# ON TIGHTLY $\kappa$-FILTERED BOOLEAN ALGEBRAS

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ABSTRACT. In this article we study the notion of tight $\kappa$-filteredness of a Boolean algebra for infinite regular cardinals $\kappa$. Tight $\aleph_0$-filteredness is projectivity. We give characterizations of tightly $\kappa$-filtered Boolean algebras which generalize the internal characterizations of projectivity given by Haydon, Ščepin, and Koppelberg (see [15] or [17]). We show that for each $\kappa$ there is an rc-filtered Boolean algebra which is not tightly $\kappa$-filtered. This generalizes a result of Ščepin (see [15]). We prove that no complete Boolean algebra of size larger than $\aleph_2$ is tightly $\aleph_1$-filtered. We give a new example of a model of set theory where $\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered. We study the effect of the tight $\sigma$-filteredness of $\mathfrak{P}(\omega)$ on the automorphism group of $\mathfrak{P}(\omega)/fin$.

## 0. Introduction and plan of the paper

Koppelberg ([16]) introduced and studied the notion of tight $\sigma$-filteredness of a Boolean algebra, which generalizes projectivity. Using this notion she gave uniform proofs of several mostly known results about the existence of certain homomorphisms into countably complete Boolean algebras. In this article we study tight $\kappa$-filteredness for all infinite regular cardinals $\kappa$. Koppelberg's tight $\sigma$-filteredness is tight $\aleph_1$-filteredness. Projectivity is tight $\aleph_0$-filteredness.

Our research concerning tight $\kappa$-filteredness was initiated by a list of questions about tight $\sigma$-filteredness addressed by Fuchino. The first task was to obtain usable characterizations of tight $\kappa$-filteredness. These characterizations can be found in Section 2. Using these characterizations, in Section 3 we generalize some results of Koppelberg ([17]) on Stone spaces of projective Boolean algebras to Stone spaces of tightly $\kappa$-filtered Boolean algebras. Section 3 and the following sections are independent of each other, exept for Section 6, which uses a result from Section 5. In Section 4 we show that for all infinite regular $\kappa$ there are rc-filtered Boolean algebras that are not tightly $\kappa$-filtered. rc-filteredness is a generalization of projectivity and was shown to be strictly weaker than projectivity by Ščepin (see [15]). Our proof generalizes Ščepin's argument.

In Section 5 we show that complete Boolean algebras of size $\geq \aleph_3$ are not tightly $\sigma$-filtered. This implies that $\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered iff the size of the continuum

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is at most $\aleph_2$ and $\mathfrak{P}(\omega)$ has the socalled weak Freese-Nation property studied in [6].

Moreover, we prove that $\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered in a model of set theory obtained by adding at most $\aleph_2$ Cohen reals using the pseudo product of partial orders introduced by Fuchino, Shelah, and Soukup ([10]). By *the Cohen model* we will refer to a model of set theory obtained by adding $\aleph_2$ Cohen reals to a model of CH using finite support iteration. At least implicitly, it is well known that $\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered in the Cohen model. Explicitly, this was proved by Koppelberg in [16].

Finally, in Section 6 we discuss the effect tight $\sigma$-filteredness of $\mathfrak{P}(\omega)$ has on the automorphism group of $\mathfrak{P}(\omega)/fin$. This continues the discussion in [16]. It turns out that the tight $\sigma$-filteredness of $\mathfrak{P}(\omega)$ implies more or less all the facts about $\mathrm{Aut}(\mathfrak{P}(\omega)/fin)$ that are known to hold in the Cohen model. This together with the results from [16] and [6] shows that the statement '$\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered' captures a great deal of the combinatorics of the reals in the Cohen model.

This article is based on a part of the author's Ph.D. thesis, but contains some new results, especially Theorem 5.2 and Corollary 5.4.

## 1. Preparation

Throughout this article let $\kappa$ be an infinite regular cardinal.

### 1.1. $\kappa$-embeddings.

**Definition 1.1.** Let $A$ and $B$ be Boolean algebras with $A \leq B$. For $x \in B$ let $A \restriction x$ denote the ideal $\{a \in A : a \leq x\}$ of $A$. $A$ is called a *$\kappa$-subalgebra* of $B$ iff for each $x \in B$, $A \restriction x$ has cofinality $< \kappa$. In this case we write $A \leq_\kappa B$. $A$ is a *$\sigma$-subalgebra* (*relatively complete* subalgebra) of $B$ iff $A \leq_{\aleph_1} B$ ($A \leq_{\aleph_0} B$). In this case we write $A \leq_\sigma B$ ($A \leq_{\mathrm{rc}} B$). A relatively complete subalgebra is also called an *rc-subalgebra*. An isomorphism between a Boolean algebra $A$ and a $\kappa$-subalgebra (rc-subalgebra, $\sigma$-subalgebra) $A'$ of a Boolean algebra $B$ is called a $\kappa$-embedding (rc-embedding, $\sigma$-embedding). □

Note that $A \leq_\kappa B$ iff $A \leq B$ and for every ideal $I$ of $B$ which has cofinality $< \kappa$ the ideal $I \cap A$ also has cofinality $< \kappa$. Also note that $A \leq_{\mathrm{rc}} B$ iff for every $x \in B$ the ideal $A \restriction x$ is generated by a single element.

The following lemma collects some frequently used facts on $\leq_\kappa$.

**Lemma 1.2.** *Let $A$, $B$, and $C$ be Boolean algebras.*

*a) $A \leq_\kappa B \leq_\kappa C \Rightarrow A \leq_\kappa C$.*

*b) If $B$ is the union of a family $\mathcal{B}$ of subalgebras of $B$ and $A \leq_\kappa B'$ for every $B' \in \mathcal{B}$, then $A \leq_\kappa B$.*

*c) If $(A_\alpha)_{\alpha<\lambda}$ is an ascending chain of $\kappa$-subalgebras of $B$ and $\mathrm{cf}(\lambda) < \kappa$, then $\bigcup_{\alpha\leq\lambda} A_\alpha \leq_\kappa B$.*

*d) $A \leq_\kappa B$, $X \in [B]^{<\kappa} \Rightarrow A(X) \leq_\kappa B$.*

*Proof.* a) and b) are easy. For c) let $C := \bigcup_{\alpha<\lambda} A_\alpha$. Fix a cofinal set $X \subseteq \lambda$ of size $< \kappa$. For $b \in B$ and $\alpha \in X$ let $Y_\alpha^b$ be a cofinal subset of $A_\alpha \upharpoonright b$ of size $< \kappa$. Then $\bigcup_{\alpha\in X} Y_\alpha^b$ is cofinal in $C \upharpoonright b$ and has size $< \kappa$ by regularity of $\kappa$.

d) was shown by Koppelberg for $\kappa \leq \aleph_1$ ([16]). The proof of the general case is the same. □

1.2. **$\kappa$-filtrations.** A Boolean algebra is $\kappa$-filtered iff it has many $\kappa$-subalgebras. In order to give a precise formulation of 'many', we introduce various notions of skeletons.

**Definition 1.3.** Let $\mathcal{S}$ be a family of subalgebras of a Boolean algebra $A$. $\mathcal{S}$ is called a $<\kappa$*-skeleton* of $A$ iff the following conditions hold:

(i) $\mathcal{S}$ is closed under unions of subchains.
(ii) For every subalgebra $B$ of $A$ there are $\mu < \kappa$ and $C \in \mathcal{S}$ such that $B \subseteq C$ and $|C| \leq |B| + \mu$.

$\mathcal{S}$ is called a $\kappa$*-skeleton* of $A$ iff $\mathcal{S}$ satisfies (i) as above and instead of (ii) the following holds:

(ii)′ Every subalgebra $B$ of $A$ is included in a member $C$ of $\mathcal{S}$ such that $|C| = |B| + \kappa$.

$\mathcal{S}$ is called a *skeleton* iff it is an $\aleph_0$-skeleton. □

The exact definition of $\kappa$-filteredness is the following:

**Definition 1.4.** A Boolean algebra $A$ is $\kappa$-filtered iff it has a $\kappa$-skeleton $\mathcal{S}$ consisting of $\kappa$-subalgebras. $A$ is $\sigma$*-filtered* iff it is $\aleph_1$-filtered. $A$ is *rc-filtered* iff it is $\aleph_0$-filtered. In some part of the literature rc-filtered Boolean algebras are called *openly generated*. □

The main notion that will be investigated in this article is *tight* $\kappa$-filteredness. While $\kappa$-filteredness and tight $\kappa$-filteredness seem to be unrelated at first sight, it will turn out later that tight $\kappa$-filteredness is stronger than $\kappa$-filteredness.

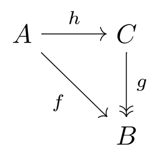
**Definition 1.5.** Let $A$ be a Boolean algebra and $\delta$ an ordinal. A continuous ascending chain $(A_\alpha)_{\alpha<\delta}$ of subalgebras of $A$ such that $A = \bigcup_{\alpha<\delta} A_\alpha$ is called a (wellordered) *filtration* of $A$.

A filtration $(A_\alpha)_{\alpha<\delta}$ is called *tight* iff $A_0 = 2$ and there is a sequence $(x_\alpha)_{\alpha<\delta}$ in $A$ such that $A_{\alpha+1} = A_\alpha(x_\alpha)$ holds for all $\alpha < \delta$.

A filtration $(A_\alpha)_{\alpha<\delta}$ is called a $\kappa$*-filtration* (*rc-filtration*, $\sigma$*-filtration*) iff $A_\alpha \leq_\kappa A_{\alpha+1}$ ($A_\alpha \leq_{\mathrm{rc}} A_{\alpha+1}$, $A_\alpha \leq_\sigma A_{\alpha+1}$) holds for all $\alpha < \delta$. $A$ is *tightly $\kappa$-filtered* iff it has a tight $\kappa$-filtration. □

1.3. **Universal properties.** This subsection will not really be needed for the rest of this article, but it provides some motivation for studying tight $\kappa$-filteredness. Tightly $\kappa$-filtered Boolean algebras have properties similar to projectivity. While no infinite complete Boolean algebra is projective, in some models of set theory interesting complete Boolean algebras are, for example, tightly $\sigma$-filtered. This has nice applications concerning the existence of certain homomorphisms.

**Definition 1.6.** A Boolean algebra $A$ is *projective* iff for any two Boolean algebras $B$ and $C$, every epimorphism $g : C \to B$, and every homomorphism $f : A \to B$ there is a homomorphism $h : A \to C$ such that




commutes. □

While this definition works in every category, the following characterization due to Halmos (see [17]) provides more insight into the structure of projective Boolean algebras.

**Definition and Lemma 1.7.** *$A$ is a* retract *of $B$ iff there are homomorphisms $e : A \to B$ and $p : B \to A$ such that $p \circ e = \mathrm{id}_A$. A Boolean algebra $A$ is projective iff it is a retract of a free Boolean algebra.*

*Proof.* Abstract nonsense. □

This lemma is true in every category with sufficiently many free objects. However, there are categories in which this lemma does not hold since there are non-trivial projective objects, but no non-trivial free objects. (See [12] for an example.)

