A NOTE ON MINIMAL DYNAMICAL SYSTEMS

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Abstract. Let G be a topological group acting continuously on an infinite compact space X. Suppose the dynamical system (X, G) is minimal, i.e., suppose that every point in X has a dense G-orbit. We show that X is coabsolute with a Cantor space if G is ω-bounded. This generalizes a theorem of Balcar and Błaszczyk [1].

1. Introduction

Let G be a topological group and X a compact space. (Here compact means Hausdorff and quasi-compact.) An *action* of G on X is a homomorphism

$$\pi : G \to \mathrm{Aut}(X)$$

where $\mathrm{Aut}(X)$ is the group of autohomeomorphisms of X.

$\mathrm{Aut}(X)$ carries a natural topology, the *compact open topology* (see [3] for the definition and the properties of this topology). The action π is *continuous* if it is continuous with respect to this topology on $\mathrm{Aut}(X)$. It is important to notice that an action

$$\pi : G \to \mathrm{Aut}(X)$$

is continous if and only if the mapping

$$G \times X \to X; (g, x) \mapsto \pi(g)(x)$$

is continuous. This is the crucial feature of the compact open topology on $\mathrm{Aut}(X)$.

If π is a continuous action of G on X, then the triple (X, G, π) is a *dynamical system* or, in the context of this paper, simply a *system*. Usually the action π will be clear from the context and we do not mention it, i.e., we write (X, G) instead of (X, G, π) and for $g \in G$ and $x \in X$ we write gx instead of $\pi(g)(x)$. Similarly, for every $Y \subseteq X$ and every $g \in G$ we write gY for the set $\pi(g)[Y]$. For every $x \in X$ the set $Gx = \{gx : g \in G\}$ is the *orbit* of x. The space X is the *phase space* of the system. To avoid trivialities, we assume all the phase spaces under consideration to be either infinite or singletons.

The system (X, G) is *minimal* if every orbit is dense in X. It is easily checked that (X, G) is minimal if and only X has no proper non-empty closed subset Y which is *G-invariant*, i.e., for which $GY = \{gy : g \in G, y \in Y\}$ is equal to Y. (Hence the term minimal.) Moreover, (X, G) is minimal if and only if for all non-empty open $O \subseteq X$ there is a finite set $F \subseteq G$ such that $FO = \{gx : g \in F, x \in O\} = X$.

For dynamical systems (X, G) and (Y, G), a continuous map $h : X \to Y$ is a *homomorphism* if it commutes with the actions, i.e., if for all $x \in X$ and all $g \in G$ we have $gh(x) = h(gx)$. A homomorphism h of dynamical systems is an *isomorphism* if it is a homeomorphism. A minimal system (X, G) is *universal* if for every other minmal system (Y, G) there is a homomorphism $h : X \to Y$. It is known that every topological group has a universal minimal system, which is unique up to isomorphism.

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The phase spaces of universal minimal systems for discrete groups are *extremely disconnected*, that is, they are Stone spaces of complete Boolean algebras. This is where Cohen algebras enter the picture. A complete Boolean algebra is *Cohen* if it is the completion of a free Boolean algebra. Balcar and Błaszczyk [1] showed that extremely disconnected phase spaces of minimal systems for countable groups are Stone spaces of Cohen algebras. In particular, phase spaces of universal minimal systems for countable discrete groups are Stone spaces of Cohen algebras. (For more informations about universal minimal systems see [2] and the references therein.)

The following is easily verified: for every discrete group G and every dynamical system (X,G) there is a canonical action of G on the *absolute* (or *Gleason space*) $E(X)$ of X. $E(X)$ is the Stone space of the complete Boolean algebra of regular open subsets of X. There is a canonical continuous map from $E(X)$ onto X. This map turns out to be a homomorphism of dynamical systems. Moreover, if (X,G) is minimal, then so is $(E(X),G)$.

