|x_1 - x_2\| \,,$$

and T is even uniformly continuous on E_1. (b) trivially implies (c).
(c) $\to$ (a). Suppose T is continuous at x_0. To each $\varepsilon > 0$ one can find $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $\|T(x) - T(x_0)\| < \varepsilon$. Let $y = x - x_0$. In other words $\|y\| < \delta$ implies

$$\|T(y + x_0) - T(x_0)\| = \|T(y)\| < \varepsilon.$$

Suppose $z \in E_1$, $\|z\| \leq 1$. Then $\|\delta/2z\| \leq \delta/2 < \delta$; hence $\|T(\delta/2z)\| < \varepsilon$. By linearity of T, $\|T(z)\| < 2\varepsilon/\delta$. This shows $\|T\| \leq 2\varepsilon/\delta$. ∎

Q 28. $T\colon E_1 \to E_2$ is bounded if and only if for any sequence $x_n \longrightarrow 0$, $T(x_n)$ is bounded.
Proof. If T is bounded, then T is continuous at 0; hence $T(x_n) \longrightarrow 0$ which is a bounded sequence. Conversely, suppose that the condition is satisfied. We prove continuity at 0. Suppose that $x_n \neq 0$ for all n and put $\lambda_n = \|x_n\|$ which tends to 0. Then $\left(\frac{1}{\sqrt{\lambda_n}} x_n\right)$ still converges to 0 since $\left\|\frac{x_n}{\sqrt{\lambda_n}}\right\| = \sqrt{\lambda_n} \longrightarrow 0$. By assumption, the image sequence is bounded, say

$$\left\|T\left(\frac{1}{\sqrt{\lambda_n}} x_n\right)\right\| = \frac{1}{\sqrt{\lambda_n}} \|T(x_n)\| \leq C.$$

Hence,

$$\|T(x_n)\| \leq C\sqrt{\lambda_n} \longrightarrow 0$$

which proves continuity at 0. ∎

Definition 11.13 Let E and F be normed linear spaces. Let $\mathcal{L}(E, F)$ denote the set of all bounded linear maps from E to F. In case $E = F$ we simply write $\mathcal{L}(E)$ in place of $\mathcal{L}(E, F)$.

Proposition 11.19 *Let E and F be normed linear spaces. Then $\mathcal{L}(E, F)$ is a normed linear space if we define the linear structure by*

$$(S + T)(x) = S(x) + T(x), \quad (\lambda T)(x) = \lambda \, T(x)$$

for $S, T \in \mathcal{L}(E, F)$, $\lambda \in \mathbb{K}$. The operator norm $\|T\|$ makes $\mathcal{L}(E, F)$ a normed linear space.

Note that $\mathcal{L}(E,F)$ is complete if and only if F is complete.

Example 11.6 (a) Recall that $\mathcal{L}(\mathbb{K}^n,\mathbb{K}^m)$ is a normed vector space with $\|A\| \le \left(\sum_{i,j} |\, a_{ij}\,|^2\right)^{\frac{1}{2}}$, where $A=(a_{ij})$ is the matrix representation of the linear operator A, see Proposition 8.1
(b) The space $E' = \mathcal{L}(E,\mathbb{K})$ of continuous linear functionals on E.
(c) $H = \mathrm{L}^2((0,1))$, $g \in \mathrm{C}([0,1])$,

$$T_g(f)(t) = g(t)f(t)$$

defines a bounded linear operator on H. (see homework 39.5)
(d) $H = \mathrm{L}^2((0,1))$, $k(s,t) \in \mathrm{L}^2([0,1]\times[0,1])$. Then

$$(Kf)(t) = \int_0^1 k(s,t)f(s)\,\mathrm{d}s, \quad f \in H = \mathrm{L}^2([0,1])$$

defines a continuous linear operator $K \in \mathcal{L}(H)$. We have

$$\begin{aligned}
|\,(Kf)(t)\,|^2 &= \left|\int_0^1 k(s,t)f(s)\,\mathrm{d}s\right|^2 \le \left(\int_0^1 |\,k(s,t)\,|\,|\,f(s)\,|\,\mathrm{d}s\right)^2\\
&\underset{\text{C-S-I}}{\le} \int_0^1 |\,k(s,t)\,|^2\,\mathrm{d}s \int_0^1 |\,f(s)\,|^2\,\mathrm{d}s\\
&= \int_0^1 |\,k(s,t)\,|^2\,\mathrm{d}s\,\|f\|_H^2 .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|K(f)\|_H^2 &\le \int_0^1 \left(\int_0^1 |\,k(s,t)\,|\,\mathrm{d}s\right)\mathrm{d}t\,\|f\|_h^2\\
\|K(f)\|_H &\le \|k\|_{\mathrm{L}^2([0,1]\times[0,1])}\,\|f\|_H .
\end{aligned}$$

This shows $Kf \in H$ and further, $\|K\| \le \|k\|_{\mathrm{L}^2([0,1]^2)}$. K is called an *integral operator*; K is compact, i. e. it maps the unit ball into a set whose closur is compact.
(e) $H = \mathrm{L}^2(\mathbb{R})$, $a \in \mathbb{R}$,

$$(V_a f)(t) = f(t-a), \quad t \in \mathbb{R},$$

defines a bounded linear operator called the shift operator. Indeed,

$$\|V_a f\|_2^2 = \int_{\mathbb{R}} |\,f(t-a)\,|^2\,\mathrm{d}t = \int_{\mathbb{R}} |\,f(t)\,|^2\,\mathrm{d}t = \|f\|_2^2 ;$$

hence $\|V_a\| \le 1$. Later we will see that $\|V_a\| = 01$.
(f) $H = \ell_2$. We define the *right-shift* S by

$$S(x_1,x_2,\dots) = (0,x_1,x_2,\dots).$$

Obviously, $\|S(x)\| = \|x\| = \left(\sum_{n=1}^{\infty} |\,x_n\,|^2\right)^{\frac{1}{2}}$. Hence, $\|S\| = 1$.