By theorems of Haydon, Koppelberg, and Ščepin, the tightly rc-filtered Boolean algebras are exactly the projective Boolean algebras. (See [17] or [15].) The following theorem generalizes one direction of this to tightly $\kappa$-filtered Boolean algebras and was proved by Koppelberg ([16]) for $\kappa = \aleph_1$. Her proof works for uncountable $\kappa$ as well. However, we do not know whether the following theorem actually characterizes tight $\kappa$-filteredness. It probably does not. Let us introduce some additional notions first.

**Definition 1.8.** A Boolean algebra $A$ has the *$\kappa$-separation property* ($\kappa$-s.p. for short) iff for any two subsets $S$ and $T$ of $A$ of size $< \kappa$ with $S \cdot T := \{s \cdot t : s \in S \wedge t \in T\} = \{0\}$ there is $a \in A$ such that $s \leq a$ for all $s \in S$ and $t \leq -a$ for all $t \in T$. An ideal $I$ of a Boolean algebra $A$ is *$\kappa$-directed* iff every subset of $I$ of size $< \kappa$ has an upper bound in $I$. □

In particular, every $\kappa$-complete Boolean algebra has the $\kappa$-s.p. Similarly, every $\kappa$-ideal, i.e., every ideal which is closed under sums of less than $\kappa$ elements, is $\kappa$-directed.

**Theorem 1.9.** *Let $A$ be a tightly $\kappa$-filtered Boolean algebra. If $B$ and $C$ are Boolean algebras, $C$ has the $\kappa$-s.p., $g : C \to B$ is an epimorphism such that the kernel of $g$ is $\kappa$-directed, and $f : A \to B$ is a homomorphism, then there is a homomorphism $h : A \to C$ such that $g \circ h = f$.* □

The proof uses

**Lemma 1.10.** *Let $A$ and $A'$ be Boolean algebras such that $A'$ is a simple extension of $A$, i.e., $A' = A(x)$ for some $x \in A'$. Assume that $A \leq_\kappa A(x)$, $B$ and $C$ are Boolean algebras, $C$ has the $\kappa$-s.p., $g : C \to B$ is an epimorphism with $\kappa$-directed kernel, $f : A' \to B$ is a homomorphism, and $h : A \to C$ is a homomorphism such that $g \circ h = f \restriction A$. Then there is an extension $h' : A' \to C$ of $h$ such that $g \circ h' = f$, i.e.,*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle \leq_\kappa}\downarrow & \overset{h'}{\nearrow} & \downarrow\!\!\downarrow {\scriptstyle g} \\ A(x) & \xrightarrow[f]{} & B \end{array}$$

*commutes.*

*Proof.* The proof is the same as the one given by Koppelberg ([16]) for $\kappa = \aleph_1$. □

*Proof of the theorem.* Fix a tight $\kappa$-filtration of $A$ and construct $h$ by transfinite induction along this filtration, using Lemma 1.10 at the successor stages. □

In particular, this theorem gives that if $A$ has the $\kappa$-s.p., $f : A \to B$ is an epimorphism with $\kappa$-directed kernel, and $B$ is tightly $\kappa$-filtered, then there is an homomorphism $h : B \to A$ such that $f \circ h = \mathrm{id}_B$. $h$ is called a *lifting* for $f$. Note that $h$ is injective.

**Definition 1.11.** Let $\mathcal{M}$ be the ideal of meager subsets of the Cantor space ${}^\omega 2$ and let $\mathcal{N}$ be the ideal of subsets of ${}^\omega 2$ of measure zero. Here the measure on ${}^\omega 2$ is just the product measure induced by the measure on 2 mapping the singletons to $\frac{1}{2}$. Let $\mathrm{Bor}({}^\omega 2)$ be the $\sigma$-algebra of Borel subsets of ${}^\omega 2$ and let $\mathbb{C}(\omega) := \mathrm{Bor}({}^\omega 2)/\mathcal{M}$ and $\mathbb{R}(\omega) := \mathrm{Bor}({}^\omega 2)/\mathcal{N}$. $\mathbb{C}(\omega)$ is the *Cohen algebra* or *category algebra* and $\mathbb{R}(\omega)$ is the *measure algebra* or *random algebra*. Let $p : \mathrm{Bor}({}^\omega 2) \to \mathbb{R}(\omega)$ and $q : \mathrm{Bor}({}^\omega 2) \to \mathbb{C}(\omega)$ be the quotient mappings. A lifting for $p$ is a *Borel lifting for measure* and a lifting for $q$ is a *Borel lifting for category.* □

Using her version of Theorem 1.9, Koppelberg gave uniform proofs of several mostly known results about the existence of certain homomorphism into Boolean algebras with the countable separation property. Among other things, she observed that under CH and in the Cohen model, $\mathbb{C}(\omega)$ and $\mathbb{R}(\omega)$ are tightly $\sigma$-filtered. This implies the existence of Borel liftings for measure and category in the respective models (see [16]). Originally, the results on Borel liftings in these models were obtained by von Neumann, Stone, Carlson, Frankiewicz, and Zbierski.

One may ask whether the existence of a Borel lifting implies the existence of a tight $\sigma$-filtration of the respective algebra. At least for measure, this it not the case. According to Burke ([2]), Veličkovič has shown that after adding $\aleph_2$ random reals to a model of CH, there is a Borel lifting for measure. However, it follows from the results in [6] and [7] that in that model $\mathbb{R}(\omega)$ fails to have the weak Freese-Nation property, which will be introduced below and which is equivalent to $\sigma$-filteredness. It follows from Lemma 2.5 that $\mathbb{R}(\omega)$ is not tightly $\sigma$-filtered if it is not $\sigma$-filtered.

Shelah proved that it is consistent that there is no Borel lifting for measure, respectively for category ([22]). Apart from the results mentioned above, little is known about the existence of Borel liftings for measure or category. For example, it is an open question whether it is consistent that the continuum is $\aleph_3$ and there is a Borel lifting for measure or category. There is some connection to Theorem 5.4 here. The theorem says that no complete Boolean algebra of size $> \aleph_2$ is tightly $\sigma$-filtered. This means that the known arguments giving the existence of a Borel lifting for measure, respectively category, in some models of set theory do not work if the continuum is $\geq \aleph_3$.

We do not know whether tight $\kappa$-filteredness can be characterized by some property like the one in Theorem 1.9. However, there will be several internal characterizations of tight $\kappa$-filteredness in the next section.

1.4. **The $\kappa$-Freese-Nation property.** In this subsection we introduce the $\kappa$-Freese-Nation property, which turns out to be equivalent to $\kappa$-filteredness. The $\kappa$-Freese-Nation property has been studied by Fuchino, Koppelberg, Shelah, and Soukup ([8], [9], and [11]).

**Definition 1.12.** A Boolean algebra $A$ has the *$\kappa$-Freese-Nation property* ($\kappa$-FN for short) iff there is a function $f : A \to [A]^{<\kappa}$ such that for all $a, b \in A$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ such that $a \leq c \leq b$. $f$ is called a *$\kappa$-FN-function* for $A$. The $\aleph_0$-FN is the original Freese-Nation property (FN), which has been used by Freese and Nation to characterize projective lattices ([3]). The $\aleph_1$-FN is called *weak* Freese-Nation property (WFN for short) and was introduced by Heindorf and Shapiro ([15]). WFN($A$) denotes the statement '$A$ has the WFN'. □

This definition works perfectly well for partial orders instead of Boolean algebras. The same is true for $\kappa$-filteredness. However, in this article we will only be interested in Boolean algebras. This is due to the fact that we do not know how to generalize the notion of tight $\kappa$-filteredness to arbitrary partial orders in a reasonable way.

It is easily seen that small Boolean algebras have the $\kappa$-FN.

**Lemma 1.13** ([8])**.** *Every Boolean algebra $A$ of size $\leq \kappa$ has the $\kappa$-FN.* □

By a result of Heindorf ([15]), a Boolean algebra is rc-filtered iff it has the FN. Similarly, in [15] it is proved that for Boolean algebras the WFN is the same as $\sigma$-filteredness. Fuchino, Koppelberg, and Shelah ([8]) gave a characterization of partial orderings with the $\kappa$-FN in terms of elementary submodels of some $H_\chi$. Their arguments implicitly show that $\kappa$-filteredness and the $\kappa$-FN are equivalent. The following lemma collects the basic observations needed for the proof that $\kappa$-filteredness and the $\kappa$-FN are equivalent:

**Lemma 1.14.** a) *(*[8]*) If $f$ is a $\kappa$-FN function for a Boolean algebra $A$ and $B \leq A$ is closed under $f$, then $B \leq_\kappa A$.*

b) *If $B$ is a $\kappa$-subalgebra of a Boolean algebra $A$ and $A$ has the $\kappa$-FN, then $B$ has the $\kappa$-FN, too.*

c) *(*[8]*) Let $\delta$ be a limit ordinal and let $(A_\alpha)_{\alpha\leq\delta}$ be in increasing continuous chain of Boolean algebras such that $A_\alpha \leq_\kappa A_\delta$ for every $\alpha<\delta$. If $A_\alpha$ has the $\kappa$-FN for every $\alpha<\delta$, then $A_\delta$ has the $\kappa$-FN as well.*

*Proof.* Only b) has not been proved in [8]. Let $f$ be a $\kappa$-FN-function for $A$. For each $a\in A$ fix $X_a\in[B]^{<\kappa}$ such that $X_a$ is cofinal in $B\restriction a$. For each $b\in B$ let $g(b):=\bigcup_{a\in f(b)}X_a$. $g$ is a $\kappa$-FN-function for $B$: By regularity of $\kappa$, $|g(b)|<\kappa$ for every $b\in B$. Let $b,c\in B$ be such that $b\leq c$. Now there is $a\in f(b)\cap f(c)$ such that $b\leq a\leq c$. Let $a'\in X_a$ be such that $b\leq a'\leq a$. Now $b\leq a'\leq c$ and $a'\in g(b)\cap g(c)$. □

**Theorem 1.15.** *A Boolean algebra $A$ has the $\kappa$-FN iff it is $\kappa$-filtered.*

*Proof.* First let $f$ be a $\kappa$-FN-function for $A$. The family of those subalgebras of $A$ which are closed under $f$ is easily seen to be a $\kappa$-skeleton of $A$. By part a) of Lemma 1.14, it consists of $\kappa$-subalgebras of $A$.

For the other direction let $\mathcal{S}$ be a $\kappa$-skeleton of $A$ consisting of $\kappa$-subalgebras. Let $B\in\mathcal{S}$ be of minimal size such that $B$ does not have the $\kappa$-FN. If such a $B$ does not exist, we are done since $A$ itself is an element of $\mathcal{S}$.