Thus, from the Balcar-Błaszczyk theorem it follows that for every countable discrete group G and every minimal dynamical system (X,G), the absolute of X is the Stone space of a Cohen algebra or, in other words, X is coabsolute with a generalized Cantor space. Here two compact spaces X and Y are *coabsolute* if their absolutes are homeomorpic. We show that for every ω-bounded group G and every minimal dynamical system (X,G), X is coabsolute with a generalized Cantor space. Here a topological group G is *ω-bounded* if for every non-empty open set $O\subseteq G$, G is covered by countably many translates of O. Thus, the class of ω-bounded groups includes important uncountable examples such as $\mathbb{R}$ and the unit circle. Our argument is a variation of Uspenskij's proof of the fact that compact groups are Dugundji [7].

2. Compact spaces coabsolute with Cantor spaces

The key for proving that the phase spaces of minimal dynamical systems for ω-bounded groups are coabsolute with generalized Cantor spaces is Shapiro's [6] characterization of compact spaces that are coabsolute with generalized Cantor spaces. Koppelberg [5] reformulated Shapiro's characterization in terms of Boolean algebras and simplified it a bit.

Definition 2.1. Let A and B be Boolean algebras such that $A\leq B$, i.e., such that A is a subalgebra of B. Then A is a *regular* subsalgebra of B if every maximal antichain in A is maximal in B. The *weight* $\mathrm{w}(B/A)$ of B over A is the least size of a set $C\subseteq B$ such that B is generated by $A\cup C$. The *π-weight* $\pi\mathrm{w}(B)$ of B is the least size of a dense subset of B. For $b\in B$ let $B\restriction b:=\{b'\in B: b'\leq b\}$. B is *homogeneous* in π-weight if for all $b\in B^+=B\setminus\{0\}$, $\pi\mathrm{w}(B\restriction b)=\pi\mathrm{w}(B)$. The *$\pi$-weight* $\pi\mathrm{w}(B/A)$ of B over A is the least size of a set $C\subseteq B$ such that the subalgebra of B generated by $A\cup C$ is dense in B.

We use a condensed version of Koppelberg's characterization of Cohen algebras extracted from [4].

Theorem 2.2. *Let B be a complete Boolean algebra which is homogeneous in π-weight. The following are equivalent:*

(1) *B is the completion of a free Boolean algebra.*
(2) *B is the union of an increasing chain $(B_\alpha)_{\alpha<\delta}$ of regular subalgebras of B such that $B_0=\{0,1\}$, for every limit ordinal $\gamma<\delta$, $B_\gamma=\bigcup_{\alpha<\gamma}B_\alpha$, and for all $\alpha<\delta$, $\mathrm{w}(B_{\alpha+1}/B_\alpha)\leq\aleph_0$.*
(3) *B has a dense subalgebra that is the union of an increasing chain $(B_\alpha)_{\alpha<\delta}$ of regular subalgebras of B such that $B_0=\{0,1\}$, for all limit ordinals $\gamma<\delta$, $\bigcup_{\alpha<\gamma}B_\alpha$ is dense in B_γ, and for all $\alpha<\delta$, $\pi\mathrm{w}(B_{\alpha+1}/B_\alpha)\leq\aleph_0$.*

From this characterization we derive a criterion for when a compact space is coabsolute with a generalized Cantor space. Our criterion seems to be more applicable than Shapiro's original characterization.

As in Shapiro's characterization, we use inverse systems. Our notation for inverse systems follows [3].

Definition 2.3. Let δ be an ordinal. An inverse system $\{X_\alpha, \pi^\beta_\alpha, \delta\}$ of compact spaces is *continuous* if for every limit ordinal $\gamma < \delta$, X_γ is the limit of the inverse system $\{X_\alpha, \pi^\beta_\alpha, \delta\}$.

A continuous map $f : X \to Y$ is *semi-open* if for every non-empty open set $O \subseteq X$, $f[O]$ has a non-empty interior. The *weight* $\mathrm{w}(f)$ of f is the least cardinal κ such that there is a family $\mathcal{O}$ of size κ of open sets in X such that

$$\mathcal{O} \cup \{f^{-1}[U] : U \subseteq Y \text{ is open}\}$$

forms a subbase of X.

The π-weight of a topological space X is the least size of a family $\mathcal{O}$ of non-empty open subsets of X such that for every non-empty open set $U \subseteq X$ there is $O \in \mathcal{O}$ such that $O \subseteq U$. X is homogeneous in π-weight if all the non-empty open subsets of X are of the same π-weight.

