BASIS THEOREMS FOR CONTINUOUS n-COLORINGS

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ABSTRACT. This article is devoted to the study of continuous colorings of the n-element subsets of a Polish space.

The *homogeneity number* $\mathfrak{hm}(c)$ of an n-coloring $c : [X]^n \to 2$ is the least size of a family of c-homogeneous sets that covers X. An n-coloring is *uncountably homogeneous* if $\mathfrak{hm}(c) > \aleph_0$. Answering a question of B. Miller, we show that for every $n > 1$ there is a finite family $\mathcal{B}$ of continuous n-colorings on 2^ω such that every uncountably homogeneous, continuous n-coloring on a Polish space contains a copy of one of the colorings from $\mathcal{B}$. We also give upper and lower bounds for the minimal size of such a *basis* $\mathcal{B}$.

1. INTRODUCTION

For a class $\mathcal{A}$ of structures with a reasonable notion of embeddability we say that $\mathcal{B} \subseteq \mathcal{A}$ is a *basis* for $\mathcal{A}$ if for every $A \in \mathcal{A}$ there is B is $\mathcal{B}$ such that B embeds into A. A classical example is the class of infinite linear orders and its basis $\{\omega, \omega^*\}$, where ω^* denotes the set ω of natural numbers with the reversed ordering.

Determining bases for classes of infinite or uncountable structures is often difficult. A recent example is the consistency of a finite basis for the class of uncountable linear orders due to Moore [16]. A question that is still open is whether the class of infinite compact spaces can consistently have a 2-element basis. This problem is known as *Efimov's problem* (see [4]).

Kechris, Solecki, and Todorcevic showed that there is a closed graph $\mathcal{G}_0$ on the Cantor space 2^ω such that for every analytic graph $\mathcal{G}$ on a Hausdorff space, either $\mathcal{G}$ admits a Borel measurable coloring with countably many colors or there is a graph homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$ [12]. This result can be considered as a basis theorem for analytic graphs that have an uncountable Borel chromatic number. However, Lecomte [13] proved that in this dichotomy, the graph homomorphisms from $\mathcal{G}_0$ cannot necessarily be chosen to be 1-1 and that a basis in the strict sense, i.e., with

respect to graph embeddings, for the class of uncountably Borel chromatic, analytic graphs has to be of size $2^{\aleph_0}$.

If we consider graphs of a descriptive complexity that is even lower than that of $\mathcal{G}_0$, namely, if we consider open graphs on Polish spaces, we first observe that such a graph admits a Borel measurable coloring with countably many colors if and only if it admits any coloring with countably many colors. As shown in [5], an open graph on a Polish space either contains a clique that is homeomorphic to 2^ω or admits a coloring with countably many colors. In other words, the class of uncountably chromatic, open graphs on Polish spaces has a one element basis, consisting of the complete graph on the Cantor space. But this shows that in the infinite, both the chromatic and the Borel chromatic number of open graphs on Polish spaces are degenerate. A related cardinal invariant is the *co-chromatic* or *homogeneity number* of a graph, i.e., the smallest size of a family F consisting of cliques and independent subsets of the graph such that every vertex is contained in at least one member of F. In [9] it was shown that there is an uncountably homogeneous, clopen graph on 2^ω that embeds into every uncountably homogeneous, clopen graph on any Polish space. It follows that the class of uncountably homogeneous, clopen graphs on Polish spaces has a one-element basis.

A natural generalization of clopen graphs to higher dimensions are continuous colorings of the n-element subsets of a Polish space X, or continuous n-colorings for short. Ben Miller asked whether the class of uncountably homogeneous, continuous 3-colorings on Polish spaces has a finite basis. In Section 5 we answer this question by showing that for every $n > 1$, the class of uncountably homogeneous, continuous n-colorings has a finite basis. The proof of this result requires a generalization of a theorem of Blass [2] about continuous n-colorings on 2^ω to n-colorings on m^ω, which we provide in Sections 3 and 4. Rather than generalizing Blass' original proof, we derive our generalization from Milliken's result about partitions of sets of strongly embedded subtrees of infinite trees [15][1].

In contrast to the situation for uncountably homogeneous, continuous 2-colorings where we have a one-element basis, a basis for the uncountably homogeneous, continuous 3-colorings has at least six elements. This is shown in Section 6. Lower bounds for the size of a basis in higher dimensions are provided in Section 7. The proofs in Section 7 rely on a number of results that say that whenever e is a topological embedding from a perfect set $P \subseteq 2^\omega$ into 2^ω, then there is a perfect set $Q \subseteq P$ on which e behaves very nicely.

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[1]After this article was submitted for publication, the authors got hold of a copy of Todorcevic's recent book on Ramsey spaces [19]. It turns out that our Theorem 4 is an easy corollary of Todorcevic's much more general Theorem 6.13. We will, however, present our original proof.

2. Prelimininaries

Let X be a Hausdorff space. Then the set $[X]^n$ of n-element subsets of X carries a natural topology, namely the topology generated by the sets of the form

$$[U_1, \ldots, U_n] = \{\{x_1, \ldots, x_n\} : (x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n\}$$

where $U_1, \ldots, U_n$ are pairwise disjoint open subsets of X. A *continuous n-coloring on X with k colors* is a continuous map $c : [X]^n \to k$. If we do not mention k, that usually means that we are considering only 2 colors, the simplest nontrivial case. Also, abusing notation, we frequently write $c(x_1, \ldots, x_n)$ instead of $c(\{x_1, \ldots, x_n\})$.

Given an n-coloring $c : [X]^n \to k$, $H \subseteq X$ is *c-homogeneous* if c is constant on $[H]^n$. If c is clear from the context, we write just "homogeneous" instead of "c-homogeneous". The *homogeneity number* $\mathfrak{hm}(c)$ of an n-coloring $c : [X]^n \to k$ is the least size of a family of c-homogeneous sets that covers the underlying set X. The coloring c is *uncountably homogeneous* if $\mathfrak{hm}(c) > \aleph_0$.

Given two continuous n-colorings $c : [X]^n \to 2$ and $d : [Y]^n \to 2$ we say that c *embeds into* d ($c \leq d$) if there a topological embedding $e : X \to Y$ such that for all $\{x_1, \ldots, x_n\} \in [X]^n$, $c(x_1, \ldots, x_n) = d(e(x_1), \ldots, e(x_n))$. Note that $c \leq d$ implies $\mathfrak{hm}(c) \leq \mathfrak{hm}(d)$.

Continuous 2-colorings on Polish spaces came up in the context of convexity numbers of closed subsets of $\mathbb{R}^2$ in [9]. In that article it was shown that there is an uncountably homogeneous, continuous coloring $c_{\min} : [2^\omega]^2 \to 2$ such that for every uncountably homogeneous, continuous coloring c on a Polish space we have $c_{\min} \leq c$ [9, Theorem 10]. In the same paper it was proved that in the so-called *Sacks model*, for every continuous 2-coloring c on a Polish space X we have $\mathfrak{hm}(c) \leq \aleph_1$ while $2^{\aleph_0} = \aleph_2$. A more systematic investigation of continuous colorings on Polish followed in [8].

The results for continuous 2-colorings do not easily generalize to continuous n-colorings with $n > 2$. One of the problems is that there are continuous 3-colorings on 2^ω without uncountable homogeneous sets. Namely, given $\{x, y, z\} \in [2^\omega]^3$ with x, y, z lexicographically increasing, let type($\{x, y, z\}$) = 0 if $\Delta(x, y) < \Delta(y, z)$ and type($\{x, y, z\}$) = 1, otherwise. Here $\Delta(x, y)$ denotes the least $n \in \omega$ such that $x(n) \neq y(n)$. It is easily checked that every uncountable subset of 2^ω contains 3-element sets of both types.

Given a coloring $c : [2^\omega]^3 \to 2$ we call a set $H \subseteq 2^\omega$ *weakly homogeneous* if for all $\{x, y, z\} \in [H]^3$, $c(x, y, z)$ only depends on type(x, y, z). Galvin showed that for every continuous $c : [2^\omega]^3 \to 2$ there is a (nonempty) perfect set $P \subseteq 2^\omega$ that is weakly homogeneous (see [11, Theorem 19.7]). This theorem was later generalized by Blass to continuous n-colorings on 2^ω [2].