11.2.2 The Adjoint Operator

In this subsection H is a Hilbert space and $\mathcal{L}(H)$ the space of bounded linear operators on H.
Let $T \in \mathcal{L}(H)$ be a bounded linear operator and $y \in H$. Then $F(x) = \langle T(x), y\rangle$ defines a continuous linear functional on H. Indeed,

$$| F(x) | = | \langle T(x), y\rangle | \underset{\text{CSI}}{\leq} \|T(x)\| \, \|y\| \leq \underbrace{\|T\| \, \|y\|}_{C} \|x\| \leq C \, \|x\| .$$

Hence, F is bounded and therefore continuous. in particular,

$$\|F\| \leq \|T\| \, \|y\|$$

By Riesz's representation theorem, there exists a unique vector $z \in H$ such that

$$\langle T(x), y\rangle = F(x) = \langle x, z\rangle .$$

Note that by homework 39.1 and the above inequality

$$\|z\| = \|F\| \leq \|T\| \, \|y\| . \tag{11.8}$$

Suppose y_1 is another element of H which corresponds to $z_1 \in H$ with

$$\langle T(x), y_1\rangle = \langle x, z_1\rangle .$$

Finally, let $u \in H$ be the element which corresponds to $y + y_1$,

$$\langle T(x), y + y_1\rangle = \langle x, u\rangle .$$

Since the element u which is given by Riesz's representation theorem is unique, we have $u = z + z_1$. Similarly,

$$\langle T(x), \lambda y\rangle = F(x) = \langle x, \lambda z\rangle$$

shows that λz corresponds to λy.

Definition 11.14 The above correspondence $y \mapsto z$ is linear. Define the linear operator T^* by $z = T^*(y)$. By definition,

$$\langle T(x), y\rangle = \big\langle x, T^*(y)\big\rangle , \quad x, y \in H. \tag{11.9}$$

T^* is called the *adjoint operator to* T.

Proposition 11.20 *Let* $T, T_1, T_2 \in \mathcal{L}(H)$. *Then* T^* *is a bounded linear operator with* $\|T^*\| = \|T\|$. *We have*

(a) $(T_1 + T_2)^* = T_1^* + T_2^*$ *and*

(b) $(\lambda T)^* = \overline{\lambda} T^*$.

(c) $(T_1 T_2)^* = T_2^* \, T_1^*$.

(d) *If* T *is invertible in* $\mathcal{L}(H)$, *so is* T^*, *and we have* $(T^*)^{-1} = (T^{-1})^*$.

(e) $(T^*)^* = T$.

Proof. Inequality (11.8) shows that

$$\left\| T^*(y) \right\| \le \|T\| \, \|y\| \,, \quad y \in H.$$

By definition, this implies

$$\left\| T^* \right\| \le \|T\|$$

and T^* is bounded. Since

$$\left\langle T^*(x), y \right\rangle = \overline{\langle y, T^*(x) \rangle} = \overline{\langle T(y), x \rangle} = \langle x, T(y) \rangle \,,$$

we get $(T^*)^* = T$. We conclude $\|T\| = \left\| T^{**} \right\| \le \left\| T^* \right\|$; such that $\left\| T^* \right\| = \|T\|$.
(a). For $x, y \in H$ we have

$$\begin{aligned} \langle (T_1 + T_2)(x), y \rangle &= \langle T_1(x) + T_2(x), y \rangle = \langle T_1(x), y \rangle + \langle T_2(x), y \rangle \\ &= \left\langle x, T_1^*(y) \right\rangle + \left\langle x, T_2^*(y) \right\rangle = \left\langle x, (T_1^* + T_2^*)(y) \right\rangle ; \end{aligned}$$

which proves (a).
(c) and (d) are left to the reader. ■

A mapping $*\colon A \to A$ such that the above properties (a), (b), and (c) are satisfied is called an *involution*. An algebra with involution is called a $*$-*algebra*.
We have seen that $\mathcal{L}(H)$ is a (non-commutative) $*$-algebra. An example of a commutative $*$-algebra is $\mathrm{C}(K)$ with the involution $f^*(x) = \overline{f(x)}$.

Example 11.7 (Example 11.6 continued)
(a) $H = \mathbb{C}^n$, $A = (a_{ij}) \in \mathrm{M}(n \times n, \mathbb{C})$. Then $A^* = (b_{ij})$ has the matrix elements $b_{ij} = \overline{a_{ji}}$.
(b) $H = \mathrm{L}^2([0,1])$, $T_g^* = T_{\overline{g}}$.
(c) $H = \mathrm{L}^2(\mathbb{R})$, $V_a(f)(t) = f(t-a)$ (Shift operator), $V_a^* = V_{-a}$.
(d) $H = \ell_2$. The *right-shift* S is defined by $S((x_n)) = (0, x_1, x_2, \dots)$. We compute the adjoint S^*.

$$\langle S(x), y \rangle = \sum_{n=2}^{\infty} x_{n-1} y_n = \sum_{n=1}^{\infty} x_n y_{n+1} = \langle (x_1, x_2, \dots), (y_2, y_3, \dots) \rangle \,.$$

Hence, $S^*((y_n)) = (y_2, y_3, \dots)$ is the *left-shift*.

11.2.3 Classes of Bounded Linear Operators

H is a Hilbert space.

(a) Self-Adjoint and Normal Operators

Definition 11.15 An operator $T \in \mathcal{L}(H)$ is called

(a) *self-adjoint*, if $A^* = A$,

(b) *normal* , if $A^* A = A \, A^*$,

A self-adjoint operator A is called *positive*, if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. We write $A \geq 0$. If A and B are self-adjoint, we write $A \geq B$ if $A - B \geq 0$.

A crucial role in proving the simplest properties plays the *polarization identity* which generalizes the polarization identity from Section 11.1.2.

$$4\langle Ax, y\rangle = \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle + \mathrm{i}\langle A(x+\mathrm{i}y), x+\mathrm{i}y\rangle - \mathrm{i}\langle A(x-\mathrm{i}y), x-\mathrm{i}y\rangle .$$

We use the identity as follows

$$\langle A(x), x\rangle = 0 \quad \text{for all } x \in H \text{ implies} \quad A = 0.$$

Indeed, by the polarization identity, $\langle A(x), y\rangle = 0$ for all $x, y \in H$. In particular $y = A(x)$ yields $A(x) = 0$ for all x; thus, $A = 0$.