Lemma 2.4. *Let X be a compact space homogeneous in π-weight. Suppose there is an ordinal δ such that X is the limit of a continuous inverse system $\{X_\alpha, \pi^\beta_\alpha, \delta\}$ such that for all $\alpha < \delta$, $\pi^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$ is onto, semi-open, and of countable weight. Then X is coabsolute with a generalized Cantor space.*

The proof of this Lemma uses

Lemma 2.5. *Suppose X and Y are compact spaces and $f : X \to Y$ is continuous, semi-open, and onto. Then*

$$f^* : \mathrm{ro}(Y) \to \mathrm{ro}(X); O \mapsto \mathrm{int}\,\mathrm{cl}(f^{-1}[O])$$

is a complete embedding.

Moreover,

$$\pi\mathrm{w}(\mathrm{ro}(X)/f^*[\mathrm{ro}(Y)]) \leq \mathrm{w}(f).$$

Proof. First we have to check that f^* is a Boolean homomorphism at all. It is clear that f^* is monotone and preserves intersections. Now let $S \subseteq \mathrm{ro}(Y)$. Put $O = \mathrm{int}\,\mathrm{cl}(\bigcup S)$. We show that $f^*(O) = \mathrm{int}\,\mathrm{cl}(\bigcup\{f^*(U) : U \in S\})$.

By the monotonicity of f^*,

$$f^*(O) \supseteq \mathrm{int}\,\mathrm{cl}(\bigcup\{f^*(U) : U \in S\}).$$

Suppose

$$f^*(O) \neq \mathrm{int}\,\mathrm{cl}(\bigcup\{f^*(U) : U \in S\}).$$

Then there is a non-empty open set $V \subseteq f^*(O)$ which is disjoint from $\mathrm{cl}(\bigcup\{f^*(U) : U \in S\})$. In particular, $f[V]$ is disjoint from $\bigcup S$. Since f is semi-open, $W = \mathrm{int}(f[V])$ is non-empty. But W is disjoint from $\mathrm{cl}(\bigcup S)$, contradicting the fact

$$W \subseteq O = \mathrm{int}\,\mathrm{cl}(\bigcup S).$$

This shows that f^* preserves complements and is complete. Clearly, the kernel of f^* is trivial. Thus, f^* is a complete embedding.

Now let $\mathcal{O}$ be a family of size $\mathrm{w}(f)$ of open subsets of X such that

$$\mathcal{O} \cup \{f^{-1}[U] : U \subseteq Y \text{ is open}\}$$

is a subbase for X. Then clearly, $F^*[\mathrm{ro}(Y)] \cup \{\mathrm{int}\,\mathrm{cl}(O) : O \in \mathcal{O}\}$ generates a dense subalgebra of $\mathrm{ro}(X)$. $\square$

Proof of Lemma 2.4. Let $B = \mathrm{ro}(X)$. The homogeneity of X in π-weight implies that B is homogeneous in π-weight. Let $\{X_\alpha, \pi_\alpha^\beta, \delta\}$ be an inverse system as in Lemma 2.4. For all $\alpha < \delta$ let $\pi_\alpha : X \to X_\alpha$ be the canonical map. Put $B_\alpha = \pi_\alpha^*[\mathrm{ro}(X_\alpha)]$.

We show that $\bigcup_{\alpha<\delta} B_\alpha$ is dense in B. It is sufficient to show that for every non-empty open subset O of X there are $\alpha < \delta$ and a non-empty open set $U \subseteq X_\alpha$ such that $\pi_\alpha^{-1}[U] \subseteq O$.

Let $x \in O$. For every $y \in X \setminus O$ choose an ordinal $\alpha_y < \gamma$ and disjoint open sets $V_y, W_y \subseteq X$ such that $x \in V_y$, $y \in W_y$, and V_y and W_y are preimages under π_{α_y} of open subsets of X_{α_y}. It follows from the properties of limits of inverse systems that this can be done.

Since $X \setminus O$ is compact, it is covered by finitely many W_y, say

$$X \setminus O \subseteq W_{y_1} \cup \cdots \cup W_{y_n}.$$

Let $\alpha = \max\{\alpha_{y_1}, \dots, \alpha_{y_n}\}$ and $V = V_{y_1} \cap \cdots \cap V_{y_n}$. Then V is the preimage under π_α of an open subset U of X_α. Moreover, $V \subseteq O$.