Let $\lambda:=|B|$. By Lemma 1.13, $\lambda>\kappa^+$. Let $(b_\alpha)_{\alpha<\lambda}$ be an enumeration of $B$. Inductively pick a continuously increasing sequence $(B_\alpha)_{\alpha<\lambda}$ in $\mathcal{S}$ such that for each $\alpha<\lambda$, $|B_\alpha|<\lambda$ and $\{b_\beta:\beta<\alpha\}\subseteq B_\alpha$. This is possible by the properties of $\mathcal{S}$. Now $B':=\bigcup_{\alpha<\lambda}B_\alpha\in\mathcal{S}$ and $B\subseteq B'$. By part c) of Lemma 1.14, $B'$ has the $\kappa$-FN. Since $B\leq_\kappa B'$, it follows from part b) of Lemma 1.14 that $B$ has the $\kappa$-FN. A contradiction. □

The following lemma comes in handy when one wants to find out whether or not certain complete Boolean algebras have the $\kappa$-FN. The $\kappa$-FN does not reflect to subalgebras in general, but to subalgebras which are retracts.

**Lemma 1.16.** *(*[8]*) Let $A$ and $B$ be Boolean algebras. If $A$ is a retract of $B$ and $B$ has the $\kappa$-FN, then $A$ has the $\kappa$-FN.* □

Since $\mathfrak{P}(\omega)$ embeds into every infinite complete Boolean algebra and is complete, $\mathfrak{P}(\omega)$ is a retract of every infinite complete Boolean algebra. Thus, $\mathfrak{P}(\omega)$ has the $\kappa$-FN iff any infinite complete Boolean algebra does. The most interesting case seems to be $\kappa=\aleph_1$. Fuchino, Koppelberg, and Shelah ([8]) noticed that every complete Boolean algebra with the WFN satisfies the c.c.c. As mentioned earlier, for every Boolean algebra $A$ of size $\aleph_1$, WFN($A$) holds. Thus CH implies WFN($\mathfrak{P}(\omega)$). It is possible to enlarge the continuum by adding Cohen reals without destroying WFN($\mathfrak{P}(\omega)$). Here adding $\kappa$ Cohen reals means forcing with $\mathrm{Fn}(\kappa,2)$. In [8] and [11] the following facts about WFN($\mathfrak{P}(\omega)$) were established:

**Theorem 1.17.** *a)(*[8]*) Adding less than $\aleph_\omega$ Cohen reals to a model of CH gives a model of* WFN($\mathfrak{P}(\omega)$)*.*

*b)([11]) Adding any number of Cohen reals to a model of CH+$\neg 0^\sharp$ gives a model of* WFN($\mathfrak{P}(\omega)$). □

In [6] it was shown that the universe must be quite similar to a model obtained by adding Cohen reals to a model of CH if WFN($\mathfrak{P}(\omega)$) holds, at least as far as the reals are concerned. Note that the Cohen algebra $\mathbb{C}(\omega)$ and $\mathfrak{P}(\omega)$ both are retracts of each other. Therefore one of them has WFN if the other one does. This was noticed by Koppelberg ([16]). In [7] it was proved that $\mathfrak{P}(\omega)$ has the WFN iff the measure algebra $\mathbb{R}(\omega)$ does.

## 2. Characterizations of tightly $\kappa$-filtered Boolean algebras

In this section we give characterizations of tightly $\kappa$-filtered Boolean algebras which are similar to the characterizations known for projective Boolean algebras. For these characterizations we have to assume that $\kappa$ is uncountable, simply because some of the proofs given below do not work for $\kappa = \aleph_0$. However, some of the characterizations given below are parallel to those of projective Boolean algebras. The main difference to the projective case is that projective Boolean algebras are exactly the retracts of free Boolean algebras. A similar characterization of tightly $\kappa$-filtered Boolean algebras does not seem to be available. For the characterization of tightly $\kappa$-filtered Boolean algebras we will use the concept of commuting subalgebras of a Boolean algebra.

**Definition 2.1.** Let $A$ and $B$ be subalgebras of the Boolean algebra $C$. Then $A$ and $B$ *commute* iff for every $a \in A$ and every $b \in B$ such that $a \cdot b = 0$ there is $c \in A \cap B$ such that $a \leq c$ and $b \leq -c$.

A family $\mathcal{F}$ of subsets of a Boolean algebra $A$ is called *commutative* iff it consists of pairwise commuting subalgebras. □

The connection between $\kappa$-subalgebras and commutative families is given by

**Lemma 2.2.** *Let $\mathcal{F}$ be a commutative family of subalgebras of $A$ such that every $a \in A$ is contained in some $B \in \mathcal{F}$ of size $< \kappa$. Then $\mathcal{F}$ consists of $\kappa$-subalgebras of $A$.*

*Proof.* Let $C \in \mathcal{F}$ and $a \in A$. Then there is $B \in \mathcal{F}$ such that $a \in B$. We *claim* that $B$ contains a cofinal subset of $C \restriction a$. Let $c \in C \restriction a$. Now $-a \cdot c = 0$. Since $B$ and $C$ commute, there is $b \in B \cap C$ such that $c \leq b$ and $-a \leq -b$. But now $c \leq b \leq a$. □

This lemma is implicitly contained in the book by Heindorf and Shapiro ([15]) for the case $\kappa = \aleph_1$.

It turns out that additivity of skeletons is what separates tight $\kappa$-filteredness from $\kappa$-filteredness.

**Definition 2.3.** A $< \kappa$-skeleton (respectively $\kappa$-skeleton) $\mathcal{S}$ of a Boolean algebra $A$ is called *additive* iff for every $T \subseteq \mathcal{S}$ the Boolean algebra $\langle \bigcup T \rangle$ generated in $A$ by $\bigcup T$ is a member of $\mathcal{S}$. □

In order to make the similarities between projective Boolean algebras and tightly $\kappa$-filtered Boolean algebras apparent, we quote the following from Heindorf and Shapiro ([15]):

**Theorem 2.4.** *The following are equivalent for a Boolean algebra $A$:*

(i) *$A$ is projective.*
(ii) *For some ordinal $\delta$, $A$ is the union of a continuous chain $(A_\alpha)_{\alpha<\delta}$ consisting of rc-subalgebras such that $A_{\alpha+1}$ is countably generated over $A_\alpha$ for every $\alpha < \delta$ and $A_0$ is countable.*
(iii) *$A$ has a tight rc-filtration.*
(iv) *$A$ has an additive commutative skeleton.*
(v) *$A$ has an additive skeleton consisting of rc-embedded subalgebras.*
(vi) *$A$ is the union of a family $\mathcal{C}$ of countable subsets of $A$ such that $\langle\bigcup S\rangle \leq_{\mathrm{rc}} A$ for every $S \subseteq \mathcal{C}$.* □

We have the following characterizations of tightly $\kappa$-filtered Boolean algebras:

**Theorem 2.5.** *Let $\kappa$ be an uncountable regular cardinal. The following are equivalent for a Boolean algebra $A$:*

(i) *For some ordinal $\delta$, $A$ is the union of a chain $(A_\alpha)_{\alpha<\delta}$ of $\kappa$-subalgebras which is continous at limit ordinals of cofinality $\geq \kappa$ such that $A_{\alpha+1}$ is $\leq \kappa$-generated over $A_\alpha$ for every $\alpha < \delta$ and $A_0$ has size $\leq \kappa$.*
(ii) *$A$ has a tight $\kappa$-filtration.*
(iii) *$A$ has an additive commutative $< \kappa$-skeleton.*
(iv) *$A$ has an additive $< \kappa$-skeleton consisting of $\kappa$-embedded subalgebras.*
(v) *$A$ has an additive $\kappa$-skeleton consisting of $\kappa$-embedded subalgebras.*
(vi) *$A$ is the union of a family $\mathcal{C}$ of subsets of size $< \kappa$ of $A$ such that for all $S, T \subseteq \mathcal{C}$ the algebras $\langle\bigcup S\rangle$ and $\langle\bigcup T\rangle$ commute.*
(vii) *$A$ is the union of a family $\mathcal{C}$ of subsets of size $< \kappa$ of $A$ such that for every $S \subseteq \mathcal{C}$, $\langle\bigcup S\rangle \leq_\kappa A$.*
(viii) *$A$ is the union of a family $\mathcal{C}$ of subsets of size $\leq \kappa$ of $A$ such that for every $S \subseteq \mathcal{C}$, $\langle\bigcup S\rangle \leq_\kappa A$.*

*Proof.* (i)⇒(ii) was proved by Koppelberg ([16]) for $\kappa = \aleph_1$. The proof for arbitrary regular $\kappa$ is exactly the same.

(iii)⇒(iv) follows from Lemma 2.2.

(iv)⇒(v) is trivial.

(iii)⇒(vi), (iv)⇒(vii), and (v)⇒(viii) can be seen using the same argument: Let the $\mathcal{C}$ consist of the elements of the $< \kappa$-skeleton ($\kappa$-skeleton) of size $< \kappa$ (of size $\leq \kappa$). Then additivity of the $< \kappa$-skeleton ($\kappa$-skeleton) yields the desired property of $\mathcal{C}$.

(vi)⇒(vii) follows from Lemma 2.2 applied to the family $\mathcal{F}$ of all subalgebras of $A$ generated by a union of elements of $\mathcal{C}$.

(vii)⇒(i) and (viii)⇒(i) are easily seen using the following argument: Let $A = \{a_\alpha : \alpha <|A|\}$. For every $\alpha <|A|$ choose $B_\alpha \in \mathcal{C}$ such that $a_\alpha \in B_\alpha$. Let $A_\alpha := \langle\bigcup_{\beta<\alpha} B_\beta\rangle$ for every $\alpha <|A|$. $(A_\alpha)_{\alpha<|A|}$ works for (i).

(ii)⇒(iii) is the only part that requires some work. Let $(x_\alpha)_{\alpha<\delta} \in {}^\delta A$ be such that $(\langle\{x_\beta : \beta < \alpha\}\rangle)_{\alpha<\delta}$ is a tight $\kappa$-filtration of $A$. For every $S \subseteq \delta$ let $A_S := \langle\{x_\beta : \beta \in S\}\rangle$. With this notation the filtration is simply $(A_\alpha)_{\alpha<\delta}$. Choose $f : \delta \longrightarrow [\delta]^{<\kappa}$ such that for every $\alpha < \delta$ the ideals $A_\alpha \restriction x_\alpha$ and $A_\alpha \restriction -x_\alpha$ are generated by $(A_\alpha \restriction x_\alpha) \cap A_{f(\alpha)}$ and $(A_\alpha \restriction -x_\alpha) \cap A_{f(\alpha)}$ respectively and such that $f(\alpha) \subseteq \alpha$. Let $\mathcal{S} := \{A_T : T \subseteq \delta \wedge \bigcup f[T] \subseteq T\}$. $\mathcal{S}$ is an additive $< \kappa$-skeleton:

Clearly, every subset of $A$ of size at least $\kappa$ is included in a member of $\mathcal{S}$ of the same size. Moreover, any subset of $A$ of size $< \kappa$ is included in an element of $\mathcal{S}$ of size $< \kappa$. Suppose $\mathcal{T} \subseteq \mathcal{S}$. Let $\mathcal{U} \subseteq \mathfrak{P}(\delta)$ be such that $\mathcal{T} = \{A_T : T \in \mathcal{U}\}$ and every $T \in \mathcal{U}$ is closed under $f$. Then $\langle\bigcup\mathcal{T}\rangle = A_{\bigcup\mathcal{U}} \in \mathcal{S}$ since $\bigcup\mathcal{U}$ is closed under $f$. In particular, $\mathcal{S}$ is closed under unions of subchains.