Remarks 11.3 (a) A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$. Indeed, if A is normal, then for all $x \in H$ we have $\langle A^*A(x), x\rangle = \langle A\,A^*(x), x\rangle$ which imply $\|A(x)\|^2 = \langle A(x), A(x)\rangle = \langle A^*(x), A^*(x)\rangle = \|A^*(x)\|^2$. On the other hand, the polarizarion identity and $\langle A^*A(x), x\rangle = \langle A\,A^*(x), x\rangle$ implies $\langle (A^*A - A\,A^*)(x), x\rangle = 0$ for all x; hence $A^*A - A\,A^* = 0$ which proves the claim.
(b) Sums and real scalar multiples of self-adjoint operators are self-adjoint.
(c) The product AB of self-adjoint operators is self-adjoint if and only if A and B commute with each other, $AB = BA$.
(d) A is self-adjoint if and only if $\langle Ax, x\rangle$ is real for all $x \in H$.
Proof. Let $A^* = A$. Then $\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle}$ is real; for the opposite direction $\langle A(x), x\rangle = \langle x, A(x)\rangle$ and the polarization identity yield $\langle A(x), y\rangle = \langle x, A(y)\rangle$ for all x, y; hence $A^* = A$. ■

(b) Unitary and Isometric Operators

Definition 11.16 Let $T \in \mathcal{L}(H)$. Then T is called

(a) *unitary*, if $\quad T^*T = I = T\,T^*$.

(b) *isometric*, if $\quad \|T(x)\| = \|x\|$ for all $x \in H$.

Proposition 11.21 (a) *T is isometric if and only if $T^*T = I$ and if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in H$.*
(b) *T is unitary, if and only if T is isometric and surjective.*
(c) *If S, T are unitary, so are ST and T^{-1}. The unitary operators of $\mathcal{L}(H)$ form a group.*

Proof. (a) T isometric yields $\langle T(x), T(x)\rangle = \langle x, x\rangle$ and further $\langle (T^*T - I)(x), x\rangle = 0$ for all x. The polarization identity implies $T^*T = I$. This implies $\langle (T^*T - I)(x), y\rangle = 0$, for all $x, y \in H$. Hence, $\langle T(x), T(y)\rangle = \langle x, y\rangle$. Inserting $y = x$ shows T is isometric.
(b) Suppose T is unitary. $T^*T = I$ shows T is isometric. Since $T\,T^* = I$, T is surjective. Suppose now, T is isometric and surjective. Since T is isometric, $T(x) = 0$ implies $x = 0$;

hence, T is bijective with an inverse operator T^{-1}. Isert $y = T^{-1}(z)$ into $\langle T(x), T(y)\rangle = \langle x, y\rangle$. This gives

$$\langle T(x), z\rangle = \left\langle x, T^{-1}(z)\right\rangle, \quad x, z \in H.$$

Hence $T^{-1} = T^*$ and therefore $T^*T = TT^* = I$.
(c) is easy (see homework 40.4). ∎

Note that an isometric operator is injective with norm 1 (since $\|T(x)\| / \|x\| = 1$ for all x). In case $H = \mathbb{C}^n$, the unitary operators on $\mathbb{C}^n$ form the *unitary group* $\mathrm{U}(n)$. In case $H = \mathbb{R}^n$, the unitary operators on H form the *orthogonal group* $\mathrm{O}(n)$.

Example 11.8 (a) $H = \mathrm{L}^2(\mathbb{R})$. The shift operator V_a is unitary since $V_aV_b = V_{a+b}$. The multiplication operator $T_g f = gf$ is unitary if and only if $|\, g\,| = 1$. T_g is self-adjoint (resp. positive) if and only if g is real (resp positive).
(b) $H = \ell_2$, the right-shift $S((x_n)) = (0, x_1, x_2, \dots)$ is isometric but not unitary since S is not surjective. S^* is not isometric since $S^*(1, 0, \dots) = 0$; hence S^* is not injective.
(c) **Fourier transform.** For $f \in \mathrm{L}^1(\mathbb{R})$ define

$$(\mathcal{F}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx} f(x)\,\mathrm{d}x.$$

Let $\mathcal{S}(\mathbb{R}) = \{f \in \mathrm{C}^\infty(\mathbb{R}) \mid \sup_{t\in\mathbb{R}} \left| t^n f^{(k)}(t) \right| < \infty,\ \forall\, n, k \in \mathbb{Z}_+\}$. $\mathcal{S}(\mathbb{R})$ is called the *Schwartz space* after Laurent Schwartz. We have $\mathcal{S}(\mathbb{R}) \subseteq \mathrm{L}^1(\mathbb{R})$, $\mathrm{e}^{-x^2} \in \mathcal{S}(\mathbb{R})$. One can show that $\mathcal{F}\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is injective, $\|\mathcal{F}(f)\|_{\mathrm{L}^2(\mathbb{R})} = \|f\|_{\mathrm{L}^2(\mathbb{R})}$, $f \in \mathcal{S}(\mathbb{R})$. $\mathcal{F}$ has a unique extension to a unitary operator on $\mathrm{L}^2(\mathbb{R})$. The inverse Fourier transform is

$$(\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x, \quad f \in \mathcal{S}(\mathbb{R}).$$

11.2.4 Orthogonal Projections

(a) Riesz's First Theorem—revisited

Let H_1 be a closed linear subspace. By Theorem 11.7 any $x \in H$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in H_1$ and $x_2 \in H_1^\perp$. The map $P_{H_1}(x) = x_1$ is a linear operator from H to H, (see homework 39.1). P_{H_1} is called the *orthogonal projection* from H onto the closed subspace H_1. Obviously, H_1 is the image of P_{H_1}; in particular, P_{H_1} is surjective if and only if $H_1 = H$. In this case, $P_H = I$ is the identity. Since

$$\|P_{H_1}(x)\|^2 = \|x_1\|^2 \le \|x_1\|^2 + \|x_2\|^2 = \|x\|^2$$

we have $\|P_{H_1}\| \le 1$. If $H_1 \neq \{0\}$, there exists a non-zero $x_1 \in H_1$ such that $\|P_{H_1}(x_1)\| = \|x_1\|$. This shows $\|P_{H_1}\| = 1$.