3. Strong types and trees

In this section we prove a Ramsey theorem for continuous n-colorings on closed subsets of ω^ω. This theorem is a generalization of Blass' result on continuous n-colorings on 2^ω [2].

Blass defines a notion of *type* of certain n-element subsets of 2^ω and then shows that for every continuous coloring $c : [2^\omega]^n \to 2$ there is a perfect set $P \subseteq 2^\omega$ such that every n-element set $\{x_1, \ldots, x_n\} \subseteq P$ has a type and $c(x_1, \ldots, x_n)$ only depends on the type of $\{x_1, \ldots, x_n\}$.

In the introduction we have stated a simplified definition of the type of a 3-element subset of 2^ω that is equivalent to Blass' definition for the case $n = 3$. The total number of types of n-element subsets of 2^ω is $(n-1)!$. Another way of stating Blass' theorem is that for every $k \in \omega$ and every continuous coloring $c : [2^\omega]^n \to k$ there is a perfect set $P \subseteq 2^\omega$ such that $|c[[P]^n]| \leq (n-1)!$ (see [18]).

The proof of our generalization of Blass' theorem is based on Milliken's theorem on partitions of the collection of strongly embedded subtrees of a fixed finite height of a finitely splitting tree of height ω. Let us first fix some notation concerning trees.

Definition 1. Given a partial order $(P, \leq)$, we use $<$ to denote the relation $\leq \setminus =$. For the purpose of this article, a *tree* is a partial order $(T, \leq)$ with a unique minimal element r, the *root* of T, such that for all $t \in T$ the set $\{s \in T : s < t\}$ is finite and linearly ordered by $\leq$.

The ordertype of $\{s \in T : s < t\}$ is the *height* $\text{height}_T(t)$ of t in the tree T. Since for all $t \in T$ the set $\{s \in T : s < t\}$ is assumed to be finite, its ordertype is simply its size. The *height* of T is the ordinal

$$\text{height}(T) = \sup\{\text{height}_T(t) + 1 : t \in T\},$$

which is always $\leq \omega$ for the trees that we consider here.

A *branch* of a tree T is a maximal linearly ordered subset of T.

Now let $\alpha \leq \omega$. We say that T is an α-*tree* if all branches of T are of length α. Note that every α-tree is of height α. For every $n \in \mathbb{N}$ and every tree T let $T(n)$ denote the set of elements of T of height n.

Let S and T be trees. S is a *subtree* of T if $S \subseteq T$, the order on S is the restriction of the order on T, and S is downward closed in T. S is an α-*subtree* of T if S is both a subtree of T and an α-tree. Given $t \in T$, let T_t be the subtree of T consisting of all $s \in t$ such that $s \leq t$ or $t \leq s$. Let $\text{succ}_T(t)$ denote the set of immediate successors of t in T.

Now let T be an ω-tree and let $\alpha \leq \omega$ be the height of S. S is a *strongly embedded subtree* or, more precisely, a *strongly embedded α-subtree* of T if the following hold:

(1) $S \subseteq T$,

(2) the order on S is the restriction of the order on T,

(3) there is a function $f : \alpha \to \omega$ such that for all $k < \alpha$, $S(k) \subseteq T(f(k))$, and

(4) for every nonmaximal $s \in S$ and every $t \in \mathrm{succ}_T(s)$, $|\mathrm{succ}_S(s) \cap T_t| = 1$.

Here condition (4) says that S reflects the splitting behaviour of T in the sense that for every node $s \in S$ that has any successors in S at all, every immediate successor of s in T is less or equal to exactly one immediate successor in S. In particular, such a node s has the same number of immediate successors in S as in T.

S is a *weakly embedded α-subtree of T* if it satisfies the conditions (1)–(3) above and additionally,

(4)$'$ for every $s \in S$ and every $t \in \mathrm{succ}_T(s)$, $|\mathrm{succ}_S(s) \cap T_t| \leq 1$.

In (4)$'$ we demand that every immediate successor of s in S corresponds in a natural 1-1 way to an immediate successor of s in T but not necessarily every immediate successor in T corresponds to one in S. Since this is vacuously true if s has no successors in S, we do not have assume that s is nonmaximal in S as in (4).

We point out that while the notions of *subtree*, *weakly*, and *strongly embedded subtree* defined above do agree with large parts of the literature, they are slightly confusing since a weakly or strongly embedded subtree is not necessarily a subtree of a given tree.

All the trees considered in this article are subtrees of $\omega^{<\omega}$ or strongly embedded subtrees of subtrees of $\omega^{<\omega}$. Here $\omega^{<\omega}$ is the set of all finite sequences of natural numbers ordered by set-theoretic inclusion.

Definition 2. Given a set $T \subseteq \omega^{<\omega}$, let

$$[T] = \{x \in \omega^\omega : \forall k \in \omega \exists \ell > k (x \restriction \ell \in T)\}.$$

It is wellknown that the map $T \mapsto [T]$ is a bijection between the ω-subtrees of $\omega^{<\omega}$ and the closed subsets of ω^ω. We intend to use the notation $[T]$ also for subsets T of $\omega^{<\omega}$ that are not necessarily ω-subtrees.

As in the case of Blass' theorem on continuous n-colorings on the Cantor space, we need to consider some kind of type of n-element subsets of ω^ω. For our purposes, the types of n-element subsets of ω^ω have to carry more information than in the case of Blass' theorem.

Definition 3. Recall from the introduction that for two distinct points $x, y \in \omega^\omega$,

$$\Delta(x, y) = \min\{k \in \omega : x(k) \neq y(k)\}.$$

Given distinct elements $x_0, \ldots, x_{n-1}$ of ω^ω, let

$$\Delta(x_0, \ldots, x_{n-1}) = \{\Delta(x_i, x_j) : \{i, j\} \in [n]^2\}.$$

Let $\mu : \Delta(x_0, \ldots, x_{n-1}) \to \omega$ be the *Mostowski collapse* of $\Delta(x_0, \ldots, x_{n-1})$, i.e., the unique increasing bijection between the set $\Delta(x_0, \ldots, x_{n-1})$ and its cardinality. For all $i \in n$ and all $k \in \mu[\Delta(x_0, \ldots, x_{n-1})]$ let $s_i(k) = x_i(\mu^{-1}(k))$. In other words, s_i records the behaviour of x_i at all the coordinates where two distinct x_j separate. We define the *strong type* of $\{x_0, \ldots, x_{n-1}\}$ to be

$$\text{stype}(x_0, \ldots, x_{n-1}) = \{s_0, \ldots, s_{n-1}\}.$$

Note that the map stype $: [\omega^\omega]^n \to [\omega^{<\omega}]^n$ is continuous since the strong type of $\{x_0, \ldots, x_{n-1}\}$ is determined by sufficiently long initial segments of $x_0, \ldots, x_{n-1}$. Also, observe that for all $m \in \omega$ there are only finitely many strong types of n-element subsets of m^ω. Finally, whenever $1 \le m \le \omega$ and T is a strongly embedded subtree of $m^{<\omega}$, then all strong types of n-element subsets of m^ω occur as strong types of n-element subsets of $[T]$.

We are now ready to state the main theorem of this section.

Theorem 4. *Let $m \in \omega \setminus 2$, $j \in \omega \setminus 1$, and let $c : [m^\omega]^n \to j$ be continuous. If $\{s_1, \ldots, s_n\}$ is the strong type of an n-element subset of m^ω, then $m^{<\omega}$ has a strongly embedded ω-subtree R such that c is constant on the set of n-element subsets of $[R]$ of strong type $\{s_1, \ldots, s_n\}$.*

We derive this theorem from Milliken's theorem on partitions of the collection of strongly embedded k-subtrees of a finitely splitting ω-tree.

Theorem 5 (Milliken [15]). *Let T be a finitely splitting ω-tree and $k \in \omega$. If the set of strongly embedded k-subtrees of T is partitioned into finitely many classes, then there is a strongly embedded ω-subtree S of T, all whose strongly embedded k-subtrees lie in the same class of the partition.*

Another ingredient of our proof is the following observation about continuous n-colorings on the set of branches of a finitely splitting ω-tree.

Lemma 6. *Let $c : [m^\omega]^n \to j$ be a continuous coloring as in Theorem 4. Then there is a strongly embedded ω-subtree T of $m^{<\omega}$ such that for all n-element sets $\{x_1, \ldots, x_n\} \subseteq [T]$, $c(x_1, \ldots, x_n)$ only depends on*

$$\{x_1 \restriction k, \ldots, x_n \restriction k\}$$

where $k = \max(\Delta(x_1, \ldots, x_n)) + 1$.