It remains to show that $\mathcal{S}$ is commutative.

Suppose $S, T \subset \kappa$ are closed under $f$. It is sufficient to show that $A_{S\cap\alpha}$ and $A_{T\cap\alpha}$ commute for every $\alpha < \delta$. We will do so by induction on $\alpha$. The limit stages of the induction are trivial. Suppose $\alpha = \beta+1$. W.l.o.g. we may assume $\beta \in S$. Let $u \in A_{S\cap\alpha}$ and $v \in A_{T\cap\alpha}$ be such that $u \cdot v = 0$. W.l.o.g. we may assume that $u$ is of the form $a \cdot x_\beta$ for some $a \in A_{S\cap\beta}$. The case $u = a - x_\beta$ is completely analogous. Only the following cases are interesting:

I. $v = b - x_\beta$ for some $b \in A_{T\cap\beta}$ and $\beta \in T$. Then $x_\beta \in A_S \cap A_T$, $u \leq x_\beta$ and $v \leq -x_\beta$.
II. $v = b\cdot x_\beta$ for some $b \in A_{T\cap\beta}$ and $\beta \in T$. Then $a\cdot b\cdot x_\beta = 0$. Hence $a\cdot b \leq -x_\beta$. Take $c \in A_{f(\beta)}$ such that $a\cdot b \leq c \leq -x_\beta$. Then $(a-c)\cdot(b-c) = 0$, $a\cdot x_\beta \leq a-c$ and $b \cdot x_\beta \leq b - c$. Now $a - c \in A_{S\cap\beta}$ and $b - c \in A_{T\cap\beta}$. By hypothesis, there is $r \in A_{T\cap\beta} \cap A_{S\cap\beta}$ such that $a - c \leq r$ and $b - c \leq -r$. $r$ is as required.
III. $v \in A_{T\cap\beta}$. Then $a \cdot v \leq -x_\beta$. Choose $c \in A_{f(\beta)}$ such that $a \cdot v \leq c \leq -x_\beta$. Then $a \cdot v - c = 0$ and $u = a \cdot x_\beta \leq a - c$. Since $a - c \in A_{S\cap\beta}$, there is $r \in A_{S\cap\beta} \cap A_{T\cap\beta}$ such that $a - c \leq r$ and $v \leq -r$.

This completes the induction and (ii)⇒(iii) of the theorem follows. □

**Remark 2.6.** It follows from the proof of the last theorem that $A$ is tightly $\kappa$-filtered iff it has a tight $\kappa$-filtration indexed by $|A|$. □

The assumption $\kappa > \aleph_0$ was only needed for this theorem. From now on we only assume $\kappa$ to be regular and infinite. The following corollary is very useful when one wants to show that some Boolean algebra is not tightly $\kappa$-filtered.

**Corollary 2.7.** *Let $\kappa$ be an infinite regular cardinal. If a Boolean algebra $A$ is tightly $\kappa$-filtered, then there is a function $f : A \to [A]^{<\kappa}$ such that for any two sets $X, Y \subseteq A$ which are closed under $f$, $\langle X \cup Y\rangle \leq_\kappa A$.*

*Proof.* By Theorem 2.5 respectively Theorem 2.4, there is a subset $\mathcal{C}$ of $[A]^{<\kappa}$ such that $A = \bigcup \mathcal{C}$ and for each $\mathcal{S} \subseteq \mathcal{C}$, $\langle \bigcup S \rangle \leq_\kappa A$. For each $a \in A$ choose $f(a) \in \mathcal{C}$ such that $a \in f(a)$. $f$ works for the corollary. □

Note that the function $f$ constructed in the proof above has the following property: Whenever $\mathcal{F}$ is a family of subsets of $A$ which are closed under $f$, then $\langle \bigcup \mathcal{F} \rangle \leq_\kappa A$. The existence of such a function characterizes tight $\kappa$-filteredness since the family $\mathcal{C}$ of subsets of $A$ which are closed under $f$ and of size $\leq \kappa$ works for (viii) in Theorem 2.5. It would be interesting to know whether the existence of a function as in Corollary 2.7 already characterizes tight $\kappa$-filteredness.

Theorem 2.5 also gives

**Corollary 2.8.** *a) Every Boolean algebra $A$ of size $\kappa$ is tightly $\kappa$-filtered.*

*b) Every Boolean algebra of size $\kappa^+$ which has the $\kappa$-FN is tightly $\kappa$-filtered.*

*c) Every tightly $\kappa$-filtered Boolean algebra has the $\kappa$-FN.*

*d) If a Boolean algebra $A$ is a retract of a tightly $\kappa$-filtered Boolean algebra $B$, then $A$ is tightly $\kappa$-filtered, too.*

*Proof.* a) follows immediately from (i) in Theorem 2.5 respectively from (ii) in Theorem 2.4.

For b) let $A$ be a Boolean algebra of size $\kappa^+$ which has the $\kappa$-FN. By Lemma 1.15, $A$ is $\kappa$-filtered. Let $\mathcal{S}$ be a $\kappa$-skeleton of $A$ consisting of $\kappa$-subalgebras. In $\mathcal{S}$ choose a strictly increasing sequence $(A_\alpha)_{\alpha<\kappa^+}$ such that $A = \bigcup_{\alpha<\kappa^+} A_\alpha$ and for all $\alpha < \kappa^+$, $|A_\alpha| = \kappa$. By (i) of Theorem 2.5 respectively (ii) of Theorem 2.4, $A$ is tightly $\kappa$-filtered.

c) follows easily from (v) of Theorem 2.5, respectively (v) of Theorem 2.4.

For d) let $p : B \to A$ and $e : A \to B$ be homomorphisms such that $p \circ e = \mathrm{id}_A$. By Theorem 2.5 respectively Theorem 2.4, $B$ has an additive $\kappa$-skeleton $\mathcal{T}$ consisting of $\kappa$-subalgebras. Let $\mathcal{T}'$ be the set of those elements of $\mathcal{T}$ which are closed under $e \circ p$. It is easy to see that $\mathcal{T}'$ is an additive $\kappa$-skeleton for $B$ as well. Now let

$$\mathcal{S} := \{p[C] : C \in \mathcal{T}'\}.$$

Again, it is easy to see that $\mathcal{S}$ is an additive $\kappa$-skeleton for $A$. We *claim* that $\mathcal{S}$ consists of $\kappa$-subalgebras of $A$.

Let $C \in \mathcal{T}'$ and $a \in A$. Let $Y$ be a cofinal subset of $C \upharpoonright e(a)$ of size $< \kappa$. Then $p[Y]$ is a cofinal subset of $p[C] \upharpoonright a$ of size $< \kappa$. This proves the claim.

By Theorem 2.5, respectively Theorem 2.4, $A$ is tightly $\kappa$-filtered. □

## 3. Stone spaces of tightly $\kappa$-filtered Boolean algebras

The implication (i)⇒(viii) and the proof of (viii)⇒(i) of Theorem 2.5 show that for a tightly $\kappa$-filtered Boolean algebra there is a lot of freedom in the choice of a tight $\kappa$-filtration of $A$. This fact allows it to generalize some results of Koppelberg ([17]) on Stone spaces of projective Boolean algebras to Stone spaces of tightly $\kappa$-filtered Boolean algebras. Koppelberg used her results to show that for each regular uncountable cardinal $\lambda$ there are only $2^{<\lambda}$ pairwise non-isomorphic projective Boolean algebras of size $\lambda$. However, this does not work for tightly $\kappa$-filtered Boolean algebras for $\kappa > \aleph_0$. In [5] it will be proved that for every regular $\lambda$ there are $2^\lambda$ pairwise non-isomorphic Boolean algebras of size $\lambda$ which are tightly $\sigma$-filtered. Recall that tightly $\sigma$-filtered Boolean algebras are tightly $\kappa$-filtered for every uncountable regular $\kappa$.

Let $A$ be a tightly $\kappa$-filtered Boolean algebra of size $\lambda$ and $X$ its Stone space. We are interested in the subspace of $X$ of points of small character.

**Definition 3.1.** Let $M_\lambda$ be the subspace of $X$ that consists of the ultrafilters of $A$ which have character $< \lambda$. For Boolean algebras $B \leq C$ an ultrafilter $p$ of $B$ *splits* in $C$ iff there are distinct ultrafilters $q$ and $r$ of $C$ both extending $p$. □

Note that $p$ splits in $C$ iff there is $c \in C$ such that $p \cup \{c\}$ and $p \cup \{-c\}$ both have the finite intersection property.

**Theorem 3.2.** *Let $A$ be a tightly $\kappa$-filtered Boolean algebra of size $\lambda$ where $\kappa < \lambda$, $\lambda$ is regular, and $|\delta|^{<\kappa} < \lambda$ holds for every $\delta < \lambda$. Let $X$ and $M_\lambda$ be as above. Then $M_\lambda$ is an intersection of subsets of $X$ which are unions of less than $\kappa$ clopen sets and is determined by a subalgebra $B$ of $A$ of size $< \lambda$, i.e., there is $B \leq A$ such that $|B| < \lambda$ and $p \cap B$ does not split in $A$ for any $p \in M_\lambda$.*

*Proof.* For the first assertion it is enough to show that for every point $p$ in the complement of $M_\lambda$ there is a set $a_p \subseteq X \setminus M_\lambda$ such that $p \in a_p$ and $a_p$ is the intersection of less than $\kappa$ clopen subsets of $X$.

Let $p \in X \setminus M_\lambda$. Then there is a $\kappa$-filtration $(A'_\alpha)_{\alpha<\lambda}$ of $A$ such that the following hold for all $\alpha < \lambda$:

a) $p \cap A'_\alpha$ splits in $A'_{\alpha+1}$
b) $A'_{\alpha+1}$ is $\kappa$-generated, but not $< \kappa$-generated over $A'_\alpha$.

This filtration can be constructed as in the proof of (viii)⇒(i) of Theorem 2.5 using the fact $\chi(p) = \lambda$ to get a) together with some extra care to get b). Now this filtration can easily be refined to a tight $\kappa$-filtration $(A_\alpha)_{\alpha<\lambda}$ such that $p \cap A_\alpha$ splits in $A_{\alpha+1}$ for every ordinal $\alpha < \lambda$ of cofinality $\geq \kappa$.