Proposition 11.22 *A linear operator $P \in \mathcal{L}(H)$ is an orthogonal projection if and only if $P^2 = P$ and $P^* = P$.*

Proof. "$\to$". Suppose that $P = P_{H_1}$ is the projection onto H_1. Since P is the identity on H_1, $P^2(x) = P(x_1) = x_1 = P(x)$ for all $x \in H$; hence $P^2 = P$.
Let $x = x_1 + x_2$ and $y = y_1 + y_2$ be the unique decompositions of x and y in elements of H_1 and $H_1^\perp$, respectively. Then

$$\langle P(x), y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle + \underbrace{\langle x_1, y_2\rangle}_{=0} = \langle x_1, y_1\rangle = \langle x_1 + x_2, y_1\rangle = \langle x, P(y)\rangle\,,$$

that is, $P^* = P$.
"$\leftarrow$". Suppose $P^2 = P = P^*$. Let $H_1 = \{x \mid P(x) = x\}$. Since P is continuous, $H_1 = (P - I)^{-1}(\{0\})$ is a closed linear subspace of H. By Riesz's first theorem, $H = H_1 \oplus H_1^\perp$. We have to show that $P(x) = x_1$ for all x.
Since $P^2 = P$, $P(P(x)) = P(x)$ for all x; hence $P(x) \in H_1$. We show $x - P(x) \in H_1^\perp$ which completes the proof. For, let $z \in H_1$, then

$$\langle x - P(x), z\rangle = \langle x, z\rangle - \langle P(x), z\rangle = \langle x, z\rangle - \langle x, P(z)\rangle = \langle x, z\rangle - \langle x, z\rangle = 0.$$

Hence $x = P(x) + (I - P)(x)$ is the unique Riesz decomposition of x with respect to H_1 and $H_1^\perp$. ∎

(b) Properties of Orthogonal Projections

Throughout this paragraph let P_1 and P_2 be orthogonal projections on the closed subspaces H_1 and H_2, respectively.

Lemma 11.23 *The following are equivalent.*

(a) $P_1 + P_2$ *is an orthogonal projection.*
(b) $P_1 P_2 = 0$.
(c) $H_1 \perp H_2$.

Proof. (a) $\to$ (b). Let $P_1 + P_2$ be a projection. Then

$$(P_1 + P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1 + P_2 + P_1P_2 + P_2P_1 \stackrel{!}{=} P_1 + P_2,$$

hence $P_1P_2 + P_2P_1 = 0$. Multiplying this from the left by P_1 and from the right by P_1 yields

$$P_1P_2 + P_1P_2P_1 = 0 = P_1P_2P_1 + P_2P_1.$$

This implies $P_1P_2 = P_2P_1$ and finally $P_1P_2 = P_2P_1 = 0$.
(b) $\to$ (c). Let $x_1 \in H_1$ and $x_2 \in H_2$. Then

$$0 = \langle P_1P_2(x_2), x_1\rangle = \langle P_2(x_2), P_1(x_1)\rangle = \langle x_2, x_1\rangle\,.$$

This shows $H_1 \perp H_2$.
(c) $\to$ (b). Let $x, z \in H$ be arbitrary. Then

$$\langle P_1P_2(x), z\rangle = \langle P_2(x), P_1(z)\rangle = \langle x_2, z_1\rangle = 0;$$

Hence $P_1P_2(x) = 0$ and therefore $P_1P_2 = 0$. The same argument works for $P_2P_1 = 0$.
(b) → (a). Since $P_1P_2 = 0$ implies $P_2P_1 = 0$ (via $H_1 \perp H_2$),

$$(P_1 + P_2)^* = P_1^* + P_2^* = P_1 + P_2,$$
$$(P_1 + P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1 + 0 + 0 + P_2.$$

■

Lemma 11.24 *The following are equivalent*

(a) P_1P_2 *is an orthogonal projection.*
(b) $P_1P_2 = P_2P_1$.

In this case, P_1P_2 *is the orthogonal projection onto* $H_1 \cap H_2$.

Proof. (b) → (a). $(P_1P_2)^* = P_2^*\,P_1^* = P_2P_1 = P_1P_2$, by assumption. Moreover, $(P_1P_2)^2 = P_1P_2P_1P_2 = P_1P_1P_2P_2 = P_1P_2$ which completes this direction.
(a) → (b). $P_1P_2 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$.
Clearly, $P_1P_2(H) \subseteq H_1$ and $P_2P_1(H) \subseteq H_2$; hence $P_1P_2(H) \subseteq H_1 \cap H_2$. On the other hand $x \in H_1 \cap H_2$ implies $P_1P_2x = x$. This shows $P_1P_2(H) = H_1 \cap H_2$. ■

The proof of the following lemma is quite similar to that of the previous two lemmas, so we omit it (see homework 40.5).

Lemma 11.25 *The following are equivalent.*

(a) $H_1 \subseteq H_2$, (d) $P_1 \leq P_2$,
(b) $P_1P_2 = P_1$, (c) $P_2P_1 = P_1$,
(e) $P_2 - P_1$ *is an orth. projection,* (f) $\|P_1(x)\| \leq \|P_2(x)\|$, $x \in H$.

11.2.5 Spectrum and Resolvent

Let $T \in \mathcal{L}(H)$ be a bounded linear operator.

(a) Definitions

Definition 11.17 (a) The *resolvent set* of T, denoted by $\rho(T)$, is the set of all $\lambda \in \mathbb{C}$ such that there exists a bounded linear operator $R_\lambda(T) \in \mathcal{L}(H)$ with

$$R_\lambda(T)(T - \lambda I) = (T - \lambda I)R_\lambda(T) = I,$$

i. e. there $T - \lambda I$ has a bounded (continuous) inverse $R_\lambda(T)$. We call $R_\lambda(T)$ the *resolvent* of T at λ.
(b) The set $\mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T and is denoted by $\sigma(T)$.
(c) $\lambda \in C$ is called an *eigenvalue* of T if there exists a nonzero vector x, called *eigenvector*, with $(T - \lambda I)x = 0$. The set of all eigenvalues is the *point spectrum* $\sigma_{\rm p}(T)$

Remark 11.4 (a) Note that the point spectrum is a subset of the spectrum, $\sigma_{\mathrm{p}}(T) \subseteq \sigma(T)$. Suppose to the contrary, the eigenvalue λ with eigenvector y belongs to the resolvent set. Then there exists $R_\lambda(T) \in \mathcal{L}(H)$ with

$$y = R_\lambda(T)(T - \lambda I)(y) = R_\lambda(T)(0) = 0$$

which contradicts the definition of an eigenvector; hence eigenvalues belong to the spectrum.
(b) $\lambda \in \sigma_{\mathrm{p}}(T)$ is equivalent to $T - \lambda I$ not being injective. It may happen that $T - \lambda I$ is not surjective, which also implies $\lambda \in \sigma(T)$ (see Example 11.9 (b) below).