By the same argument, for every limit ordinal $\gamma < \delta$, $\bigcup_{\alpha<\gamma} B_\alpha$ is dense in B_γ.

In order to show that every B_α is a regular subalgebra of B, by Lemma 2.5 it is enough to prove that every π_α is semi-open. Let $\alpha < \delta$. By induction we show that for all $\gamma < \delta$ with $\alpha < \gamma$, π_α^γ is semi-open. The argument used in the limit steps of this inductive proof then also shows that π_α is semi-open.

The successor steps of the induction are easy. Let $\gamma < \delta$ be a limit ordinal and let O be a non-empty open subset of X_γ. Since $\bigcup_{\beta<\gamma} B_\beta$ is dense in B_γ, there are $\beta < \gamma$ and a non-empty open subset U of X_β such that $(\pi_\beta^\gamma)^{-1}[U] \subseteq O$. We may assume $\beta > \alpha$. Since π_α^β is semi-open by the inductive hypothesis, $\pi_\alpha^\gamma[O] \supseteq \pi_\alpha^\beta[U]$ has a non-empty interior. It follows that π_α^γ is semi-open.

Lemma 2.5 also implies that for every $\alpha < \delta$,

$$\pi\mathrm{w}(B_{\alpha+1}/B_\alpha) = \pi\mathrm{w}(\mathrm{ro}(X_{\alpha+1})/(\pi_\alpha^{\alpha+1})^*[\mathrm{ro}(X_\alpha)]) \leq \mathrm{w}(\pi_\alpha^{\alpha+1}) \leq \aleph_0$$

Now Theorem 2.2 implies that B is the completion of a free Boolean algebra. $\square$

We intend to use yet another criterion for when a compact space X is coabsolute with a generalized Cantor space. This criterion speaks about the Banach algebra $C(X)$ of continuous real-valued functions on X.

Definition 2.6. Let X be a compact space. For a closed subalgebra B of $C(X)$ let $\equiv_B$ denote the equivalence relation on X that identifies two points $x, y \in X$ if for all $f \in B$, $f(x) = f(y)$. (We require subalgebras of $C(X)$ to contain the constant function with value 1 and therefore all constant functions.)

Recall that, essentially by the Stone-Weierstrass theorem, closed subalgebras of $C(X)$ correspond to Hausdorff quotients of X. A closed subalgebra B corresponds to the quotient $X/\equiv_B$ or, more precisely, to the quotient map $q : X \to X/\equiv_B$. If Y is Hausdorff and $p : X \to Y$ is onto, then p corresponds to the closed subalgebra $\{f \circ p : f \in C(Y)\}$ of $C(X)$. Note that

$$C(Y) \to C(X); f \mapsto f \circ p$$

is an isometric homomorphism of real algebras.

Definition 2.7. For $a \in C(X)$ let

$$a^\perp = \{b \in C(X) : a \cdot b = 0\}.$$

A closed subalgebra B of $C(X)$ is a *regular* subalgebra of $C(X)$ if for every non-zero $a \in C(X)$ there is a non-zero $b \in B$ such that $a^\perp \cap B \subseteq b^\perp$.

Lemma 2.8. *Let X be a compact space and suppose that B is a closed subalgebra of $C(X)$. Then the canonical map $q : X \to X/\equiv_B$ is semi-open if and only if B is a regular subalgebra of $C(X)$.*

Proof. Suppose that B is a regular subalgebra of $C(X)$. Let $O \subseteq X$ be a non-empty open set. We have to show that $q[O]$ has a non-empty interior. Let $U \subseteq X$ be a non-empty open set such that $\mathrm{cl}(U) \subseteq O$. It suffices to show that $q[\mathrm{cl}(U)]$ has a non-empty interior.

Let $f \in C(X)$ be a non-zero function that vanishes outside U. Since B is regular, there is a non-zero $g \in B$ such that $f^\perp \cap B \subseteq g^\perp$. The function g is of the form $\tilde{g} \circ q$ for some $\tilde{g} \in C(X/\equiv_B)$.