A moment's reflection shows that for all $\alpha < \lambda$ the set $a_\alpha$ of ultrafilters of $A_\alpha$ which split in $A_{\alpha+1}$ is an intersection of less than $\kappa$ clopen sets in the Stone space of $A_\alpha$. More exactly: Let $x \in A_{\alpha+1}$ be such that $A_\alpha(x) = A_{\alpha+1}$. An ultrafilter $q$ of $A_\alpha$ splits in $A_{\alpha+1}$ iff $q \cup \{x\}$ and $q \cup \{-x\}$ both are centered. Let $I_x$ and $I_{-x}$ be cofinal subsets of size $< \kappa$ of $A_\alpha \restriction x$ and $A_\alpha \restriction -x$ respectively. Now $q \cup \{x\}$ and $q \cup \{-x\}$ both are centered iff $q$ is disjoint from $I_x \cup I_{-x}$. But this holds iff the point $q$ in the Stone space of $A_\alpha$ is contained in the intersection of the clopen sets corresponding to complements of elements of $I_x \cup I_{-x}$.

For every $\alpha < \lambda$ let $I_\alpha$ be a subset of $A_\alpha$ of size $< \kappa$ which generates the filter corresponding to $a_\alpha$.

W.l.o.g. we may assume that the underlying set of $A$ is $\lambda$. Let

$$S := \{\alpha < \lambda : \alpha \text{ is a limit ordinal of cofinality } \geq \kappa \text{ and the underlying set of } A_\alpha \text{ is } \alpha\}.$$

Since $\lambda$ is a regular cardinal larger than $\kappa$, $S$ is a stationary subset of $\lambda$. Let $f : \lambda \longrightarrow \lambda$ be the mapping which assigns to each $\alpha < \lambda$ the least upper bound of $I_\alpha$. Then $f$ is regressive on $S$. Hence there is a stationary subset $T$ of $S$ such that $f$ is constant on $T$. Let $\delta$ be the value of $f$ on $T$. Since $\delta$ has less than $\lambda$ subsets of size $< \kappa$, there is a stationary subset $U$ of $T$ such that the mapping $F : \alpha \longmapsto I_\alpha$ is constant on $U$. Let $I$ be the value of $F$ on $U$ and let $a_p$ be the corresponding closed subset of $X$ which is an intersection of less than $\kappa$ clopen sets. For every ultrafilter $q \in a_p$ and every $\alpha \in U$, $q \cap A_\alpha$ splits in $A_{\alpha+1}$. Therefore each $q \in a_p$ has character $\lambda$. Hence $a_p \subseteq X \setminus M_\lambda$. Finally, $p \in a_p$ by construction. This proves the first assertion of the theorem.

For the second assertion suppose that $M_\lambda$ is not determined by a subalgebra of $A$ of size less than $\lambda$. By a similar argument as above, get a tight $\kappa$-filtration $(A_\alpha)_{\alpha<\lambda}$ such that for every ordinal $\alpha < \lambda$ of cofinality $\geq \kappa$ there is an ultrafilter $p \in M_\lambda$ such that $p \cap A_\alpha$ splits in $A_{\alpha+1}$. As above, there is a stationary subset $U$ of $\lambda$ consisting of ordinals of cofinality $\geq \kappa$ and a subset $I$ of $A$ of size $< \kappa$ such that for every $\alpha \in U$ the filter generated by $I$ in $A_\alpha$ corresponds to the closed subset of the Stone space of $A_\alpha$ of those ultrafilters which split in $A_{\alpha+1}$. Let $a$ be the closed subset of $X$ corresponding to $I$. $a$ is an intersection of less than $\kappa$ clopen sets. By construction, $a \cap M_\lambda$ is non-empty. But all points in $M_\lambda$ have character less than $\lambda$ and all points in $a$ have character $\lambda$ because $\lambda$ is regular. Thus $M_\lambda$ and $a$ are disjoint. This contradicts the choice of the filtration. □

## 4. Boolean algebras that are rc-filtered, but not tightly $\kappa$-filtered

In this section the arguments will be mainly topological. Let us collect some topological characterizations of the Stonean duals of $\kappa$-embeddings.

**Lemma 4.1.** *Let $A$ be a subalgebra of the Boolean algebra $B$. Let $X$ and $Y$ be the Stone spaces of $A$ and $B$ respectively. Let $\phi : Y \to X$ be the Stonean dual of the inclusion of $A$ into $B$. The following statements are equivalent:*

(i) $A \leq_\kappa B$
(ii) *For each clopen set $b \subseteq Y$, $\chi(\phi[b], X) < \kappa$.*
(iii) *For each closed set $b \subseteq Y$ such that $\chi(b, Y) < \kappa$, $\chi(\phi[b], X) < \kappa$.*

*Proof.* Stone duality. □

Recall that for a closed subset $a$ of topological space $X$ the pseudo-character of $a$ is the minimal size of an open family $\mathcal{F}$ in $X$ such that $\bigcap \mathcal{F} = a$. For a Boolean space it sufficient to consider clopen families $\mathcal{F}$. The pseudo-character of $a$ equals the character of $a$ if $X$ is compact.

The concept of a symmetric power of a topological space was used by Ščepin in order to get an openly generated space that is not Dugundji or, in terms of Boolean algebras, to get a Boolean algebra that is rc-filtered but not projective. We will give a slight generalizion of his result.

**Definition 4.2.** Let $X$ be a topological space. Let $\sim_X$ be the equivalence relation on $X^2$ that identifies $(x,y)$ and $(y,x)$ for all $x,y \in X$. Let $\mathrm{SP}^2(X) := X^2/\sim_X$. If $X$ is the Stone space of the Boolean algebra $A$, then $\mathrm{SP}^2(X)$ is also a Boolean space and the algebra of clopen subsets of $\mathrm{SP}^2(X)$ corresponds to the subalgebra $\mathrm{SP}^2(A)$ of $A \oplus A$ consisting of those elements which are fixed by the automorphism of $A \oplus A$ that interchanges the two copies of $A$. □

**Lemma 4.3.** *(Ščepin, see* [15]*)* $\mathrm{SP}^2$ *is a covariant functor from the category of Boolean algebras into itself where the definition of* $\mathrm{SP}^2$ *on homomorphisms is the natural one. Let $A$ be a Boolean algebra. Then the embedding* $\mathrm{SP}^2(A) \to A \oplus A$ *is relatively complete.* $\mathrm{SP}^2$ *is continuous, i.e., if $(A_\alpha)_{\alpha<\lambda}$ is an ascending chain of subalgebras of $A$, then*

$$\mathrm{SP}^2(\bigcup_{\alpha<\lambda} A_\alpha) = \bigcup_{\alpha<\lambda} \mathrm{SP}^2(A_\alpha).$$

$\mathrm{SP}^2$ *preserves cardinalities, i.e., if $A$ is infinite, then* $|A|=|\mathrm{SP}^2(A)|$. $\mathrm{SP}^2(A)$ *is rc-filtered provided that $A$ is.* □

It turns out that $\mathrm{SP}^2(\mathrm{Fr}(\lambda))$ is not tightly $\kappa$-filtered if $\lambda$ is large enough. This will follow easily from

**Lemma 4.4.** *Let $A$, $B$, and $C$ be infinite Boolean algebras such that the Stone space of $A$ has character $\geq \kappa$.*

*Then*

$$\langle \mathrm{SP}^2(A \oplus B) \cup \mathrm{SP}^2(A \oplus C)\rangle \not\leq_\kappa \mathrm{SP}^2(A \oplus B \oplus C).$$

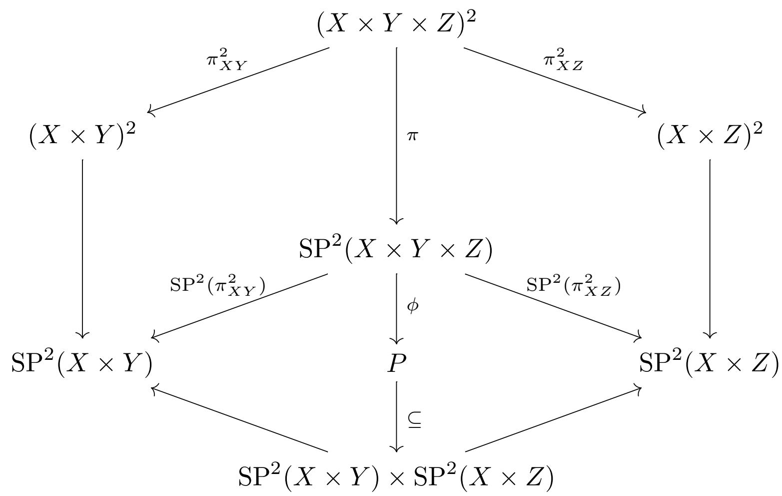
*Proof.* We prove the topological dual. Let $X$, $Y$, and $Z$ be the Stone spaces of $A$, $B$, and $C$ respectively. To commence we introduce names for several mappings. Let $\pi^2_{XY}$ and $\pi^2_{XZ}$ denote the projections of $(X \times Y \times Z)^2$ onto $(X \times Y)^2$ and $(X \times Z)^2$ respectively. Let $\pi$ denote the quotient map from $(X \times Y \times Z)^2$ onto $\mathrm{SP}^2(X \times Y \times Z)$. It follows from Lemma 4.3 that $\pi$ is open. Let $\pi_{XY}$ and $\pi_{XZ}$ denote the projections of $X \times Y \times Z$ onto $X \times Y$ and $X \times Z$ respectively. Now $\mathrm{SP}^2(\pi_{XY})$ and $\mathrm{SP}^2(\pi_{XZ})$ are also defined. Let

$$\phi : \mathrm{SP}^2(X \times Y \times Z) \to \mathrm{SP}^2(X \times Y) \times \mathrm{SP}^2(X \times Z);$$
$$p \mapsto (\mathrm{SP}^2(\pi_{XY})(p), \mathrm{SP}^2(\pi_{XZ})(p))$$

and $P := \mathrm{Im}\,\phi$. Note that $\phi$ is the Stonean dual of the inclusion from

$$\langle \mathrm{SP}^2(A \oplus B) \cup \mathrm{SP}^2(A \oplus C)\rangle$$

into $\mathrm{SP}^2(A \oplus B \oplus C)$. The picture looks like this:




Here the mappings that are not labeled are the natural ones.

Now let $U_1, U_2 \subseteq Y$ and $V_1, V_2 \subseteq Z$ be non-empty, clopen, and disjoint.

*Claim 1:* $\pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$ is clopen in $\mathrm{SP}^2(X \times Y \times Z)$ but $(\phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$ has character $\geq \kappa$ in $P$.

This claim together with Lemma 4.1 proves the lemma. For its proof we need

*Claim 2:*

$$\begin{aligned} W &:= (\phi^{-1} \circ \phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2] \\ &= \pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2] \cup \bigcup_{x \in X} \pi[\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1]. \end{aligned}$$

Proof of Claim 2: Let $(a_1, b_1, c_1, a_2, b_2, c_2)$ be such that $\pi(a_1, b_1, c_1, a_2, b_2, c_2)$ is contained in $W$ but not in $\pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$. Then there is $(a_1', b_1', c_1', a_2', b_2', c_2') \in X \times U_1 \times V_1 \times X \times U_2 \times V_2$ such that

$$(\phi \circ \pi)(a_1, b_1, c_1, a_2, b_2, c_2) = (\phi \circ \pi)(a_1', b_1', c_1', a_2', b_2', c_2').$$

We may assume $a_1 = a_1'$ and $a_2 = a_2'$. Now the following holds: $\{b_1, b_2\} = \{b_1', b_2'\}$, $\{c_1, c_2\} = \{c_1', c_2'\}$, $b_1' \neq b_2'$, $c_1' \neq c_2'$, and hence $c_1 \neq c_2$ and $b_1 \neq b_2$.