Example 11.9 (a) $H = \mathbb{C}^n$, $A \in \mathrm{M}(n \times n, \mathbb{C})$. Since in finite dimensional spaces $T \in \mathcal{L}(H)$ is injective if and only if T is surjective, $\sigma(A) = \sigma_{\mathrm{p}}(A)$.
(b) $H = \mathrm{L}^2([0,1])$. $(Tf)(x) = xf(x)$. We have

$$\boldsymbol{\sigma_{\mathrm{p}}(T) = \varnothing.}$$

Indeed, suppose λ is an eigenvalue and $f \in \mathcal{L}^2([0,1])$ an eigenfunction to T, that is $(T - \lambda I)(f) = 0$; hence $(x - \lambda)f(x) \equiv 0$ a. e. on $[0,1]$. Since $x - \lambda$ is nonzero a. e. , $f = 0$ a. e. on $[0,1]$. That is $f = 0$ in H which contradicts the definition of an eigenvector. We have

$$\boldsymbol{\mathbb{C} \setminus [0,1] \subseteq \rho(T).}$$

Suppose $\lambda \notin [0,1]$. Since $x - \lambda \neq 0$ for all $x \in [0,1]$, $g(x) = \dfrac{1}{x - \lambda}$ is a continuous (hence bounded) function on $[0,1]$. Hence,

$$(R_\lambda f)(x) = \frac{1}{x - \lambda}\, f(x)$$

defines a bounded linear operator which is invers to $T - \lambda I$ since

$$(T - \lambda I)\left(\frac{1}{x - \lambda}\, f(x)\right) = (x - \lambda)\left(\frac{1}{x - \lambda}\, f(x)\right) = f(x).$$

We have

$$\boldsymbol{\sigma(T) = [0,1].}$$

Suppose to the contrary that there exists $\lambda \in \rho(T) \cap [0,1]$. Then there exists $R_\lambda \in \mathcal{L}(H)$ with

$$R_\lambda(T - \lambda I) = I. \tag{11.10}$$

By homework 39.5 (a), the norm of the multiplication operator T_g is less that or equal to $\|g\|_\infty$ (the supremum norm of g). Choose $f_\varepsilon = \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}$. Since $\chi_M = \chi_M^2$,

$$\|(T - \lambda I)f_\varepsilon\| = \left\|(x - \lambda)\chi_{U_\varepsilon(\lambda)}(x) f_\varepsilon(x)\right\| \leq \sup_{x \in [0,1]} \left| (x - \lambda)\chi_{U_\varepsilon(\lambda)}(x) \right| \|f_\varepsilon\| \,.$$

However,

$$\sup_{x\in[0,1]} \left| (x-\lambda)\chi_{U_\varepsilon(\lambda)}(x) \right| = \sup_{x\in U_\varepsilon(\lambda)} |\, x-\lambda \,| = \varepsilon.$$

This shows

$$\|(T-\lambda I)f_\varepsilon\| \le \varepsilon \, \|f_\varepsilon\| .$$

Inserting f_ε into (11.10) we obtain

$$\|f_\varepsilon\| = \|R_\lambda (T-\lambda I)f_\varepsilon\| \le \|R_\lambda\| \, \|(T-\lambda I)f_\varepsilon\| \le \|R_\lambda\| \; \varepsilon \; \|f_\varepsilon\|$$

which implies $\|R_\lambda\| \ge 1/\varepsilon$. This contradicts the boundedness of R_λ since $\varepsilon > 0$ was arbitrary.

(b) Properties of the Spectrum

Lemma 11.26 *Let $T \in \mathcal{L}(H)$. Then*

$$\sigma(T^*) = \sigma(T)^*, \quad \textit{(complex conjugation)} \quad \rho(T^*) = \rho(T)^*.$$

Proof. Suppose that $\lambda \in \rho(T)$. Then there exists $R_\lambda(T) \in \mathcal{L}(H)$ such that

$$\begin{aligned} R_\lambda(T)(T-\lambda I) = (T-\lambda I)R_\lambda(T) &= I \\ (R_\lambda(T)(T-\lambda I))^* = ((T-\lambda I)R_\lambda)^* &= I \\ (T^* - \overline{\lambda} I)R_\lambda(T)^* = R_\lambda(t)^*(T^* - \overline{\lambda} I) &= I. \end{aligned}$$

This shows $R_{\overline{\lambda}}(T^*) = R_\lambda(T)^*$ is again a bounded linear operator on H. Hence, $\rho(T^*) \subseteq (\rho(T))^*$. Since $*$ is an involution ($T^{**} = T$), the opposite inclusion follows. Since $\sigma(T)$ is the complement of the resolvent set, the claim for the spectrum follows as well. ∎

For λ, μ, T and S we have

$$\begin{aligned} &R_\lambda(T) - R_\mu(T) = (\lambda-\mu)R_\lambda(T)R_\mu(T) = (\lambda-\mu)R_\mu(T)R_\lambda(T), \\ &R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(S-T)R_\lambda(S). \end{aligned}$$

Proposition 11.27 (a) *$\rho(T)$ is open and $\sigma(T)$ is closed.*
(b) *If $\lambda_0 \in \rho(T)$ and $|\, \lambda - \lambda_0 \,| < \|R_{\lambda_0}(T)\|^{-1}$ then $\lambda \in \rho(T)$ and*

$$R_\lambda(T) = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n R_{\lambda_0}(T)^{n+1}.$$

(c) *If $|\, \lambda \,| > \|T\|$, then $\lambda \in \rho(T)$ and*

$$R_\lambda(T) = -\sum_{n=0}^{\infty} \lambda^{-n-1} T^n.$$

Proof. (a) follows from (b).
(b) For brevity, we write R_{λ_0} in place of $R_{\lambda_0}(T)$. With $q = |\,\lambda-\lambda_0\,|\,\|R_{\lambda_0}(T)\|$, $q\in(0,1)$ we have

$$\sum_{n=0}^{\infty}|\,\lambda-\lambda_0\,|^n\,\|R_{\lambda_0}\|^{n+1} = \sum_{n=0}^{\infty} q^n\ \|R_{\lambda_0}\| = \frac{\|R_{\lambda_0}\|}{1-q} \quad \text{converges.}$$