Proof. To the coloring c we assign the *modulus of continuity* $\mu_c : \omega \to \omega$ as follows:

By the continuity of c, for all n-element sets $\{x_1, \ldots, x_n\} \subseteq m^\omega$ there is $\ell \in \omega$ such that $c(x_1, \ldots, x_n)$ only depends on $\{x_1 \restriction \ell, \ldots, x_n \restriction \ell\}$. An easy compactness argument shows that for all $k \in \omega$ and all pairwise distinct $t_1, \ldots, t_n \in m^k$ there is

$\ell \in \omega$ such that for all $\{x_1, \ldots, x_n\} \in [m^\omega]^n$ with $t_1 \subseteq x_1, \ldots, t_n \subseteq x_n$, $c(x_1, \ldots, x_n)$ only depends on $\{x_1 \upharpoonright \ell, \ldots, x_n \upharpoonright \ell\}$. Since m^k is finite, we can actually choose ℓ so that for all $\{x_1, \ldots, x_n\} \in [m^\omega]^n$ with $\max(\Delta(x_1, \ldots, x_n)) < k$, $c(x_1, \ldots, x_n)$ only depends on $\{x_1 \upharpoonright \ell, \ldots, x_n \upharpoonright \ell\}$. Let $\mu_c(k)$ be the least such ℓ.

Now choose a strongly embedded ω-subtree T of $m^{<\omega}$ such that whenever T contains an element of m^k, then T has no element of a height in the interval $[k+1, \mu_c(k))$. Now, if $\{x_1, \ldots, x_n\} \in [T]$ and $k = \max(\Delta(x_1, \ldots, x_n)) + 1$, then for each $i \in \{1, \ldots, n\}$, $x_i \upharpoonright k$ already determines $x_i \upharpoonright \mu_c(k)$. It follows that on $[T]$, $c(x_1, \ldots, x_n)$ only depends on $\{x_1 \upharpoonright k, \ldots, x_n \upharpoonright k\}$. □

Proof of Theorem 4. Let m, n, j, c, and $\{s_1, \ldots, s_n\}$ be as in Theorem 4. Let T be a strongly embedded ω-subtree of $m^{<\omega}$ as guaranteed by Lemma 6. Let $k = |s_1|$. We define a map $\bar{c}$ that assigns a color in j to each strongly embedded k-subtree of T.

Let S be a strongly embedded k-subtree of T. There is a unique isomorphism $\gamma : S \to m^{<k+1}$ of trees that is monotone with respect to the lexicographic ordering on S and $m^{<k+1}$. For each $i \in \{1, \ldots, n\}$ let $t_i = \gamma^{-1}(s_i)$. By the properties of the strongly embedded subtree T of $m^{<\omega}$, for all $x_1, \ldots, x_n \in [T]$ with $t_i \subseteq x_i$, $i \in \{1, \ldots, n\}$, $c(x_1, \ldots, x_n)$ has the same value. Let $\bar{c}(S)$ be this unique color.

By Milliken's theorem, there is a strongly embedded ω-subtree R of T such that $\bar{c}$ is constant on the collection of strongly embedded k-subtrees of R. It follows that c is constant on the collection of n-element subsets of $[R]$ of strong type $\{s_1, \ldots, s_n\}$. □

Corollary 7. *Let m, n, j, and c be as in the Theorem 4. Then there is a strongly embedded ω-subtree R of $m^{<\omega}$ such that for all n-element sets $\{x_1, \ldots, x_n\} \subseteq [R]$ the color $c(x_1, \ldots, x_n)$ only depends on the strong type of $\{x_1, \ldots, x_n\}$.*

Proof. The corollary is obtained by iterating Theorem 4 through all the finitely many strong types of n-element subsets of m^ω. □

Lemma 8. *There are no more than $\binom{m^{n-1}}{n}$ strong types of n-element subsets of $m^{<\omega}$.*

Proof. The strong type of an n-element set $\{x_1, \ldots, x_n\} \subseteq m^\omega$ is determined by the values of $x_1, \ldots, x_n$ on the coordinates in the set $\Delta(x_1, \ldots, x_n)$. If $x_1, \ldots, x_n$ are lexicographically increasing, then, as is easily checked,

$$\Delta(x_1, \ldots, x_n) = \{\Delta(x_1, x_2), \ldots, \Delta(x_{n-1}, x_n)\}.$$

It follows that $\Delta(x_1, \ldots, x_n)$ is of size at most $n - 1$. Hence, there are not more than m^{n-1} possibilities for $x_i \upharpoonright \Delta(x_1, \ldots, x_n)$. There are $\binom{m^{n-1}}{n}$ ways of choosing

an n-element subset of these m^{n-1} possibilities. Hence, there are at most $\binom{m^{n-1}}{n}$ strong types of n-elements subsets of m^ω. $\qquad\square$

Corollary 9. *For every continuous coloring $c : [m^\omega]^n \to j$ there is a strongly embedded ω-subtree R of $m^{<\omega}$ such that on $[[R]]^n$, c assumes no more than $\binom{m^{n-1}}{n}$ different colors.*

4. Types and m-perfect sets

We will now derive a strengthening of the theorems of Blass and Galvin from Theorem 4. We need another notion of type.

Definition 10. Let $n > 1$ and let $\{x_0, \ldots, x_{n-1}\}$ be an n-element subset of ω^ω. Choose $k \in \omega$ with $|\{x_0 \upharpoonright k, \ldots, x_{n-1} \upharpoonright k\}| = n$. Let

$$T = \{x_i \upharpoonright \ell : \ell \le k \wedge i < n\}.$$

T is a finite subtree of $\omega^{<\omega}$.

Let $\mu : \Delta(x_0, \ldots, x_{n-1}) \to \omega$ be the Mostowski-collapse of the set $\Delta(x_0, \ldots, x_{n-1})$, as defined in Definition 3. For each $i < n$ and each $j \in \mu[\Delta(x_0, \ldots, x_{n-1})]$ let

$$s_i(j) = \begin{cases} x_i(\mu^{-1}(j)), & \text{if } x_i \upharpoonright \mu^{-1}(j) \in \operatorname{split}(T) \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

We define the *type* of the set $\{x_0, \ldots, x_{n-1}\}$ to be

$$\operatorname{type}(x_0, \ldots, x_{n-1}) = \{s_0, \ldots, s_{n-1}\}.$$

Note that our notion of type is very similar to that of a strong type, only that we forget the information about immediate successors of non-splitting nodes in T that is recorded in the strong type.

Now let $m > 1$. Given a coloring $c : [m^\omega]^n \to 2$, we call a subset H of m^ω *weakly homogeneous* if for every type $\{s_0, \ldots, s_{n-1}\}$ of an n-element subset of m^ω the coloring c is constant on the collection of all n-element subsets $\{x_0, \ldots, x_{n-1}\}$ of H with $\operatorname{type}(x_0, \ldots, x_{n-1}) = \{s_0, \ldots, s_{n-1}\}$.

Our version of Blass' theorem talks about m-perfect trees.

Definition 11. Let $n > 0$. A tree $T \subseteq \omega^{<\omega}$ is *m-perfect* if all $s \in T$ have an extension $t \in T$ that has at least m immediate successors. An m-perfect tree T is *normal* if each $s \in T$ has either exactly one immediate successor or exactly m. A nonempty closed set $X \subseteq \omega^\omega$ is *m-perfect* if the tree

$$T(X) = \{s \in \omega^{<\omega} : \exists x \in X(s \subseteq x)\}$$

is.

We are now ready to state the generalization of Galvin's theorem to continuous n-colorings on m^ω. For $m = 2$ this is just Blass' theorem from [2].