We claim that the non-empty open set

$$V = \{y \in X/\equiv_B : \tilde{g}(y) \neq 0\}$$

is a subset of $q[\mathrm{cl}(U)]$. For let $y \in V$. Suppose $y \notin q[\mathrm{cl}(U)]$. Since $q[\mathrm{cl}(U)]$ is closed, there is a function $h \in C(X/\equiv_B)$ such that $h(y) \neq 0$ but h vanishes on $q[\mathrm{cl}(U)]$. Thus, $h \circ q \in f^\perp \cap B$. Now from the choice of g it follows that $h \circ q \in g^\perp$. But $h(y) \cdot \tilde{g}(y) \neq 0$. A contradiction.

For the other implication suppose that q is semi-open. We show that B is a regular subalgebra of $C(X)$. Let $f \in C(X)$ be non-zero. Put

$$O = \{x \in X : f(x) \neq 0\}.$$

O is a non-empty open set. Since q is semi-open, there is a non-empty open set $U \subseteq q[O]$. Let $g \in C(X/\equiv_B)$ be non-zero such that g vanishes outside U. Put $h = g \circ q$. Now $g \in B$ and $f^\perp \cap B \subseteq g^\perp$. $\square$

Lemma 2.9. *Let X be a compact space which is homogeneous in π-weight. Suppose there is an increasing sequence $(B_\alpha)_{\alpha<\delta}$ of regular closed subalgebras of $C(X)$ such that*

(i) *$\bigcup_{\alpha<\delta} B_\alpha$ is dense in $C(X)$,*
(ii) *for every limit ordinal $\gamma < \delta$, $\bigcup_{\alpha<\gamma} B_\alpha$ is dense in B_γ, and*
(iii) *for every $\alpha < \delta$, $B_{\alpha+1}$ is countably generated (as a closed subalgebra of $C(X)$) over B_α.*

Then X is coabsolute with a generalized Cantor space.

Proof. For every $\alpha < \delta$ let $X_\alpha = X/\equiv_{B_\alpha}$. For $\alpha, \beta < \delta$ with $\alpha < \beta$ let $\pi^\beta_\alpha : X_\beta \to X_\alpha$ be the canonical map.

Now X is the limit of the continuous inverse system $\{X_\alpha, \pi^\beta_\alpha, \delta\}$. By Lemma 2.8, the π^β_α are semi-open. By Lemma 2.4 it remains to show that for every $\alpha < \delta$, $\pi^{\alpha+1}_\alpha$ is of countable weight.

Let $F \subseteq C(X)$ be a countable set such that $B_{\alpha+1}$ the closed subalgebra of $C(X)$ that is generated by $B_\alpha \cup F$. Let A be the closed subalgebra of $C(X)$ generated by F. Note that A is separable. Therefore $X/\equiv_A$ is a compact metric space.

Let

$$h : X_{\alpha+1} \to X_\alpha \times X/\equiv_A$$

be the canonical embedding and let

$$p : X_\alpha \times X/\equiv_A \to X_\alpha$$

be the projection to the first coordinate. Now $\pi^{\alpha+1}_\alpha = p \circ h$. Since $X/\equiv_A$ is of countable weight, so is $\pi^{\alpha+1}_\alpha$. $\square$

3. Minimal systems for ω-bounded groups

The crucial fact about homomorphisms of minimal dynamical systems is the following well known

Lemma 3.1. *Let G be a group and (X,G) and (Y,G) minimal dynamical systems. If $h : X \to Y$ is a homomorphism of dynamical systems, then h is onto and semi-open.*

Proof. The set $h[X]$ is a non-empty closed subspace of Y that is G-invariant. By the minimality of (Y,G), $h[X] = Y$, i.e., h is onto.