By homework 38.4, $\sum x_n$ converges if $\sum \|x_n\|$ converges. Hence,

$$B = \sum_{n=0}^{\infty}(\lambda-\lambda_0)^n R_{\lambda_0}^{n+1}$$

converges in $\mathcal{L}(H)$ with respect to the operator norm. Moreover,

$$\begin{aligned}(T-\lambda I)B &= (T-\lambda_0 I)B - (\lambda-\lambda_0)B\\ &= \sum_{n=0}^{\infty}(\lambda-\lambda_0)^n(T-\lambda_0 I)R_{\lambda_0}^{n+1} - \sum_{n=0}^{\infty}(\lambda-\lambda_0)^{n+1}R_{\lambda_0}^{n+1}\\ &= \sum_{n=0}^{\infty}(\lambda-\lambda_0)^n R_{\lambda_0}^{n} - \sum_{n=0}^{\infty}(\lambda-\lambda_0)^{n+1}R_{\lambda_0}^{n+1}\\ &= (\lambda-\lambda_0)^0 R_{\lambda_0}^0 = I.\end{aligned}$$

Similarly, one shows $B(T-\lambda I) = I$. Thus, $R_\lambda(T) = B$.
(c) Since $|\,\lambda\,| > \|T\|$, the series converges with respect to operator norm, say

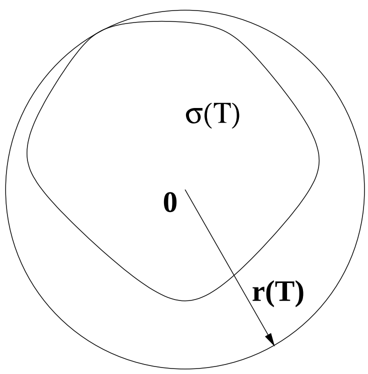
$$C = -\sum_{n=0}^{\infty}\lambda^{-n-1}T^n.$$

We have

$$(T-\lambda I)C = -\sum_{n=0}^{\infty}\lambda^{-n-1}T^{n+1} + \sum_{n=0}^{\infty}\lambda^{-n}T^n = \lambda^0 T^0 = I.$$

Similarly, $C(T-\lambda I) = I$; hence $R_\lambda(T) = C$. ∎

Remarks 11.5 (a) By (b), $R_\lambda(T)$ is a holomorphic (i. e. complex differentiable) function in the variable λ with values in $\mathcal{L}(H)$. One can use this to show that the spectrum is non-empty, $\sigma(T)\neq\varnothing$.
(b) If $\|T\| < 1$, $T - I$ is invertible with invers $-\sum_{n=0}^{\infty} T^n$.



(c) Proposition 11.27 (c) means: If $\lambda\in\sigma(T)$ then $|\,\lambda\,|\le\|T\|$. However, there is, in general, a smaller disc around 0 which contains the spectrum. By definition, the *spectral radius* $r(T)$ of T is the smallest non-negative number such that the spectrum is completely contained in the disc around 0 with radius $r(T)$:

$$r(T) = \sup\{|\,\lambda\,|\mid \lambda\in\sigma(T)\}.$$

(d) $\lambda \in \sigma(T)$ implies $\lambda^n \in \sigma(T^n)$ for all non-negative integers. Indeed, suppose $\lambda^n \in \rho(T^n)$, that is $B(T^n - \lambda^n) = (T^n - \lambda^n)B = I$ for some bounded B. Hence,

$$B \sum_{k=0}^{n} T^k \lambda^{n-1-k} (T - \lambda) = (T - \lambda) CB = I;$$

thus $\lambda \in \rho(T)$.
We shall refine the above statement and give a better upper bound for $\{|\lambda| \mid \lambda \in \sigma(T)\}$ than $\|T\|$.

Proposition 11.28 *Let $T \in \mathcal{L}(H)$ be a bounded linear operator. Then the spectral radius of T is*

$$r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}. \tag{11.11}$$

The proof is in the appendix.

11.2.6 The Spectrum of Self-Adjoint Operators

Proposition 11.29 *Let $T = T^*$ be a self-adjoint operator in $\mathcal{L}(H)$. Then $\lambda \in \rho(T)$ if and only if there exists $C > 0$ such that*

$$\|(T - \lambda I)x\| \geq C \|x\|.$$

Proof. Suppose that $\lambda \in \rho(T)$. Then there exists (a non-zero) bounded operator $R_\lambda(T)$ such that

$$\|x\| = \|R_\lambda(T)(T - \lambda I)x\| \leq \|R_\lambda(T)\| \|(T - \lambda I)x\|.$$

Hence,

$$\|(T - \lambda I)x\| \geq \frac{1}{\|R_\lambda(T)\|} \|x\|, \quad x \in H.$$

We can choose $C = 1/\|R_\lambda(T)\|$ and the condition of the proposition is satisfied.
Suppose, the condition is satisfied. We prove the other direction in 3 steps, i. e. $T - \lambda_0 I$ has a bounded inverse operator which is defined on the whole space H.
Step 1. $T - \lambda I$ is injective. Suppose to the contrary that $(T - \lambda)x_1 = (T - \lambda)x_2$. Then

$$0 = \|(T - \lambda)(x_1 - x_2)\| \geq C \|x_1 - x_2\|,$$

and $\|x_1 - x_2\| = 0$ follows. That is $x_1 = x_2$. Hence, $T - \lambda I$ is injective.
Step 2. $H_1 = (T - \lambda I)H$, the range of $T - \lambda I$ is closed. Suppose that $y_n = (T - \lambda I)x_n$, $x_n \in H$, converges to some $y \in H$. We want to show that $y \in H_1$. Clearly (y_n) is a Cauchy sequence such that $\|y_m - y_n\| \to 0$ as $m, n \to \infty$. By assumption,

$$\|y_m - y_n\| = \|(T - \lambda I)(x_n - x_m)\| \geq C \|x_n - x_m\|.$$

Thus, (x_n) is a Cauchy sequence in H. Since H is complete, $x_n \to x$ for some $x \in H$. Sinnce $T - \lambda I$ is continuous,

$$y_n = (T - \lambda I)x_n \underset{n \to \infty}{\longrightarrow} (T - \lambda I)x.$$

Hence, $y = (T - \lambda I)x$ and H_1 is a closed subspace.
Step 3. $H_1 = H$. By Riesz first theorem, $H = H_1 \oplus H_1^\perp$. We have to show that $H_1^\perp = \{0\}$. Let $u \in H_1^\perp$, that is, since $T^* = T$,

$$0 = \langle (T - \lambda I)x, u\rangle = \left\langle x, (T - \overline{\lambda} I)u\right\rangle, \quad \text{for all} \quad x \in H.$$