Theorem 12. *Let $n, m > 1$ and $j > 0$. Then for every continuous coloring $c :$ $[m^\omega]^n \to j$ there is an m-perfect set $P \subseteq m^\omega$ that is weakly homogeneous.*

Proof. By Corollary 7, there is a strongly embedded ω-subtree T of $m^{<\omega}$ such that on $[T]$, the color $c(x_1, \ldots, x_n)$ of an n-element set only depends on the strong type of $\{x_1, \ldots, x_n\}$. By thinning out T we obtain a weakly embedded ω-subtree R of T with the following properties:

(1) For all $n \in \omega$, $R(n) \subseteq T(n)$,
(2) R is m-perfect and for all $n \in \omega$, $|\operatorname{split}(R) \cap R(n)| = 1$, and
(3) for all $s \in R \setminus \operatorname{split}(R)$ and all $t \in R$ with $s \subseteq t$, $t(|s|) = 0$.

In other words, we remove nodes of T in order to obtain a weakly embedded ω-subtree R of T that is m-perfect, lives on the same levels of T, and has exactly one splitting node on each level, while all the non-splitting nodes of R are continued in R by the digit 0. Making sure that R is m-perfect requires some bookkeeping during the construction.

Claim 13. *On $[R]$, the strong type of an n-element set is identical to its type.*

The claim follows immediately from the definition of type and from the construction of R. By the claim, on $[R]$, the color of an n-element set only depends on its type. Hence $[R]$ is weakly homogeneous. Since R is m-perfect, also the tree $T([R])$, the subtree of $m^{<\omega}$ generated by R, is m-perfect. It follows that $[R]$ is an m-perfect, weakly c-homogeneous subset of m^ω, finishing the proof of the theorem. $\square$

5. UNCOUNTABLY HOMOGENEOUS, CONTINUOUS n-COLORINGS

Definition 14. A *partial n-coloring* on a Polish space X is a map $c : A \to 2$ where $A \subseteq [X]^n$. If c and d are partial n-colorings on Polish spaces X and Y, we write $c \leq d$ if there is a topological embedding $e : X \to Y$ such that for all $\{x_1, \ldots, x_n\}$ in the domain of c we have $d(e(x_1), \ldots, e(x_n)) = c(x_1, \ldots, x_n)$.

The partial n-coloring $c_{\min}^n$ is defined on all n-element sets $\{x_1, \ldots, x_n\} \subseteq n^\omega$ such that any two distinct x_i and x_j separate at the same level m, i.e., such that for all $\{i, j\} \in [\{1, \ldots, n\}]^2$, $\Delta(x_i, x_j) = m$. If $\{x_1, \ldots, x_n\} \subseteq n^\omega$ is an n-element set such that any two distinct elements separate at level m, let

$$c_{\min}^n(x_1, \ldots, x_n) = m \mod 2.$$

Note that the domain of $c_{\min}^n$ is a clopen subset of $[n^\omega]^n$.

Lemma 15. *Let X be a Polish space and let $c : [X]^n \to 2$ be continuous. Then c is uncountably homogeneous iff $c_{\min}^n \leq c$.*

Proof. For closed sets $C \subseteq X$ we define the *derivative $D(C)$* as follows:

Let U be the union of all open sets $O \subseteq X$ such that $c \upharpoonright (C \cap O)$ is countably homogeneous. Since the topology on X has a countable basis, U is the union of countably many basic open sets O with $\mathfrak{hm}(c \upharpoonright (C \cap O)) \leq \aleph_0$. It follows that $c \upharpoonright (C \cap U)$ is countably homogeneous. Let $D(C) = C \setminus U$.

Using standard arguments from descriptive set theory we see that there is a (nonempty) perfect set $P \subseteq X$ such that $D(P) = P$. Now for every nonempty set $O \subseteq P$ that is relatively open in P, $\mathfrak{hm}(c \upharpoonright O) > \aleph_0$. In particular, no nonempty, relatively open subset of P is c-homogeneous. Without loss of generality we may assume that $P = X$.

Now, whenever $O \subseteq X$ is open and nonempty, for all $i \in 2$ there are distinct points $x_1, \ldots, x_n \in O$ such that $c(x_1, \ldots, x_n) = i$. By the continuity of c, there are disjoint open neighborhoods $O_1, \ldots, O_n \subseteq O$ of the points $x_1, \ldots, x_n$ such that for all $y_1, \ldots, y_n \in [O_1, \ldots, O_n]$, $c(y_1, \ldots, y_n) = c(x_1, \ldots, x_n)$. Using this argument, we construct an *n-perfect scheme* $(O_\sigma)_{\sigma \in n^{<\omega}}$ of nonempty open subsets of X with the following properties:

(1) If $\sigma \in n^m$, then O_σ is of diameter at most 2^{-m}. (Here the diameter refers to some fixed complete metric on X that generates the topology.)

(2) For all $\sigma \in n^m$, c is constant on $[O_{\sigma \frown 1}, \ldots, O_{\sigma \frown n}]$ with constant value m mod 2.

(3) For all $\sigma \in n^{<\omega}$ and distinct $i, j < n$, $\mathrm{cl}(O_{\sigma \frown i}) \cap \mathrm{cl}(O_{\sigma \frown j}) = \emptyset$ and $\mathrm{cl}(O_{\sigma \frown i}) \subseteq O_\sigma$.

For each $x \in n^\omega$ let $e(x)$ be the unique element of $\bigcap_{m \in \omega} \mathrm{cl}(O_{x \upharpoonright m})$. It is easily checked that $e : n^\omega \to X$ is a topological embedding that works for the lemma. $\square$

Definition 16. Let $n > 1$. We define a partial n-coloring $d_{\min}^n$ on $(2n-1)^\omega$ as follows:

Let $\{x_1, \ldots, x_n\} \in [(2n-1)^\omega]^n$ be such that for some $m \in \omega$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we have $\Delta(x_i, x_j) = m$. Then we put

$$d_{\min}^n(x_1, \ldots, x_n) = \begin{cases} 0, \text{ if } \{x_1(m), \ldots, x_n(m)\} = \{0, \ldots, n-1\}, \\ 1, \text{ if } \{n, \ldots, 2n-2\} \subseteq \{x_1(m), \ldots, x_n(m)\}, \text{ and} \\ \text{undefined, otherwise.} \end{cases}$$

Note that the domain of $d_{\min}^n$ is clopen in $(2n-1)^\omega$. The crucial properties of $d_{\min}^n$ are stated in the following lemma.

Lemma 17. *a) $d_{\min}^n \leq c_{\min}^n$ and $c_{\min}^n \leq d_{\min}^n$.*

b) For every $(2n-1)$-perfect set $P \subseteq (2n-1)^\omega$, $d_{\min}^n \leq d_{\min}^n \upharpoonright P$. In particular, $d_{\min}^n \upharpoonright P$ is uncountably homogeneous.

Proof. In order to show a), we give explicit definitions of embeddings witnessing $c_{\min}^n \leq d_{\min}^n$ and $d_{\min}^n \leq c_{\min}^n$.

Let $f : n^\omega \to (2n-1)^\omega$ be defined by letting $f(x)(i) = x(i)$ for even $i \in \omega$ and $f(x)(i) = x(i) + n - 1$ for odd i. It is easily checked that f witnesses $c_{\min}^n \leq d_{\min}^n$.

Let $g : (2n-1)^\omega \to n^\omega$ be defined as follows: for every $x \in (2n-1)^\omega$ and for every $i \in \omega$ let $g(x)(0) = 0$,

$$g(x)(2i+1) = \begin{cases} 0, & \text{if } x(i) < n, \text{ and} \\ x(i) - n + 1, & \text{otherwise} \end{cases}$$

and

$$g(x)(2i+2) = \begin{cases} x(i), & \text{if } x(i) < n, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

A straight forward computation shows that g witnesses $d_{\min}^n \leq c_{\min}^n$.