Suppose $a_1 \neq a_2$. In this case

$$((a_1, b_1), (a_2, b_2)) \sim_{X \times Y} ((a_1', b_1'), (a_2', b_2'))$$

and

$$((a_1, c_1), (a_2, c_2)) \sim_{X \times Z} ((a_1', c_1'), (a_2', c_2')).$$

Moreover, $b_i = b_i'$ and $c_i = c_i'$ for $i = 1, 2$, and hence

$$\pi(a_1, b_1, c_1, a_2, b_2, c_2) \in \pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2],$$

a contradiction. Thus, $a_1 = a_2$. Since $\{b_1, b_2\} = \{b_1', b_2'\}$ and $\{c_1, c_2\} = \{c_1', c_2'\}$,

$$(a_1, b_1, c_1, a_2, b_2, c_2) \sim_{X \times Y \times Z} (a_1', b_1', c_2', a_2', b_2', c_1').$$

Therefore

$$(a_1,b_1,c_1,a_2,b_2,c_2)\in\bigcup_{x\in X}\pi[\{x\}\times U_1\times V_2\times\{x\}\times U_2\times V_1].$$

Conversely, let $a\in X$, $b_i\in U_i$, and $c_i\in V_i$ for $i=1,2$. Now

$$\begin{aligned}(\phi\circ\pi)(a,b_1,c_2,a,b_2,c_1)&=(\phi\circ\pi)(a,b_1,c_1,a,b_2,c_2)\\&\in(\phi\circ\pi)[X\times U_1\times V_1\times X\times U_2\times V_2].\end{aligned}$$

This finishes the proof of Claim 2.

Proof of Claim 1: $\pi[X\times U_1\times V_1\times X\times U_2\times V_2]$ is clopen in $\mathrm{SP}^2(X\times Y\times Z)$ since

$$\begin{aligned}&(\pi^{-1}\circ\pi)[X\times U_1\times V_1\times X\times U_2\times V_2]\\&\qquad=(X\times U_1\times V_1\times X\times U_2\times V_2)\cup(X\times U_2\times V_2\times X\times U_1\times V_1)\end{aligned}$$

is clopen in $(X\times Y\times Z)^2$.

For the character part of Claim 1 let $\Delta^2[X]$ be the diagonal $\{(x,x):x\in X\}$ of $X^2$. Now

$$\begin{aligned}&\chi((\phi\circ\pi)[X\times U_1\times V_1\times X\times U_2\times V_2],P)\\&\geq\chi\left(\bigcup_{x\in X}\pi[\{x\}\times U_1\times V_2\times\{x\}\times U_2\times V_1],\mathrm{SP}^2(X\times Y\times Z)\right)\\&\geq\chi\Bigg(\bigcup_{x\in X}(\{x\}\times U_1\times V_2\times\{x\}\times U_2\times V_1)\\&\qquad\cup\bigcup_{x\in X}(\{x\}\times U_2\times V_1\times\{x\}\times U_1\times V_2),(X\times Y\times Z)^2\Bigg)\\&\geq\chi(\Delta^2[X],X^2)\geq\chi(X).\end{aligned}$$

Here the last inequality can be seen as follows. Let $\mu:=\chi(\Delta^2[X],X^2)$ and let $\{U^\alpha:\alpha<\mu\}$ be a local base at $\Delta^2[X]$. For each $x\in X$ and each $\alpha<\mu$ pick an open set $U^\alpha_x\subseteq X$ containing $x$ such that $(U^\alpha_x)^2\subseteq U^\alpha$. Now $(\bigcap_{\alpha<\mu}U^\alpha_x)^2=\bigcap_{\alpha<\mu}(U^\alpha_x)^2\subseteq\Delta^2[X]$. Hence $\bigcap_{\alpha<\mu}U^\alpha_x=\{x\}$. Thus $x$ has pseudo-character $\leq\mu$. Since $X$ is compact, $x$ has character $\leq\mu$. □

Now we are ready to prove a theorem which yields the promised examples of rc-filtered Boolean algebras which are not tightly $\kappa$-filtered.

**Theorem 4.5.** *Let $\kappa$ and $\lambda$ be regular.* $\mathrm{SP}^2(\mathrm{Fr}\,\lambda)$ *is tightly $\kappa$-filtered iff $\lambda\leq\kappa^+$.*

*Proof.* $A:=\mathrm{SP}^2(\mathrm{Fr}\,\lambda)$ is rc-filtered by Lemma 4.3. In particular, $A$ is $\kappa$-filtered for every regular cardinal $\kappa$. For $\lambda\leq\kappa^+$, $|A|\leq\kappa^+$. Hence, by the characterization of tightly $\kappa$-filtered Boolean algebras, $A$ is tightly $\kappa$-filtered. This proves the easy implication of the theorem.

Now let $\lambda>\kappa^+$. Suppose $A$ is tightly $\kappa$-filtered. Then there is a function $f:A\to[A]^{<\kappa}$ as in Corollary 2.7. For $S\subseteq\lambda$ let $\mathrm{SP}(S):=\mathrm{SP}^2(\mathrm{Fr}\,S)$ and consider this algebra as a subalgebra of $A$ in the obvious way. Since $\mathrm{SP}^2$ is continuous and cardinal preserving, there are disjoint sets $S, T \in [\lambda]^{\kappa^+}$ such that $\mathrm{SP}(S)$ and $\mathrm{SP}(S \cup T)$ are closed under $f$. Choose $S' \subseteq S \cup T$ such that $\mathrm{SP}(S')$ is closed under $f$ and $|S' \cap S| = |S' \cap T| = \kappa$. Let $S_0 := S' \cap S$ and $T_0 := S' \cap T$. Finally, choose $S_1 \in [S]^\kappa$ disjoint from $S_0$ such that $\mathrm{SP}(S_0 \cup S_1)$ is closed under $f$. Since $\mathrm{SP}(S_0 \cup S_1)$ and $\mathrm{SP}(S_0 \cup T_0)$ are closed under $f$ and by the choice of $f$,

$$\langle \mathrm{SP}(S_0 \cup S_1) \cup \mathrm{SP}(S_0 \cup T_0) \rangle \leq_\kappa A.$$

This contradicts Lemma 4.4. □

Clearly, this theorem implies

**Corollary 4.6.** *For each regular cardinal $\kappa$ there is a Boolean algebra $A$ such that $A$ is rc-filtered but not tightly $\kappa$-filtered.* □

## 5. Complete Boolean algebras and tight $\sigma$-filtrations

Fuchino and Soukup ([11]) have shown that there may be arbitrarily large complete Boolean algebras which are $\sigma$-filtered. More exactly, if CH holds and $0^\sharp$ does not exist, then all complete c.c.c. Boolean algebras are $\sigma$-filtered. In this section, we look at the stronger property of having a tight $\sigma$-filtration. It turns out that no infinite complete Boolean algebra of size larger than $\aleph_2$ is tightly $\sigma$-filtered. It is sufficient to prove that the completion of the free Boolean algebra over $\aleph_3$ generators has no tight $\sigma$-filtration, since the Balcar-Franěk Theorem implies that this algebra is a retract of every complete Boolean algebra of size larger than $\aleph_2$.

**Definition 5.1.** For a set $X$ let the *Cohen algebra* $\mathbb{C}(X)$ over $X$ be the completion of the free Boolean algebra $\mathrm{Fr}(X)$ over $X$. For $X \subseteq Y$, $\mathbb{C}(X)$ will be regarded as a complete subalgebra of $\mathbb{C}(Y)$ in the obvious way. □

**Theorem 5.2.** $\mathbb{C}(\aleph_3)$ *is not tightly $\sigma$-filtered.*

The proof of this theorem uses

**Lemma 5.3.** *Let $Z \subseteq \mathbb{R}$ be uncountable, $\chi$ sufficiently large, and $M_0, M_1 \preccurlyeq H_\chi$ such that $Z \subseteq M_0 \cap M_1$ and $\aleph_3 \cap (M_1 \setminus M_0)$ and $\aleph_3 \cap (M_0 \setminus M_1)$ are infinite. Then*

$$\langle (\mathbb{C}(\aleph_3) \cap M_0) \cup (\mathbb{C}(\aleph_3) \cap M_1) \rangle \not\leq_\sigma \mathbb{C}(\aleph_3).$$

*Proof.* We may assume that $\aleph_3 \subseteq \mathbb{C}(\aleph_3)$ and the canonical complete generators of $\mathbb{C}(\aleph_3)$ are precisely the elements of $\aleph_3$. Note that $A := \mathbb{C}(\aleph_3) \cap M_0 \leq \mathbb{C}(\aleph_3 \cap M_0)$ and $B := \mathbb{C}(\aleph_3) \cap M_1 \leq \mathbb{C}(\aleph_3 \cap M_1)$. Let $R := \aleph_3 \cap M_0 \cap M_1$, $S := \aleph_3 \cap (M_0 \setminus M_1)$, and $T := \aleph_3 \cap (M_1 \setminus M_0)$. Let $A_0 := \mathrm{Fr}(S) \leq A$, $B_0 := \mathrm{Fr}(T) \leq B$, and $C_0 := \mathrm{Fr}(R) \leq A \cap B$. Fix maximal antichains $(x_q)_{q \in \mathbb{Q}} \in {}^{\mathbb{Q}}A_0$ and $(y_q)_{q \in \mathbb{Q}} \in {}^{\mathbb{Q}}B_0$. Even though $S$ and $T$ are typically not elements of $M_0$, respectively $M_1$, there are infinite sets $S' \subseteq S$ and $T' \subseteq T$ such that $S' \in M_0$ and $T' \in M_1$. For example, for $\alpha \in S$ the set $\{\alpha + n : n \in \omega\}$ is a subset of $S$ and an element of $M_0$. Therefore we may assume $(x_q)_{q \in \mathbb{Q}} \in M_0$ and $(y_q)_{q \in \mathbb{Q}} \in M_1$. Let $c := \sum\{x_p \cdot y_q : p, q \in \mathbb{Q} \wedge p \geq q\}$.

For each $r \in \mathbb{R}$ let $c_r := \sum\{x_p \cdot y_q : p, q \in \mathbb{Q} \wedge p \geq r \geq q\}$. Note that for all $r \in \mathbb{R} \cap M_0 \cap M_1$,

$$c_r = \sum\{x_p : p \in \mathbb{Q} \wedge p \geq r\} \cdot \sum\{y_q : q \in \mathbb{Q} \wedge r \geq q\} \in \langle A \cup B\rangle.$$

*Claim 1*: Suppose $r_0 < \cdots < r_n$ is a finite sequence of reals and $(a_i)_{i<n} \in {}^nA$ and $(b_i)_{i<n} \in {}^nB$ are such that $\sum_{i<n} a_i b_i \leq c$. Then $\sum_{i\leq n} c_{r_i} \not\leq \sum_{i<n} a_i b_i$.