This shows $(T - \overline{\lambda}I)u = 0$, hence $T(u) = \overline{\lambda}u$. This implies

$$\langle T(u), u\rangle = \overline{\lambda}\,\langle u, u\rangle\,.$$

However, $T = T^*$ implies that the left side is real, by Remark 11.3 (d) 1.3 xxx. Hence $\overline{\lambda} = \lambda$ is real. We conclude, $(T - \lambda I)u = 0$. By injectivity of $T - \lambda I$, $u = 0$. That is $H_1 = H$.
We have shown that there exists a linear operator $S = (T - \lambda I)^{-1}$ which is inverse to $T - \lambda I$ and defined on the whole space H. Since

$$\|y\| = \|(T - \lambda I)S(y)\| \geq C\,\|S(y)\|\,,$$

S is bounded with $\|S\| \leq 1/C$. Hence, $S = R_\lambda(T)$. ∎

Note that for any bounded real function $f(x, y)$ we have

$$\sup_{x,y} f(x, y) = \sup_x(\sup_y f(x, y)) = \sup_y(\sup_x f(x, y)).$$

In particular, $\|x\| = \sup\limits_{\|y\|\leq 1} |\,\langle x, y\rangle\,|$ since $y = x/\,\|x\|$ yields the supremum and CSI gives the upper bound. Further, $\|T(x)\| = \sup\limits_{\|y\|\leq 1} |\,\langle T(x), y\rangle\,|$ such that

$$\|T\| = \sup_{\|x\|\leq 1}\,\sup_{\|y\|\leq 1} |\,\langle T(x), y\rangle\,| = \sup_{\|x\|\leq 1,\ \|y\|\leq 1} |\,\langle T(x), y\rangle\,|\,\sup_{\|y\|\leq 1}\,\sup_{\|x\|\leq 1} |\,\langle T(x), y\rangle\,|$$

In case of self-adjoint operators we can generalize this.

Proposition 11.30 *Let $T = T^* \in \mathcal{L}(H)$. Then we have*

$$\|T\| = \sup_{\|x\|\leq 1} |\,\langle T(x), x\rangle\,|\,. \tag{11.12}$$

Proof. Let $C = \sup\limits_{\|x\|\leq 1} |\,\langle T(x), x\rangle\,|$. By Cauchy–Schwarz inequality, $|\,\langle T(x), x\rangle\,| \leq \|T\|\,\|x\|^2$ such that $C \leq \|T\|$.
For any real positive $\alpha > 0$ we have:

$$\begin{aligned}
\|T(x)\|^2 = \langle T(x), T(x)\rangle = \left\langle T^2(x), x\right\rangle &= \frac{1}{4}\left(\left\langle T(\alpha x + \alpha^{-1}T(x)), \alpha x + \alpha^{-1}T(x)\right\rangle -\right.\\
&\qquad \left. = -\left\langle T(\alpha x - \alpha^{-1}T(x)), \alpha x - \alpha^{-1}T(x)\right\rangle\right)\\
&\leq \frac{1}{4}\left(C\left\|\alpha x + \alpha^{-1}T(x)\right\|^2 + C\left\|\alpha x - \alpha^{-1}T(x)\right\|^2\right)\\
&\underset{\text{P.I.}}{=} \frac{C}{4}\left(2\,\|\alpha x\|^2 + 2\left\|\alpha^{-1}T(x)\right\|^2\right) = \frac{C}{2}\left(\alpha^2\,\|x\|^2 + \alpha^{-2}\,\|T(x)\|^2\right).
\end{aligned}$$

Inserting $\alpha^2 = \|T(x)\| \,/\, \|x\|$ we obtain

$$= \frac{C}{2}\left(\|T(x)\|\,\|x\| + \|x\|\,\|T(x)\|\right)$$

which implies $\|T(x)\| \le C\,\|x\|$. Thus, $\|T\| = C$. ∎

Let $m = \inf_{\|x\|\le 1} \langle T(x), x\rangle$ and $M = \sup_{\|x\|\le 1} \langle T(x), x\rangle$ denote the *lower* and *upper* bound of T. Then we have

$$\sup_{\|x\|\le 1} |\,\langle T(x), x\rangle\,| = \max\{m, M\} = \|T\|\,,$$

and

$$m\,\|x\|^2 \le \langle T(x), x\rangle \le M\,\|x\|^2\,, \quad \text{for all} \quad x \in H.$$

Corollary 11.31 *Let $T = T^* \in \mathcal{L}(H)$ be a self-adjoint operator. Then*

$$\sigma(T) \subset [m, M].$$

Proof. Suppose that $\lambda_0 \not\in [m, M]$. Then

$$C := \inf_{\mu\in[m,M]} |\,\lambda_0 - \mu\,| > 0.$$

Since $m = \inf_{\|x\|=1} \langle T(x), x\rangle$ and $M = \sup_{\|x\|=1} \langle T(x), x\rangle$ we have for $\|x\| = 1$

$$\|(T-\lambda_0 I)x\| = \|x\|\,\|(T-\lambda_0 I)x\| \underset{\text{CSI}}{\ge} |\,\langle (T-\lambda_0 I)x, x\rangle\,|$$

$$\left|\,\underbrace{\langle T(x), x\rangle}_{\in[m,M]} - \lambda_0\,\|x\|^2\,\right| \ge C.$$

This implies

$$\|(T-\lambda_0 I)x\| \ge C\,\|x\| \quad \text{for all} \quad x \in H.$$

By Proposition 11.29, $\lambda_0 \in \rho(T)$. ∎

Example 11.10 Let $H = \mathrm{L}^2[0,1]$, $g \in \mathrm{C}[0,1]$ a real-valued function, and $(T_g f)(t) = g(t)f(t)$. Let $m = \inf_{t\in[0,1]} g(t)$, $M = \sup_{t\in[0,1]} g(t)$. One easily proves that m and M are the lower and upper bounds of T_g such that $\sigma(T_g) \subseteq [m, M]$.

Proposition 11.32 *Let $T = T^* \in \mathcal{L}(H)$ be self-adjoint. Then all eigenvalues of T are real and eigenvectors to different eigenvalues are orthogonal to each other.*

Proof. The first statement is clear from Corollary 11.31. Suppose that $T(x) = \lambda x$ and $T(y) = \mu y$ with $\lambda \ne \mu$. Then

$$\lambda xy = \langle T(x), y\rangle = \langle x, T(y)\rangle = \overline{\mu}\,\langle x, y\rangle = \mu\,\langle x, y\rangle\,.$$

Since $\lambda \ne \mu$, $\langle x, y\rangle = 0$. ∎

The statement about orthogonality holds for arbitrary normal operators.