For b) let $P \subseteq (2n-1)^\omega$ be $(2n-1)$-perfect. Consider the $(2n-1)$-perfect tree $T = T(P)$. Thinning out T, we may assume that every $t \in T$ has either one immediate successor or $(2n-1)$-immediate successors in T. Now the tree consisting of the splitting nodes of T is isomorphic to $(2n-1)^{<\omega}$ by a lexicographically increasing isomorphism. This isomorphism of trees induces a homeomorphism between $[T]$ and $(2n-1)^\omega$ that preserves $d_{\min}^n$. It follows that $d_{\min}^n \leq d_{\min}^n \restriction P$. $\qquad\square$

Corollary 18. *Let X be a Polish space and let $c : [X]^n \to 2$ be continuous. Then c is uncountably homogeneous iff $d_{\min}^n \leq c$.*

Theorem 19. *Let $n \geq 2$. The class of uncountably homogeneous, continuous n-colorings (with two colors) on Polish spaces has a finite basis.*

Proof. Let X be a Polish space and let $c : [X]^n \to 2$ be continuous and uncountably homogeneous. By Corollary 18, there is a topological embedding $e : (2n-1)^\omega \to X$ such that for all $\{x_1, \ldots, x_n\} \in [(2n-1)^\omega]^n$ it holds that if $d_{\min}^n(x_1, \ldots, x_n)$ is defined, then $c(e(x_1), \ldots, e(x_n)) = d_{\min}^n(x_1, \ldots, x_n)$. We define a continuous coloring $d : [(2n-1)^\omega]^n \to 2$ by letting

$$d(x_1, \ldots, x_n) = c(e(x_1), \ldots, e(x_n)).$$

By the choice of e, we have

$$d_{\min}^n(x_1, \ldots, x_n) = d(x_1, \ldots, x_n)$$

whenever $d_{\min}^n(x_1, \ldots, x_n)$ is defined. In other words, d is an extension of $d_{\min}^n$.

By Theorem 12, there is a normal $(2n-1)$-perfect tree $T \subseteq (2n-1)^{<\omega}$ such that for all $\{x_1, \ldots, x_n\} \in [[T]]^n$, $d(x_1, \ldots, x_n)$ only depends on $\text{type}(x_1, \ldots, x_n)$.

Since T is $(2n-1)$-perfect and d is an extension of $d_{\min}^n$ to all of $[(2n-1)^\omega]^n$, by Lemma 17 b) we have $d_{\min}^n \leq d \restriction [T]$. In particular, $d \restriction [T]$ is uncountably

homogeneous. Since on $[T]$ the color of an n-element set only depends on its type, $d \restriction [T]$ is isomorphic to the coloring on $(2n - 1)^\omega$ that extends $d_{\min}^n$ and assigns to each n-element set $\{x_1, \ldots, x_n\} \subseteq (2n - 1)^\omega$ the color that d assigns to every n-element subset of $[T]$ of the same type. Since there are only finitely many types of n-element subsets of $(2n - 1)^\omega$, up to isomorphism there are only finitely many possibilities of assigning the colors 0 and 1 to types of n-element sets. It follows that there is a finite basis of the uncountably homogeneous, continuous n-colorings on Polish spaces. $\qquad\square$

Recall call that an *analytic Hausdorff space* is a Hausdorff space that is a continuous image of ω^ω. The proof of Lemma 15 can be easily modified to show the following:

Lemma 20. *If $c : [X]^n \to 2$ is a continuous coloring on an analytic Hausdorff space X that is not the union of countably many c-homogeneous sets, then $c_{\min}^n \leq c$.*

Using this lemma, the proof of Theorem 19 actually goes through for analytic Hausdorff spaces instead of Polish spaces.

Corollary 21. *The class of uncountably homogeneous, continuous n-colorings on analytic Hausdorff spaces has a finite basis.*

We now give an upper bound for the size of a basis of the class of uncountably homogeneous, continuous n-colorings on analytic Hausdorff spaces. Lower bounds will be provided in Sections 6 and 7.

Theorem 22. *The class of uncountably homogeneous, continuous n-colorings on analytic Hausdorff spaces has a basis consisting of no more than $2^{\left(\binom{(2n-1)^{n-1}}{n}\right)}$ elements.*

Proof. Clearly, the number of strong types of n-element sets is an upper bound for the number of types.

Hence, there are no more than $2^{\left(\binom{(2n-1)^{n-1}}{n}\right)}$ ways of assigning colors in 2 to all types of n-element subsets of $(2n - 1)^\omega$. By the argument in the proof of Theorem 19, this proves the theorem. $\qquad\square$

6. Some minimal 3-colorings

We investigate several concrete continuous 3-colorings on 2^ω.

Definition 23. For $\{x, y, z\} \in [2^\omega]^3$ let $k = \min(\Delta(x, y, z))$ and $\ell = \max(\Delta(x, y, z))$. Let

$$c_{\text{low}}(x, y, z) = k \mod 2, \ c_{\text{high}}(x, y, z) = \ell \mod 2,$$

$$\text{and } c_{\text{low,high}}(x, y, z) = (k + \ell) \mod 2.$$

For $\{x, y, z\} \in [\omega^\omega]^3$ let

$$c_{\text{loc}}^\omega(x, y, z) = \begin{cases} 0, & \text{if } \Delta(x, y) = \Delta(x, z) = \Delta(y, z), \text{ and} \\ 1, & \text{otherwise.} \end{cases}$$

Let $c_{\text{loc}}^{<\omega} = c_{\text{loc}}^\omega \upharpoonright \prod_{n \in \omega}(n + 1)$ and $c_{\text{loc}}^3 = c_{\text{loc}}^\omega \upharpoonright 3^\omega$.

For an n-coloring $c : [X]^n \to 2$ let c^* be the coloring defined by

$$c^*(x_1, \ldots, x_n) = 1 - c(x_1, \ldots, x_n).$$

We say that c^* is the *dual* of c.

Definition 24. Two continuous colorings $c : [X]^n \to 2$ and $d : [Y]^n \to 2$ on Polish spaces are *equivalent* if $c \leq d$ and $d \leq c$. A continuous n-coloring c is *minimal* if it is uncountably homogeneous and every uncountably homogeneous, continuous n-coloring $d \leq c$ is equivalent to c.

Lemma 25. *a) If a continuous n-coloring c is minimal, then so is c^*.*

b) Let c and d be uncountably homogeneous, continuous n-colorings. If c is minimal and $c \not\leq d$, then $d \not\leq c$.

Lemma 26. *a) The colorings c_{low} and c_{low}^* are equivalent. Also, c_{high} and c_{high}^* are equivalent.*

b) $c_{\text{low}}, c_{\text{high}} \not\leq c_{\text{loc}}^3, (c_{\text{loc}}^3)^*, c_{\text{low,high}}, c_{\text{low,high}}^*$

c) $c_{\text{low,high}} \not\leq c_{\text{loc}}^3, (c_{\text{loc}}^3)^*, c_{\text{low,high}}^*$ and $c_{\text{low,high}}^* \not\leq c_{\text{loc}}^3, (c_{\text{loc}}^3)^*, c_{\text{low,high}}$

d) $c_{\text{loc}}^3 \not\leq (c_{\text{loc}}^3)^*$

e) $c_{\text{high}} \not\leq c_{\text{low}}$

Proof. a) Let $e : 2^\omega \to 2^\omega$ be defined by $e(x) = 0 \frown x$. This embedding witnesses $c_{\text{low}} \leq c_{\text{low}}^*$, $c_{\text{low}}^* \leq c_{\text{low}}$, $c_{\text{high}} \leq c_{\text{high}}^*$, and $c_{\text{high}}^* \leq c_{\text{high}}$.

b) c_{low} and c_{high} have infinite homogeneous sets in both colors. On the other hand, c_{loc}^3, $(c_{\text{loc}}^3)^*$, $c_{\text{low,high}}$, and $c_{\text{low,high}}^*$ have only finite homogeneous sets in one of the colors. This shows that c_{low} and c_{high} do not embed into the other four colorings.

c) The coloring $c_{\text{low,high}}$ has homogeneous sets of color 1 and size 4. But $(c_{\text{loc}}^3)^*$ has no homogeneous sets of color 1 of size > 3. Hence $c_{\text{low,high}} \not\leq (c_{\text{loc}}^3)^*$. Also, $c_{\text{low,high}}$ has infinite homogeneous sets of color 0, but c_{loc}^3 and $c_{\text{low,high}}^*$ do not. Hence $c_{\text{low,high}} \not\leq c_{\text{loc}}^3, c_{\text{low,high}}^*$.