Proof of Claim 1: Suppose $\sum_{i\leq n} c_{r_i} \leq \sum_{i<n} a_i b_i$. Fix rational numbers $q_i$, $i < n$, such that $r_0 < q_0 < r_1 < \cdots < q_{n-1} < r_n$. For $i < n$ let $(a_i^k)_{k\in\omega} \in {}^\omega A_0$, $(d_i^k)_{k\in\omega}, (e_i^l)_{l\in\omega} \in {}^\omega C_0$, and $(b_i^l)_{l\in\omega} \in {}^\omega B_0$ be such that $a_i = \sum_{k\in\omega} a_i^k d_i^k$ and $b_i = \sum_{l\in\omega} b_i^l e_i^l$. Now $a_i \cdot b_i = \sum_{k,l\in\omega} a_i^k d_i^k e_i^l b_i^l$. Inductively define sequences $(i_j)_{j<n} \in {}^nn$ and $(e_j)_{j<n} \in {}^nC_0$ as follows:

Let $i_0 < n$ be such that $b_{i_0} \cdot y_q \neq 0$ for some $q < q_0$. $i_0$ exists since $c_{r_0} \leq \sum_{i<n} a_i b_i$. Let $l \in \omega$ be such that $b_{i_0}^l \cdot y_q \neq 0$ for some $q < q_0$ and set $e_0 := e_{i_0}^l$. Since $\sum_{k,l\in\omega} a_i^k d_i^k e_i^l b_i^l = a_i \cdot b_i \leq c$, for each $k \in \omega$ with $a_{i_0}^k \cdot x_p \neq 0$ for some $p \geq q_0$, $d_{i_0}^k \cdot e_0 = 0$. Therefore $e_0 a_{i_0} b_{i_0} \leq \sum_{p<q_0} x_p$.

Now suppose $j + 1 < n$ and $i_j$ and $e_j$ have been already defined such that

$$(*) \quad e_j \cdot (a_{i_0} b_{i_0} + \cdots + a_{i_j} b_{i_j}) \leq \sum_{p<q_j} x_p.$$

Let $i_{j+1} < n$ be such that $e_j b_{i_{j+1}} y_q \neq 0$ for some $q \in [q_j, q_{j+1})$. $i_{j+1}$ exists since $c_{r_{j+1}} \leq \sum_{i<n} a_i b_i$. By $(*)$, $i_{j+1} \notin \{i_0, \ldots, i_j\}$.

Let $l \in \omega$ be such that $e_j e_{i_{j+1}}^l b_{i_{j+1}}^l y_q \neq 0$ for some $q < q_{j+1}$ and put $e_{j+1} := e_j \cdot e_{i_{j+1}}^l$. Again, since $\sum_{k,l\in\omega} a_i^k d_i^k e_i^l b_i^l = a_i \cdot b_i \leq c$, for each $k \in \omega$ with $a_{i_{j+1}}^k x_p \neq 0$ for some $p \geq q_{j+1}$, $d_{i_{j+1}}^k \cdot e_{j+1} = 0$. Therefore, $e_{j+1} a_{i_{j+1}} b_{i_{j+1}} \leq \sum_{p<q_{j+1}} x_p$.

By construction, $(i_j)_{j<n}$ is a permutation of $n$. It follows that $e_{n-1} \cdot \sum_{i<n} a_i b_i \leq \sum_{p<q_{n-1}} x_p$, contradicting the assumption $c_{r_n} \leq \sum_{i<n} a_i b_i$. This proves Claim 1.

*Claim 2*: $\langle A \cup B\rangle \restriction c$ does not have a countable cofinal subset.

Proof of Claim 2: Let $D$ be a countable subset of $\langle A \cup B\rangle \restriction c$. Every element $d \in D$ is of the form $\sum_{i<n} a_i b_i$ for some $n \in \omega$, $(a_i)_{i<n} \in {}^nA$, and $(b_i)_{i<n} \in {}^nB$. Therefore by Claim 1, for every $d \in D$ there are only finitely many $r \in \mathbb{R}$ such that $c_r \leq d$. Thus, there is $r \in Z$ such that $c_r \not\leq d$ for every $d \in D$. Since $c_r \in \langle A \cup B\rangle$ and $c_r \leq c$, $D$ is not cofinal in $\langle A \cup B\rangle \restriction c$. This proves Claim 2 and concludes the proof of Lemma 5.3. $\square$

*Proof of Theorem 5.2.* Assume on the contrary that $\mathbb{C}(\aleph_3)$ is tightly $\sigma$-filtered. Then by Corollary 2.7, there is a function $f : \mathbb{C}(\aleph_3) \to [\mathbb{C}(\aleph_3)]^{\aleph_0}$ such that for all subalgebras $A, B \leq C(\aleph_3)$ which are closed under $f$, $\langle A \cup B\rangle \leq_\sigma \mathbb{C}(\aleph_3)$.

Let $Z \subseteq \mathbb{R}$ be of size $\aleph_1$. Let $\chi$ be sufficiently large and fix $N_0, N_1 \preccurlyeq H_\chi$ such that $f \in N_0 \cap N_1$, $Z \subseteq N_0 \cap N_1$, $N_0 \subseteq N_1$, and $|N_0 \cap \aleph_3| = |(N_1 \setminus N_0) \cap \aleph_3| = \aleph_2$. Let $M_1 \preccurlyeq N_1$ be such that $f \in M_1$, $Z \subseteq M_1$, and $|M_1 \cap N_0 \cap \aleph_3| = |M_1 \cap (N_1 \setminus N_0) \cap \aleph_3| = \aleph_1$. Finally, let $M_0 \preccurlyeq N_0$ be such that $f \in M_0$, $Z \subseteq M_0$, and $|M_0 \cap \aleph_3| = |(M_0 \setminus M_1) \cap \aleph_3| = \aleph_1$. Since $f \in M_0 \cap M_1$, $\mathbb{C}(\aleph_3) \cap M_0$ and $\mathbb{C}(\aleph_3) \cap M_1$ are closed under $f$. However, by Lemma 5.3,

$$\langle(\mathbb{C}(\aleph_3)\cap M_0)\cup(\mathbb{C}(\aleph_3)\cap M_1)\rangle\not\leq_\sigma \mathbb{C}(\aleph_3).$$

A contradiction. □

Theorem 5.2 easily implies

**Corollary 5.4.** *No complete Boolean algebra $A$ of size $\geq\aleph_3$ is tightly $\sigma$-filtered.*

*Proof.* Let $A$ be a complete Boolean algebra of size at least $\aleph_3$. By the wellknown Balcar-Franěk Theorem, $\mathrm{Fr}(\aleph_3)$ embeds into $A$. By the completeness of $A$, this embedding extends to $\mathbb{C}(\aleph_3)$. Since $\mathrm{Fr}(\aleph_3)$ is dense in $\mathbb{C}(\aleph_3)$, this extension is an embedding as well. By the completeness of $\mathbb{C}(\aleph_3)$, $\mathbb{C}(\aleph_3)$ is a retract of $A$. Thus, by Corollary 2.8, $\mathbb{C}(\aleph_3)$ is tightly $\sigma$-filtered if $A$ is. Since $\mathbb{C}(\aleph_3)$ fails to be tightly $\sigma$-filtered by Theorem 5.2, so does $A$. □

**Corollary 5.5.** *A complete Boolean algebra $A$ is tightly $\sigma$-filtered iff $A$ has the WFN and $|A|\leq\aleph_2$. In particular, $\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered iff it has the WFN and $2^{\aleph_0}\leq\aleph_2$.*

*Proof.* A Boolean algebra $A$ of size $\leq\aleph_2$ which has the WFN is tightly $\sigma$-filtered by Corollary 2.8. On the other hand, if $A$ is complete and tightly $\sigma$-filtered, then $|A|\leq\aleph_2$ by Corollary 5.4 and $A$ has the WFN by Corollary 2.8. □

5.1. **The pseudo product of Cohen forcings.** While so far the only known way to obtain a model of $\neg\,\mathrm{CH}+\mathrm{WFN}(\mathfrak{P}(\omega))$ is to add Cohen reals to a model of CH, there is some freedom in the choice of the iteration used for adding the Cohen reals. In [10] Fuchino, Shelah, and Soukup introduced a new kind of side-by-side product of partial orders.

**Definition 5.6.** Let $(P_i)_{i\in X}$ be a family of partial orders where each $P_i$ has a largest element $1_{P_i}$. As usual, for $p\in\prod_{i\in X}P_i$ let $\mathrm{supp}(p):=\{i\in X: p(i)\neq 1_{P_i}\}$ be the *support* of $p$. Let $\prod^*_{i\in X}P_i:=\{p\in\prod_{i\in X}P_i: |\mathrm{supp}(p)|\leq\aleph_0\}$ be ordered such that for all $p,q\in\prod^*_{i\in X}P_i$,

$$p\leq q\Leftrightarrow\forall i\in X(p(i)\leq q(i))\wedge|\{i\in X: p(i)\neq q(i)\neq 1_{P_i}\}|<\aleph_0.\quad\square$$

Among other things, Fuchino, Shelah, and Soukup proved the following about this product:

**Lemma 5.7.** *Let $(P_i)_{i\in X}$ be as in the definition above.*
*a) For every $Y\subseteq X$, $\prod^*_{i\in X}\cong\prod^*_{i\in Y}\times\prod^*_{i\in X\setminus Y}$.*
*b) Under CH, $\prod^*_{i\in X}\mathrm{Fn}(\omega,2)$ satisfies the $\aleph_2$-c.c. and is proper.* □

Forcing with $\prod^*_{i\in X}\mathrm{Fn}(\omega,2)$ for some uncountable set $X$ over a model of CH gives a model of the combinatorial principle ♣, as was shown in [10]. ♣ is a prediction principle on $\aleph_1$ and follows from $\diamondsuit$, but is, unlike $\diamondsuit$, consistent with ¬CH. We will show that $\mathfrak{P}(\omega)$ has the WFN after forcing with $\prod^*_{i\in X}\mathrm{Fn}(\omega,2)$ over a model of CH, provided $|X|$ is smaller than $\aleph_\omega$. We will use the well-known

**Lemma 5.8.** *Assume that the partial order $P$ is the union of an increasing chain $(P_\alpha)_{\alpha<\lambda}$ of completely embedded suborders. Let $G$ be $P$-generic over the ground model $M$ and for each $\alpha<\lambda$ let $G_\alpha := P_\alpha \cap G$. If $\lambda$ has uncountable cofinality, then for every real $x \in M[G]$ there is $\alpha<\lambda$ such that $x \in M[G_\alpha]$.* □

A proof of this lemma can be found in [1].