Now let $O \subseteq X$ be non-empty and open. Pick a non-empty open set $U \subseteq X$ such that $\mathrm{cl}(U) \subseteq O$. By minimality, X is the union of finitely many translates of $\mathrm{cl}(U)$ (even of U). It follows that Y is the union of finitely many translates of $h[\mathrm{cl}(U)]$. By the Baire category theorem, one of these translates must have a non-empty interior. Since G act on Y by homeomorphisms, $h[\mathrm{cl}(U)]$ has a non-empty interior. Hence $h[O]$ has a non-empty interior. □

Another simple observation is

Lemma 3.2. *Let G be a group and (X,G) a minimal dynamical system. Then X is homogeneous in π-weight.*

Proof. Let $U \subseteq X$ be an open set of minimal π-weight. We may assume that $\pi\mathrm{w}(U)$ is infinite. By minimality, there is a finite set $F \subseteq G$ such that $X = FU$. Let $\mathcal{O}$ be a π-base of U of size $\pi\mathrm{w}(U)$. It is easily checked that

$$\{fO : f \in F, O \in \mathcal{O}\}$$

is a π-base for X. □

We are now ready to prove

Theorem 3.3. *Let G be an ω-bounded group. Suppose that (X,G) is a minimal dynamical system. Then* $\mathrm{ro}(X)$ *is Cohen, i.e., X is coabsolute with a generalized Cantor space.*

Proof. By Lemma 3.2, it is sufficient to show that there is an increasing sequence $(B_\alpha)_{\alpha<\delta}$ of closed subalgebras of $C(X)$ as in Lemma 2.9.

The natural action of G on $C(X)$ is as follows: Let π denote the action of G on X. For all $g \in G$ and all $f \in C(X)$ let $gf = f \circ \pi(g)$. It is well known that the action of G on $C(X)$ is continuous.

Let $\kappa = |C(X)|$. Fix an enumeration $\{f_\alpha : \alpha < \kappa\}$ of $C(X)$. For every $\alpha < \kappa$ let B_α be the closed subalgebra of $C(X)$ generated by $\{gf_\beta : \beta < \alpha, g \in G\}$. Clearly, every B_α is closed under the group action.

For every $\alpha < \kappa$, $C_{\alpha+1}$ is generated over C_α by the set Gf_α. By the continuity of the action of G on $C(X)$, Gf_α is a continuous image of G. We claim that it is separable.

For this it is sufficient to show that for every $\varepsilon > 0$, Gf_α is covered by countably many open balls of diameter ε. Let U be the open ball around f_α of diameter ε. Then there is an open set $V \subseteq G$ such that $Vf_\alpha \subseteq U$. Since G is ω-bounded, there is a countable set $T \subseteq G$ such that $G = TV$. This implies $Gf_\alpha = TVf_\alpha$. It follows that $Gf_\alpha \subseteq TU$, i.e., Gf_α is covered by countably many translations of U. Since G acts on $C(X)$ by isometries, the translations of U are open balls of diameter ε.

Since Gf_α is separable, $B_{\alpha+1}$ is countably generated over B_α as a closed subalgebra of $C(X)$.

Since every B_α is closed under the group operation, for all $\alpha < \kappa$ the natural map π_α from X to $X/\equiv_{B_\alpha}$ is a homomorphism of minimal systems. It follows that π_α is semi-open. Now by Lemma 2.8, B_α is a regular subalgebra of $C(X)$. □

References

[1] B. Balcar, A. Błaszczyk, *On minimal dynamical systems on Boolean algebras*, Comment. Math. Univ. Carolinae 31 (1990), 7–11

[2] B. Balcar, F. Franek, *Structural properties of universal minimal dynamical systems for discrete semigroups*, Trans. Am. Math. Soc. 349, No.5, 1697–1724 (1997)

[3] R. Engelking, *General Topology*, Heldermann Verlag (1989)

[4] L. Heindorf, L. Shapiro, *Nearly projective Boolean algebras*, Springer, Lecture Notes in Mathematics 1596 (1994)

[5] S. Koppelberg, *Characterizations of Cohen algebras*, Andima, Susan (ed.) et al., Papers on general topology and applications, Proceedings of the seventh summer conference in honor of Mary Ellen Rudin and her work, Ann. N. Y. Acad. Sci. 704, 222–237 (1993)

[6] L.B. Shapiro, *On spaces coabsolute to dyadic compacta*, Soviet Math. Doklady 35 (1987), 434–438

[7] V.V. Uspenskij, *Why compact groups are dyadic*, General topology and its relations to modern analysis and algebra VI, Proc. 6th Symp., Prague/Czech. 1986, Res. Expo. Math. 16, 601–610 (1988)

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