The same arguments apply for the dual colorings, with the roles of the colors 0 and 1 switched.

d) The coloring c_{loc}^3 has infinite homogeneous sets of color 1, but its dual does not. Hence $c_{\text{loc}}^3 \not\leq (c_{\text{loc}}^3)^*$.

e) Suppose $e : 2^\omega \to 2^\omega$ is an embedding that witnesses $c_{\text{high}} \leq c_{\text{low}}$. Let $\{x, y_1\} \in [2^\omega]^2$. Choose $m_1 \in \omega$ such that for all $y \in 2^\omega$ with $\Delta(y_1, y) \geq m_1$ we

have
$$e(y) \restriction (\Delta(e(x), e(y_1)) + 1) = e(y_1) \restriction (\Delta(e(x), e(y_1)) + 1).$$

Let $y_2 \in 2^\omega$ be such that $\Delta(y_1, y_2)$ is even and at least m_1. Choose $m_2 > m_1$ such that for all $y \in 2^\omega$ with $\Delta(y_2, y) \geq m_2$ we have

$$e(y) \restriction (\Delta(e(y_1), e(y_2)) + 1) = e(y_2) \restriction (\Delta(e(y_1), e(y_2)) + 1).$$

Finally let $y_3 \in 2^\omega$ be such that $\Delta(y_2, y_3)$ is odd and at least m_2.

Now we have $c_{\text{high}}(x, y_1, y_2) = 0$ and $c_{\text{high}}(x, y_2, y_3) = 1$. By the choice of x, y_1, y_2, and y_3, we have $\Delta(e(x), e(y_1), e(y_2)) = \Delta(e(x), e(y_2), e(y_3))$ and therefore $c_{\text{low}}(e(x), e(y_1), e(y_2)) = c_{\text{low}}(e(x), e(y_2), e(y_3))$, contradicting the fact that e witnesses $c_{\text{high}} \leq c_{\text{low}}$. $\qquad\square$

Lemma 27. *The colorings c_{low}, c_{high}, $c_{\text{low,high}}$, $c^*_{\text{low,high}}$, c^3_{loc}, and $(c^3_{\text{loc}})^*$ are minimal.*

Proof. Let $c \in \{c_{\text{low}}, c_{\text{high}}, c_{\text{low,high}}, c^*_{\text{low,high}}\}$ and suppose that $d \leq c$ is uncountably homogeneous. Without loss of generality, d is defined on a perfect subset P of 2^ω, $d = c \restriction P$, and for every open set $O \subseteq 2^\omega$, $O \cap P$ is either empty or not d-homogeneous. Since $d = c \restriction P$, for every open set $O \subseteq 2^\omega$ with $O \cap P \neq \emptyset$, $O \cap P$ is not c-homogeneous.

It follows that each $t \in T(P)$ has extensions in $\text{split}(T(P))$ of both odd and even length. Hence there is a 1-1-map $h : 2^{<\omega} \to \text{split}(T(P))$ that is monotone with respect to $\subseteq$, preserves the parity of the length of nodes, and is such that for all $s, t, r \in 2^{<\omega}$, if s and t are incomparable and r is the longest common initial segment of s and t, then $h(s)$ and $h(t)$ are incomparable and $h(r)$ is the longest common initial segment of $h(s)$ and $h(t)$. The map h induces an embedding $e : 2^\omega \to P$ such that for all $x \in 2^\omega$, $e(x) = \bigcup_{m \in \omega} h(x \restriction m)$. The embedding e witnesses $c \leq c \restriction P$.

Now suppose that for some uncountably homogeneous, continuous n-coloring d we have $d \leq c^3_{\text{loc}}$. We may assume that for some perfect set $P \subseteq 3^\omega$, $d = c^3_{\text{loc}} \restriction P$ and that for every nonempty open set $O \subseteq 2^\omega$, either $P \cap O = \emptyset$ or $P \cap O$ is not c^3_{loc}-homogeneous.

It follows that every $t \in T(P)$ has an extension $s \in \text{split}(T(P))$ that has three immediate successors in $T(P)$. Hence there is a 1-1 map $h : 3^{<\omega} \to T(P)$ that is monotone with respect to $\subseteq$ and maps each $s \in 3^{<\omega}$ to a node with three immediate successors in $T(P)$ such that whenever s is the longest common initial segment of any two of three pairwise incomparable nodes $t_1, t_2, t_3 \in 3^{<\omega}$, then $h(s)$ is the longest common initial segment of any two of the three pairwise incomparable nodes $h(t_1), h(t_2), h(t_3) \in T(P)$. The map h induces an embedding $e : 2^\omega \to P$ such that for all $x \in 3^\omega$, $e(x) = \bigcup_{m \in \omega} h(x \restriction m)$. The embedding e witnesses $c^3_{\text{loc}} \leq c^3_{\text{loc}} \restriction P$.

The minimality of c^3_{loc} implies the minimality of $(c^3_{\text{loc}})^*$ by Lemma 25 a). $\qquad\square$

Corollary 28. *The colorings c_{low}, c_{high}, $c_{\text{low,high}}$, $c^*_{\text{low,high}}$, c^3_{loc}, and $(c^3_{\text{loc}})^*$ are pairwise incomparable with respect to $\leq$.*

Proof. By the previous Lemma together with Lemma 25 b), it is enough to show that for any two distinct colorings $c, d \in \{c_{\text{low}}, c_{\text{high}}, c_{\text{low,high}}, c^*_{\text{low,high}}, c^3_{\text{loc}}, (c^3_{\text{loc}})^*\}$ we have $c \not\leq d$ or $d \not\leq c$. But this follows from Lemma 26 b)–e). $\square$

Corollary 29. *Every basis for the class of uncountably homogeneous, continuous 3-colorings on Polish spaces has at least six elements.*

We finish this section by observing that the coloring c^ω_{loc} behaves differently from all continuous 2-colorings on Polish spaces. Clearly, a closed set $C \subseteq \omega^\omega$ is c^ω_{loc}-homogeneous of color 1 iff $T(C)$ is *binary*, i.e., if no node has more than two immediate successors in $T(C)$. Since the c^ω_{loc}-homogeneous sets of color 0 are at most countable, $\mathfrak{hm}(c^\omega_{\text{loc}})$ is the minimal size of a family of binary subtrees of $\omega^{<\omega}$ such that every element of ω^ω is a branch of one of those trees. Newelski and Rosłanowski [17] showed that for any cardinal κ, forcing with a countable support product of κ copies of Sacks forcing over a model of GCH results in a model of set theory in which there are $\aleph_1$ binary trees such that each element of ω^ω is a branch of one of these trees. In such a model, $2^{\aleph_0}$ is at least κ. The forcing does not collapse any cardinals.

In [9] it was shown that in a model of set theory obtained by forcing with a countable support product of κ copies of Sacks forcing over a model of GCH we have $\mathfrak{hm}(c) \geq \kappa$ for every uncountably homogeneous, continuous 2-coloring c on a Polish space. Hence we have the following theorem.

Theorem 30. *It is consistent that $\mathfrak{hm}(c^\omega_{\text{loc}})$ is smaller than every uncountable homogeneity number of a continuous 2-coloring on a Polish space.*

On the other hand, in [7] a model of set theory was constructed in which every continuous 2-coloring c on an Polish space has $\mathfrak{hm}(c) \leq \aleph_1$ while for every family of size $\aleph_1$ of binary subtrees of $T(\prod_{m<\omega}(m+1))$ there is an element of $\prod_{m<\omega}(m+1)$ that is not a branch of any of these trees. This yields

Theorem 31. *It is consistent that $\mathfrak{hm}(c^{<\omega}_{\text{loc}})$ is larger than every homogeneity number of a continuous 2-coloring on a Polish space.*

7. MINIMAL n-COLORINGS

In this section we prove lower bounds for the size of a basis for the class of uncountably homogeneous n-colorings on Polish spaces. Our strategy is the same as in the previous section: We define a family of minimal continuous n-colorings on 2^ω that pairwise do not embed into each other. Some of the colorings that we

are going to define are essentially generalizations of c_{low}, c_{high}, and $c_{\text{low,high}}$. While higher dimensional analogs of c_{loc}^3 clearly exist, they do not give rise to a large family of pairwise incomparable minimal colorings.

In the proof of Theorem 12 we pass from an m-perfect tree to an m-perfect subtree that has at most one splitting node at each level. Following [2], we call a subtree T of $m^{<\omega}$ *skew* if T has at most one splitting node on each level. Abusing notation, we identify a type of an n-element subset of 2^ω with the finite subtree generated by the type. Thus we can talk about splitting nodes of types and we know what it means for a type to be skew. Our skew types of n-element subsets of 2^ω are equivalent to the types in [2]. Now whenever T is a skew ω-tree, then for every n-element set $\{x_1, \ldots, x_n\} \subseteq [T]$, type$(x_1, \ldots, x_n)$ is skew.