Appendix: Compact Self-Adjoint Operator in Hilbert Space

Proof of Proposition 11.28. From the theory of power series, Theorem 2.32 2.32xxx we know that the series

$$-z \sum_{n=0}^{\infty} \|T^n\| \; z^n \tag{11.13}$$

converges if $|z| < R$ and diverges if $|z| > R$, where

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|T^n\|}}. \tag{11.14}$$

Inserting $z = 1/\lambda$ and using homework 38.4, we have

$$-\sum_{n=0}^{\infty} \lambda^{-n-1} T^n$$

diverges if $|\lambda| < \overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|T^n\|}$ (and converges if $|\lambda| > \overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|T^n\|}$). The reason for the divergence of the power series is, that the spectrum $\sigma(T)$ and the circle with radius $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|T^n\|}$ have points in common; hence

$$r(T) = \overline{\lim_{n\to\infty}} \sqrt[n]{\|T^n\|}.$$

On the other hand, by Remark 11.5 (d), $\lambda \in \sigma(T)$ implies $\lambda^n \in \sigma(T^n)$; hence, by Remark 11.5 (c),

$$|\lambda^n| \le \|T^n\| \implies |\lambda| \le \sqrt[n]{\|T^n\|}.$$

Taking the supremum over all $\lambda \in \sigma(T)$ on the left and the $\underline{\lim}$ over all n on the right, we have

$$r(T) \le \underline{\lim_{n\to\infty}} \sqrt[n]{\|T^n\|} \le \overline{\lim_{n\to\infty}} \sqrt[n]{\|T^n\|} = r(T).$$

Hence, the sequence $\sqrt[n]{\|T^n\|}$ converges to $r(T)$ as n tends to ∞. ∎

Compact operators generalize finite rank operators. Integral operators on compact sets are compact.

Definition 11.18 A linear operator $T \in \mathcal{L}(H)$ is called *compact* if the closure $\overline{T(U_1)}$ of the unit ball $U_1 = \{x \mid \|x\| \le 1\}$ is compact in H. In other words, for every sequence (x_n), $x_n \in U_1$, there exists a subsenquence such that $T(x_{n_k})$ converges.

Proposition 11.33 *For $T \in \mathcal{L}(H)$ the following are equivalent:*

(a) *T is compact.*
(b) *T^* is compact.*
(c) *For all sequences (x_n) with $(\langle x_n, y\rangle) \longrightarrow \langle x, y\rangle$ converges for all y we have $T(x_n) \longrightarrow T(x)$.*
(d) *There exists a sequence (T_n) of operators of finite rank such that $\|T - T_n\| \longrightarrow 0$.*

Definition 11.19 Let T be an operator on H and H_1 a closed subspace of H. We call H_1 an *reducing subspace* if both H_1 and $H_1^\perp$ are T-invariant, i. e. $T(H_1) \subset H_1$ and $T(H_1^\perp) \subset H_1^\perp$.

Proposition 11.34 *Let $T \in \mathcal{L}(H)$ be normal.*
(a) *The eigenspace* $\ker(T-\lambda I)$ *is a reducing subspace for T and* $\ker(T-\lambda I) = \ker(T-\lambda I)^*$.
(b) *If λ, μ are distinct eigenvalues of T,* $\ker(T - \lambda I) \perp \ker(T - \mu I)$.

Proof. (a) Since T is normal, so is $T - \lambda$. Hence $\|(T - \lambda)(x)\| = \left\|(T - \lambda)^*(x)\right\|$. Thus, $\ker(T - \lambda) = \ker(T - \lambda)^*$. In particular, $T^*(x) = \overline{\lambda}x$ if $x \in \ker(T - \lambda I)$.
We show invariance. Let $x \in \ker(T - \lambda)$; then $T(x) = \lambda x \in \ker(T - \alpha I)$. Similarly, $x \in \ker(T - \lambda I)^\perp$, $y \in \ker(T - \lambda I)$ imply

$$\langle T(x), y\rangle = \left\langle x, T^*(y)\right\rangle = \left\langle x, \overline{\lambda} y\right\rangle = 0.$$

Hence, $\ker(T - \lambda I)^\perp$ is T-invariant, too.
(b) Let $T(x) = \lambda x$ and $T(y) = \mu y$. Then (a) and $T^*(y) = \overline{\mu} y$ ... imply

$$\lambda \langle x, y\rangle = \langle T(x), y\rangle = \left\langle x, T^*(y)\right\rangle = \langle x, \overline{\mu} y\rangle = \mu \langle x, y\rangle .$$

Thus $(\lambda - \mu) \langle x, y\rangle = 0$; since $\lambda \neq \mu$, $x \perp y$. ∎

Theorem 11.35 (Spectral Theorem for Compact Self-Adjoint Operators)
Let H be an infinite dimensional separabel Hilbert space and $T \in \mathcal{L}(H)$ compact and self-adjoint.
Then there exists a real sequence (λ_n) with $\lambda_n \underset{n\to\infty}{\longrightarrow} 0$ and an CNOS $\{\mathrm{e}_n \mid n \in \mathbb{N}\} \cup \{f_k \mid k \in N \subset \mathbb{N}\}$ such that

$$T(\mathrm{e}_n) = \lambda_n \mathrm{e}_n, \quad n \in \mathbb{N} \qquad T(f_k) = 0, \quad k \in N.$$

Moreover,

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, \mathrm{e}_n\rangle \, \mathrm{e}_n, \quad x \in H. \tag{11.15}$$

Remarks 11.6 (a) Since $\{\mathrm{e}_n\} \cup \{f_k\}$ is a CNOS, any $x \in H$ can be written as its Fourier series

$$x = \sum_{n=1}^{\infty} \langle x, \mathrm{e}_n\rangle \, \mathrm{e}_n + \sum_{k\in N} \langle x, f_k\rangle \, f_k.$$

Applying T using $T(\mathrm{e}_n) = \lambda_n \mathrm{e}_n$ we have

$$T(x) = \sum_{n=1}^{\infty} \langle x, \mathrm{e}_n \rangle \lambda_n \mathrm{e}_n + \sum_{k \in N} \langle x, f_k \rangle \underbrace{T(f_k)}_{=0}$$

which establishes (11.15). The main point is the existence of a CNOS of eigenvectors $\{\mathrm{e}_n\} \cup \{f_k\}$.
(b) In case $H = \mathbb{C}^n$ ($\mathbb{R}^n$) the theorem says that any hermitean (symmetric) matrix A is diagonalizable with only real eigenvalues.

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