**Theorem 5.9.** *Let $\lambda < \aleph_\omega$ be an uncountable cardinal and suppose CH holds. Let $P := \prod^*_{\alpha<\lambda} \mathrm{Fn}(\omega,2)$. Then*

$$\Vdash_P \mathrm{WFN}(\mathfrak{P}(\omega)) \text{ and } 2^{\aleph_0} = \lambda.$$

*Proof.* Let $M$ be the ground model satisfying CH and let $G$ be $P$-generic over $M$. It follows from Lemma 5.7 that $P$ is cardinal preserving and that the continuum is $\lambda$ in $M[G]$. Throughout this proof we will use Lemma 5.7 without referring to it anymore. For each $X \subseteq \lambda$ with $X \in M$ consider $P_X := \prod^*_{\alpha \in X} \mathrm{Fn}(\omega,2)$ as a suborder of $P$ in the obvious way and let $G_X := P_X \cap G$ and $\mathfrak{P}_X := (\mathfrak{P}(\omega))^{M[G_X]}$. $(\mathfrak{P}_\alpha)_{\alpha\leq\lambda}$ is continuous at limit ordinals of uncountable cofinality by Lemma 5.8.

*Claim.* In $M[G]$: For each $\alpha<\lambda$, $\mathfrak{P}_\alpha \leq_\sigma \mathfrak{P}(\omega)$.

Proof of the claim: We argue in $M[G]$. Let $\alpha<\lambda$. Let $x \in \mathfrak{P}(\omega)$. By $\aleph_2$-c.c. of $P$, in $M$ there is a subset $X$ of $\lambda$ of size $<\aleph_2$ such that $x \in \mathfrak{P}_X$. By Lemma 5.8, in $M$ there is a countable subset $Y$ of $X \setminus \alpha$ such that $x \in M[G_\alpha][G_Y]$. The set $D := \{p \in P_Y : \mathrm{supp}(p) = Y\}$ is dense in $P_Y$. Thus there is $p \in G_Y \cap D$. It is easy to see that $P_Y \downarrow p := \{q \in P_Y : q \leq p\}$ is isomorphic to $\mathrm{Fn}(\omega,2)$. Therefore, there is a Cohen real $r$ over $M[G_\alpha]$ in $M[G]$ such that $x \in M[G_\alpha][r]$. It was shown in [8] and in [23] that

$$M[G_\alpha][r] \models (\mathfrak{P}(\omega) \cap M[G_\alpha]) \restriction x \text{ has countable cofinality.}$$

By properness of $P$, $\mathfrak{P}_\alpha \restriction x$ has countable cofinality in $M[G]$. This finishes the proof of the claim.

Now it follows by induction on the size of $\lambda$ that $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds in $M[G]$. The induction uses Lemma 1.14 and the fact that $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds under CH. □

Applying part b) of Corollary 2.8 we get

**Corollary 5.10.** *Forcing with $\prod^*_{\alpha<\aleph_2} \mathrm{Fn}(\omega,2)$ over a model of CH gives a model of set theory where $\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered.* □

## 6. Automorphisms of $\mathfrak{P}(\omega)/fin$

In [23] Shelah and Steprans showed that in the Cohen model there is a non-trivial automorphism of $\mathfrak{P}(\omega)/fin$, that is, an automorphism which is not induced by a bijection between two cofinite subsets of $\omega$. Koppelberg ([16]) indicated how this result can be proved using the tight $\sigma$-filteredness of $\mathfrak{P}(\omega)/fin$ in the Cohen model.

However, it turns out that it is not necessary to assume that we are working in the Cohen model. Let MA(countable) denote the statement 'Martin's Axiom holds for all countable partial orders'. In [6] the following is proved:

**Lemma 6.1.** *If* $2^{\aleph_0} < \aleph_\omega$*, then* $\mathrm{WFN}(\mathfrak{P}(\omega))$ *implies* MA(*countable*). □

MA(countable) allows it to extend isomorphisms between small $\sigma$-subalgebras of $\mathfrak{P}(\omega)/fin$. More exactly, the following holds:

**Lemma 6.2.** *Let $A$ be a $\sigma$-subalgebra of $\mathfrak{P}(\omega)/fin$ of size $< 2^{\aleph_0}$, $f : A \to \mathfrak{P}(\omega)$ an embedding, and $x, y \in \mathfrak{P}(\omega)$. Then* MA(*countable*) *implies that $f$ extends to an embedding $\overline{f} : A(x) \to \mathfrak{P}(\omega)/fin$ such that $\overline{f}(x) \neq y$.*

*Proof.* This lemma seems to be well known. Except for the '$\overline{f}(x) \neq y$'-part, a proof is contained in [16]. It is easy to get an $\overline{f}$ with $\overline{f}(x) \neq y$ from Koppelberg's argument as well. □

Recall that an ultrafilter $x \subseteq \mathfrak{P}(\omega)/fin$ is a p-point if for every countable set $C \subseteq x$ there is $a \in x$ such that $a \leq b$ for every $b \in C$. Using Lemma 6.2 together with Theorem 2.5, we get

**Theorem 6.3.** *Suppose that $\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered. Then*

a) *$\mathfrak{P}(\omega)/fin$ has $2^{2^{\aleph_0}}$ automorphisms. In particular, there are non-trivial automorphisms of $\mathfrak{P}(\omega)/fin$.*
b) *the automorphism group of $\mathfrak{P}(\omega)/fin$ is simple.*
c) *for any two p-points $x, y \subseteq \mathfrak{P}(\omega)/fin$ there is an automorphism $h$ of $\mathfrak{P}(\omega)/fin$ such that $h[x] = y$.*
d) *for every antichain $(a_n)_{n\in\omega}$ in $\mathfrak{P}(\omega)/fin$, all $n \in \omega$, and all automorphisms $h_n$ of $\mathfrak{P}(\omega)/fin \restriction a_n$ there is an automorphism $h$ of $\mathfrak{P}(\omega)/fin$ such that for every $n \in \omega$, $h \restriction (\mathfrak{P}(\omega)/fin \restriction a_n) = h_n$.*

*Proof.* Assume $\mathfrak{P}(\omega)$ is tightly $\sigma$-filtered. By Corollary 5.4, the continuum is at most $\aleph_2$. By Lemma 2.8, a Boolean algebra of size $\leq \aleph_2$ is tightly $\sigma$-filtered iff it has the WFN. Using the countability of $fin$, it is easily seen that $\mathrm{WFN}(\mathfrak{P}(\omega))$ and $\mathrm{WFN}(\mathfrak{P}(\omega)/fin)$ are equivalent. It follows that $\mathfrak{P}(\omega)/fin$ is tightly $\sigma$-filtered.

For a) note that there are only $2^{\aleph_0}$ trivial automorphism of $\mathfrak{P}(\omega)/fin$. Thus $\mathfrak{P}(\omega)/fin$ has a non-trivial automorphism if it has more than $2^{\aleph_0}$ automorphisms. Under CH, it was essentially shown by Rudin ([18]) that $\mathfrak{P}(\omega)/fin$ has $2^{2^{\aleph_0}}$ automorphisms. Hence we may assume $2^{\aleph_0} = \aleph_2$.

We will construct a family $(h_f)_{f\in{}^{\omega_2}2}$ of pairwise distinct automorphisms of $\mathfrak{P}(\omega)$. By the tight $\sigma$-filteredness of $\mathfrak{P}(\omega)/fin$ together with Lemma 2.5, there is a continuous chain $\mathcal{C}$ of ordertype $\omega_2$ of $\sigma$-subalgebras of $\mathfrak{P}(\omega)/fin$ of size $\aleph_1$ such that $\bigcup\mathcal{C} = \mathfrak{P}(\omega)/fin$. By induction on ${}^{<\omega_2}2$ ordered by inclusion, for every $f \in {}^{<\omega_2}2$ we pick an algebra $C_f \in \mathcal{C}$ and define an automorphism $h_f$ of $C_f$ such that the following two conditions hold:

(i) For each $f \in {}^{\omega_2}2$, $(C_{f\restriction\alpha})_{\alpha<\omega_2}$ and $(h_{f\restriction\alpha})_{\alpha<\omega_2}$ are continuously increasing chains.

(ii) If $f, g \in {}^{<\omega_2}2$ and $f$ and $g$ are incomparable, then so are $h_f$ and $h_g$.

Note that by Lemma 6.1, MA(countable) holds and therefore we can apply Lemma 6.2. The limit steps of the construction are completely determined by the continuity requirements.

Now let $f \in {}^{<\omega_2}2$ and suppose $C_f$ and $h_f$ have already been defined. Using Lemma 6.2, we get two incomparable extensions $g_0^0$ and $g_0^1$ of $h_f$. Of course, typically $g_0^0$ and $g_0^1$ are not automorphisms of their domains and these domains are not elements of $\mathcal{C}$. However, we can perform an induction of length $\omega_1$ using a back-and-forth argument and some book-keeping to get continuously increasing sequences $(g_\alpha^0)_{\alpha<\omega_1}$ and $(g_\alpha^1)_{\alpha<\omega_1}$ of partial isomorphisms of $\mathfrak{P}(\omega)/fin$ such that for all $i \in 2$, $g^i := \bigcup_{\alpha<\omega_1} g_\alpha^i$ is an automorphism of its domain and $\mathrm{dom}(g^0) = \mathrm{dom}(g^1) \in \mathcal{C}$. In this induction Lemma 6.2 is used at the successor steps in order to extend the given partial isomorphisms. Recall that if $A \leq_\sigma \mathfrak{P}(\omega)/fin$, then for every $X \in [\mathfrak{P}(\omega)/fin]^{\aleph_0}$ we still have $A(X) \leq_\sigma \mathfrak{P}(\omega)/fin$ by part d) of Lemma 1.2. This keeps the induction going. For $i \in 2$ let $h_{f^\frown(i)} := g^i$ and $C_{f^\frown(i)} := \mathrm{dom}(g^i)$.

Having succeeded in the construction of $C_f$ and $h_f$ for all $f \in {}^{<\omega_2}2$, for each $f \in {}^{\omega_2}2$ let $h_f := \bigcup_{\alpha<\omega_2} h_{f\restriction\alpha}$. Since $\mathcal{C}$ has ordertype $\omega_2$, for every $f \in {}^{\omega_2}2$, $\mathrm{dom}(h_f) = \mathfrak{P}(\omega)/fin$. Clearly, the family $(h_f)_{f\in{}^{\omega_2}2}$ is as desired. This concludes the proof of part a) of the theorem.

The arguments for part b), c), and d) follow the same pattern as the argument for part a) and use some additional ingredients from the proofs that b), c), and d) hold in the Cohen model. For the Cohen model, b) is due to Fuchino ([4]) and c) and d) are due to Steprans ([24]). $\square$

Note that c) and d) together imply that every sequence $(x_n)_{n\in\omega}$ of p-points in $\mathfrak{P}(\omega)/fin$ can be mapped onto any other sequence $(y_n)_{n\in\omega}$ of p-points by an automorphism ([24]).

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