Lemma 32. *Let $n > 1$. Then there are $(n-1)!$ skew types of n-element subsets of 2^ω.*

Proof. Let $\{x_1, \ldots, x_n\} \subseteq 2^\omega$ be an n-element set of some skew type. Then for every $k < n$, type$(x_1, \ldots, x_n)$ has exactly $k+1$ nodes of height k. If $k < n-1$, then exactly one of these nodes of height k is a splitting node. To each $k < n-1$ we assign $f(k) \in k+1$ such that if $s_0, \ldots, s_k$ is the lexicographically increasing enumeration of all nodes of type$(x_1, \ldots, x_n)$ of height k, then $s_{f(k)}$ is a splitting node of type$(x_1, \ldots, x_n)$.

It is easily checked that this establishes a bijection between the set of skew types of n-element subsets of 2^ω and the set of all maps $f : n-1 \to n$ such that for each $k < n-1$, $f(k) < k+1$. It follows that there are exactly $(n-1)!$ skew types of n-element subsets of 2^ω. $\square$

Definition 33. Let T be a perfect subtree of $2^{<\omega}$. For each $s \in T$ let $\gamma(s)$ be the unique minimal $t \in \text{split}(T)$ with $s \subseteq t$.

T is a *standard skew subtree* of $2^{<\omega}$ if T is skew and for every $n \in \omega$ there is some $m \in \omega$ such that for all $s \in T(m)$ the set

$$\{t \in \text{split}(T) : t \text{ is a proper initial segment of } s\}$$

is of size n and if $s_1, \ldots, s_{2^n}$ is the lexicographically increasing enumeration of $T(m)$, then

$$|\gamma(s_1)| < |\gamma(s_2)| < \cdots < |\gamma(s_{2^n})|.$$

T is an *inhomogeneous standard skew subtree* of $2^{<\omega}$ if T is a standard skew subtree and for all $n \in \omega$ and all $s \in T$ such that the set

$$\{t \in \text{split}(T) : t \text{ is a proper initial segment of } s\}$$

is of size n, $|\gamma(s)|$ is congruent to n modulo 2.

Clearly, every perfect subtree of $2^{<\omega}$ has a standard skew subtree. It follows that if $A \subseteq 2^\omega$ is uncountable, then $T(A)$ has a standard skew subtree. Hence every uncountable set $A \subseteq 2^\omega$ has n-element subsets of all skew types. Also, every ω-subtree T of $2^{<\omega}$ such that every $t \in T$ has extensions in $\mathrm{split}(T)$ of both even and odd length has an inhomogeneous standard skew subtree.

We will be interested in maps from perfect subsets of 2^ω to 2^ω that map the set of branches of a standard skew subtree to the set of branches of a standard skew subtree. Maps that are strictly increasing with respect to the lexicographical ordering and that preserve the relation between the heights of splitting nodes will have this property. We start by considering lexicographically increasing and decreasing maps.

Lemma 34. *Let P be a perfect subset of 2^ω and let $e : P \to 2^\omega$ be a topological embedding. Then there is a perfect set $Q \subseteq P$ such that either e is strictly increasing with respect to the lexicographical ordering or e is strictly decreasing.*

Proof. This is a consequence of Galvin's theorem, i.e., of Theorem 12 for $m = n = 2$. Just color the 2-element subsets of P according to whether or not e is strictly lexicographically increasing on the set. □

Definition 35. For all $x \in 2^{\leq\omega}$ and all $n \in \omega$ let $\overline{x}(n) = 1 - x(n)$. If T is a subset of $2^{\leq\omega}$ let $\overline{T} = \{\overline{t} : t \in T\}$.

Clearly, $\overline{\cdot} : 2^\omega \to 2^\omega$ is a lexicographically decreasing auto-homeomorphism of 2^ω. Observe that for every linear type T of an n-element set, $\overline{T} \neq T$. In other words, no skew type is symmetric.

Definition 36. Let P be a perfect subset of 2^ω and let $e : P \to 2^\omega$ be a topological embedding. Then e is *height monotone* if for all $x_1, \ldots, x_4 \in [P]$ with $x_1 \neq x_2$ and $x_3 \neq x_4$,

$$\Delta(x_1, x_2) < \Delta(x_3, x_4) \quad \Longleftrightarrow \quad \Delta(e(x_1), e(x_2)) < \Delta(e(x_3), e(x_4)).$$

Lemma 37. *Let P be a perfect subset of 2^ω and let $e : P \to 2^\omega$ be a topological embedding. Then there is a perfect set $Q \subseteq P$ such that $e \restriction Q$ is height monotone.*

Proof. Let $T = T(P)$. For all $s \in T$ let

$$T_s = \{t \in T : s \subseteq t \vee t \subseteq s\}.$$

Now given $t \in T$ and $n \in \omega$, there are distinct $x, y \in [T]$ with $t \subseteq x, y$ and $\Delta(e(x), e(y)) \geq n$. Choose $m \in \omega$ such that for all $x', y' \in [T]$ with $x \restriction m = x' \restriction m$ and $y \restriction m = y' \restriction m$ we have that $e(x') \restriction \Delta(e(x), e(y)) = e(x) \restriction \Delta(e(x), e(y))$ and $e(y') \restriction \Delta(e(x), e(y)) = e(y) \restriction \Delta(e(x), e(y))$. Let $t_0 = x \restriction m$ and $t_1 = y \restriction m$. Let $T' = T_{t_0} \cup T_{t_1}$.

Iterating this basic argument above t_0 and t_1 and using some bookkeeping, we obtain a perfect subtree S of T such that for all $s, t \in S$ with $|s| < |t|$ and all $x_1, \ldots, x_4 \in [S]$ with $s ^\frown 0 \subseteq x_1$, $s ^\frown 1 \subseteq x_2$, $t ^\frown 0 \subseteq x_3$, and $t ^\frown 1 \subseteq x_4$ we have $\Delta(e(x_1), e(x_2)) < \Delta(e(x_3), e(x_4))$. In other words, e is height monotone on the perfect set $Q = [S]$.   □

Clearly, if a topological embedding e from a perfect set $P \subseteq 2^\omega$ to 2^ω is both lexicographically increasing and height monotone, then it preserves types of n-element sets and maps the set of branches of a standard skew subtree to the set of branches of a standard skew subtree. If e is height monotone and lexicographically decreasing, then it maps every n-element set of type T to an n-element set of type $\overline{T}$.

Now fix an inhomogeneous standard skew subtree S of 2^ω. We define two families of continuous n-colorings on $[S]$. Given a continuous coloring $c : [P]^n \to 2$ on some perfect set $P \subseteq 2^\omega$ and a type T of an n-element subset of 2^ω, let $c \restriction T$ be the restriction of c to all n-element sets of type T. Now, in order to define a continuous n-coloring on $[S]$ it is enough to specify $c \restriction T$ for all skew types T of n-element subsets of 2^ω.

Definition 38. Let $\mathrm{ST}(n)$ denote the set of skew types of n-element subsets of 2^ω. Fix some $T \in \mathrm{ST}(n)$. Let F_n denote the set of all functions $f : \mathrm{ST}(n) \setminus \{T, \overline{T}\} \to 2$. For all $f \in F_n$ and every n-element set $\{x_1, \ldots, x_n\} \subseteq [S]$ let

$$
c_f(x_1, \ldots, x_n) = \begin{cases} 0, \text{ if type}(x_1, \ldots, x_n) = T, \\ 1, \text{ if type}(x_1, \ldots, x_n) = \overline{T}, \\ f(\text{type}(x_1, \ldots, x_n)), \text{ otherwise.} \end{cases}
$$

We point out that while the colorings c_f depend on the choice of the inhomogenous standard skew tree, their equivalence classes with respect to equivalence of colorings as defined in Definition 24 does not.

Theorem 39. *For every $n \geq 3$, the family $(c_f)_{f \in F_n}$ is a family of size $2^{(n-1)!-2}$ of pairwise incomparable, minimal continuous n-colorings on $[S]$.*

Corollary 40. *For all $n \geq 3$, every basis of the collection of all uncountably homogeneous, continuous n-colorings on a Polish space has at least $2^{(n-1)!-2}$ elements.*

Proof of Theorem 39. We first show that each c_f is minimal. Since c_f assumes both values 0 and 1 in every open subset of $[S]$, c_f is uncountably homogeneous. Now suppose that $c \leq c_f$ is uncountably homogeneous. Up to isomorphism, c is of the form $c_f \restriction P$ for some perfect set $P \subseteq [S]$. Let R be a standard skew subtree of $T(P)$. Now c_f is isomorphic to $c_f \restriction [R]$ and thus $c_f \leq c$. This shows the minimality on c_f.

Now let $g, f \in F_n$ be distinct and suppose that $e : [S] \to [S]$ witnesses $c_g \leq c_f$. By Lemma 37 and Lemma 34, there is a perfect set $P \subseteq [S]$ such that on P, e is height monotone and either lexicographically increasing or decreasing. Let $\{x_1, \ldots, x_n\} \subseteq [S]$ be an n-element set of type T. Now $e[\{x_1, \ldots, x_n\}]$ is either of type T or of type $\overline{T}$. But if type$(e[\{x_1, \ldots, x_n\}]) = \overline{T}$, then $c_g(x_1, \ldots, x_n) = 0$ and $c_f(e(x_1), \ldots, e(x_n)) = 1$, contradicting the fact that e embeds c_g into c_f. Hence type$(e[\{x_1, \ldots, x_n\}]) = T$. But this shows that on P, e is lexicographically increasing and therefore preserves types. Since all skew types occur on P, this shows that $f = g$. Hence the colorings c_f, $f \in F_n$, are pairwise incomparable. $\qquad\square$

The colorings c_f constructed above obviously have the feature of being constant on each skew type. In other words, the whole set $[S]$ is weakly homogeneous with respect to every c_f. Since this is a somewhat pathological, we define another large family of minimal colorings that are pairwise incomparable.

Definition 41. Let $n \geq 3$. On the set 2^{n-1} we define an equivalence relation $\sim$ by letting $s \sim t$ iff $s = t$ or $s = \bar{t}$. Let $Z_n = 2^{n-1}/\sim$. We denote the equivalence class of $s \in 2^{n-1}$ by $s/\sim$.

Given an n-element set $\{x_1, \ldots, x_n\} \subseteq [S]$, let $s_{x_1,\ldots,x_n} : n - 1 \to 2$ be defined as follows. Let $\mu : \Delta(x_1, \ldots, x_n) \to n - 1$ be the Mostowski collapse. For each $i \in n - 1$ let $s_{x_1,\ldots,x_n}(i) = \mu^{-1}(i) \mod 2$.

Let G_n be the family of all functions $g : Z_n \to 2$ that are not constant. For each $g \in G_n$ let $d_g(x_1, \ldots, x_n) = g(s_{x_1,\ldots,x_n}/\sim)$. This defines a family $(d_g)_{g \in G_n}$ of continuous n-colorings on $[S]$.

As in the case of the colorings c_f, the colorings d_g depend on the particular choice of S, but their equivalence classes do not.

Theorem 42. *For every $n \geq 3$, the colorings d_g, $g \in G_n$, are minimal and pairwise incomparable.*

Corollary 43. *For every $n \geq 3$, a basis for the class of uncountably homogeneous, continuous n-colorings on a Polish space contains at least $2^{2^{n-2}} - 2$ elements.*

Proof. The corollary follows immediately from the Theorem 42, using the fact that Z_n is of size $2^{n-1}/2 = 2^{n-2}$ and thus G_n is of size $2^{2^{n-2}} - 2$. $\qquad\square$

In order to prove Theorem 42 we need yet another preservation property of embeddings.

Definition 44. Let $P \subseteq 2^\omega$ be perfect and let $e : P \to 2^\omega$ be a topological embedding. Then e is *parity preserving* if for all dinstinct $x, y \in P$ we have

$$\Delta(x, y) \mod 2 = \Delta(e(x), e(y)) \mod 2.$$

If for all distinct $x, y \in P$ we have

$$\Delta(x, y) \mod 2 = (\Delta(e(x), e(y)) + 1) \mod 2,$$

then e is *parity inverting*.

A perfect set P is *perfectly inhomogeneous* if for every nonempty subset O of P that is relatively open in P there are distinct $x, y \in O$ and distinct $x', y' \in O$ such that $\Delta(x, y)$ is even and $\Delta(x', y')$ is odd.

Proof of Theorem 42. First observe that for every nonempty open subset O of $[S]$ and every function $s : n - 1 \to 2$ there is an n-element set $\{x_1, \ldots, x_n\} \subseteq O$ with $s = s_{x_1, \ldots, x_n}$. For every $g \in G_n$, since $g : Z_n \to 2$ is not constant, no nonempty open subset of $[S]$ is g-homogeneous. It follows that every d_g is uncountably homogeneous.

In order to show the minimality of d_g, assume that $c \leq d_g$ for some uncountably homogeneous continuous n-coloring c. Up to isomorphism we may assume that c is of the form $d_g \restriction [R]$ where R is some perfect subtree of S. Since $d_g \restriction [R]$ is uncountably homogeneous, we may assume, after thinning out R if necessary, that no open subset of $[R]$ is d_g-homogeneous. But now in each nonempty open subset O of $[R]$ we find n-element sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ such that $s_{x_1, \ldots, x_n} = s_{y_1, \ldots, y_n}$. In particular, we find distinct $x, y \in O$ and distinct $x', y' \in O$ such that $\Delta(x, y)$ is even and $\Delta(x', y')$ is odd. It follows that every $t \in R$ has extensions $t_0, t_1 \in \mathrm{split}(R)$ that are of even, respectively odd length.

Hence R has an inhomogeneous standard skew subtree. We may assume that R itself is already a standard skew subtree of S. Now $d_g \restriction [R]$ is isomorphic to d_g, showing that $d_g \leq c$.

We now show that the d_g's are pairwise incomparable. Let $f, g \in G_n$ be distinct. Suppose $e : [S] \to [S]$ witnesses $d_f \leq d_g$.

Claim 45. There is a perfectly inhomogeneous set $P \subseteq [S]$ such that $e \restriction P$ is either parity preserving or parity inverting.

For the proof of the claim, we call a 2-element set $\{x, y\} \subseteq 2^\omega$ an *even pair* if $\Delta(x, y)$ is even, and otherwise an *odd pair*. Now let $O \subseteq [S]$ be a nonempty set that is relatively open. If all even pairs in O are mapped by e to an odd pair and also all odd pairs are mapped to odd pairs, then all pairs in $e[O]$ are odd. But this implies that d_g is constant on $e[O]$, which is impossible since d_f is not constant on O.

Similarly, it is not the case that all even pairs in O and all odd pairs in O are mapped to even pairs. It follows that at least one of the following holds:

   (1) There is an even pair in O that is mapped to an even pair and there is an odd pair in O that is mapped to an odd pair.

(2) There is an even pair in O that is mapped to an odd pair and there is an odd pair in O that is mapped to an even pair.

Now suppose that there is a nonempty, relatively open set $O \subseteq [S]$ such that all nonempty, relatively open sets $U \subseteq O$ satisfy (1). Then, using the usual technology, we can construct a perfectly inhomogeneous set $P \subseteq O$ on which e is parity preserving.

If there is no nonempty, relatively open set $O \subseteq [S]$ such that all nonempty, relatively open sets $U \subseteq O$ satisfy (1), then every nonempty, relatively open set $O \subseteq [S]$ satisfies (2) and we can construct a perfectly inhomogeneous set $P \subseteq [S]$ on which e is parity inverting. This shows the claim.

Claim 46. There is a perfectly inhomogeneous set $Q \subseteq P$ such that on Q, P is height monotone.

For the proof of this claim, reprove Lemma 37 starting with the perfectly inhomogeneous set $P \subseteq 2^\omega$, making sure that the resulting set $Q \subseteq P$ is height monotone as well.

Now choose an inhomogeneous standard skew subtree R of $T(Q)$. Since e is parity preserving or inverting on $[R]$ and since e is also height monotone on $[R]$, for every n-element set $\{x_1, \ldots, x_n\} \in R$, $s_{x_1,\ldots,x_n} \sim s_{e(x_1),\ldots,e(x_n)}$. It follows that for all n-element sets $\{x_1, \ldots, x_n\} \subseteq [R]$ we have

$$d_g(x_1, \ldots, x_n) = d_g(e(x_1), \ldots, e(x_n)) = d_f(x_1, \ldots, x_n).$$

But this implies $f = g$, finishing the proof of the theorem. $\square$

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