COVERING $\mathbb{R}^{n+1}$ BY GRAPHS OF n-ARY FUNCTIONS AND LONG LINEAR ORDERINGS OF TURING DEGREES

URI ABRAHAM AND STEFAN GESCHKE

ABSTRACT. A point $(x_0, \ldots, x_n) \in X^{n+1}$ is *covered* by a function $f : X^n \to X$ iff there is a permutation σ of $n+1$ such that $x_{\sigma(0)} = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

By a theorem of Kuratowski [5], for every infinite cardinal κ exactly κ n-ary functions are needed to cover all of $(\kappa^{+n})^{n+1}$. We show that for arbitrarily large uncountable κ it is consistent that the size of the continuum is κ^{+n} and $\mathbb{R}^{n+1}$ is covered by κ n-ary continuous functions.

We study other cardinal invariants of the σ-ideal on $\mathbb{R}^{n+1}$ generated by continuous n-ary functions and finally relate the question of how many continuous functions are necessary to cover $\mathbb{R}^2$ to the least size of a set of parameters such that the Turing degrees relative to this set of parameters are linearly ordered.

1. INTRODUCTION

It is obvious that $\mathbb{R}^2$ is not the union of less than $2^{\aleph_0}$ graphs of functions. However, it might be possible to cover $\mathbb{R}^2$ by a small number of graphs of functions and their reflections on the diagonal. It was noticed by several people that this requires at least $(2^{\aleph_0})^-$ functions where $(2^{\aleph_0})^-$ is the least cardinal whose successor is $\geq 2^{\aleph_0}$. (See for example [3] or [6].)

In fact, more can be said. Let us say that a point $(x_0, \ldots, x_n) \in X^{n+1}$ is *covered* by a function $f : X^n \to X$ iff there is a permutation σ of $n+1$ such that $x_{\sigma(0)} = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. A set $S \subseteq X^{n+1}$ is *covered* by a family $\mathcal{F}$ of functions from X^n to X iff every point in S is covered by some function in $\mathcal{F}$. For a cardinal κ let κ^{+n} denote its n-th successor.

Using a slightly different formulation, Kuratowski [5] proved the following theorem, which, in the case $n = 1$, was brought to the authors' attention by Ireneusz Recław.

Theorem 1.1. *For every $n \in \omega$ and every infinite cardinal κ, $(\kappa^{+n})^{n+1}$ can be covered by κ n-ary functions, but not by less.*

However, if for example $|\mathbb{R}| = \kappa^{+n}$, then the κ n-ary functions on $\mathbb{R}$ given by Theorem 1.1 are usually not reasonably definable. If we restrict our attention to nice functions such as Borel functions, continuous functions, or even smaller classes of functions, it is not at all clear that $\mathbb{R}^2$ can be covered by a small number (i.e., $< 2^{\aleph_0}$) of such functions. Since already graphs of Borel functions are small subsets of the plane in terms of measure and category, $\mathbb{R}^2$ cannot be covered by countably many Borel functions.

However, there are several consistency results saying that $\mathbb{R}^2$ or squares of related spaces can be covered by less than continuum many nice functions. Most of these results use the Sacks model, a model of set theory obtained by adding $\aleph_2$ Sacks reals

Date: July 22, 2003.

2000 *Mathematics Subject Classification.* Primary: 03E17, 03E25; Secondary: 26A99, 26B99.

Key words and phrases. continuous function, n-space, forcing extension, covering number, Turing degree.

to a model of CH using countable support iteration. The size of the continuum is $\aleph_2$ in this model. The following is known to be true in the Sacks model:

Steprans [7] proved that $\mathbb{R}^{n+1}$ can be covered by $\aleph_1$ smooth n-dimensional manifolds. It was noticed by Ciesielski and Pawlikowski [1] that Steprans' result implies that $\mathbb{R}^2$ can be covered by $\aleph_1$ continuously differentiable functions. They also showed that $\mathbb{R}^2$ can be covered by $\aleph_1$ partial smooth functions which are defined on perfect sets. Hart and van der Steeg [6] proved that $(2^\omega)^2$ can be covered by $\aleph_1$ continuous functions and in [3] it was shown that $(2^\omega)^2$ can be covered by $\aleph_1$ Lipschitz functions. The last two results actually follow from the fact that $\mathbb{R}^2$ can be covered by $\aleph_1$ continuously differentiable functions by the arguments used in [2].

All these proofs have in common that the family of ground model sets of the required type (functions or manifolds) covers the space under consideration in the extension. The sets "of the required type" are always Borel sets, and by "ground model set" we mean a Borel set that has a Borel code in the ground model. In the following we identify Borel sets in different models of set theory if they share the same Borel code.

It is a well known problem with countable support iteration that in the resulting models the continuum is not bigger than $\aleph_2$. So if we want to show the consistency of a statement like "$\mathbb{R}^2$ can be covered by $< 2^{\aleph_0}$ continuous functions" with a big continuum, we cannot simply generalize the Sacks-model arguments to higher cardinalities.

However, there is another reasonable strategy of getting models where $\mathbb{R}^2$ is covered by $< 2^{\aleph_0}$ continuous functions. Namely, we start with a model in which $2^{\aleph_0}$ has the desired size, and then add a small number of continuous real functions that will, in the final model, cover $\mathbb{R}^2$. This really works if the addition of continuous functions is organized in the right way. Moreover, the method generalizes to higher dimensions and we obtain models of set theory in which the continuum is of the form κ^{+n} for some uncountable cardinal κ and $\mathbb{R}^{n+1}$ can be covered by κ continuous n-ary functions. This is optimal by the lower bound provided by Theorem 1.1.

The approach of adding continuous functions by forcing in order to cover a square was implicitly used by Zapletal [8] who showed that under MA(σ-centered)+$\neg$CH for every set X of size $\aleph_1$ of reals there is a single real r such that the Turing degrees relative to r of the elements of X are linearly ordered, answering a question addressed by Blass. We will come back later to the connection between linear orderings of Turing degrees and coverings of squares by continuous functions.

2. Covering $(\kappa^{+n})^{n+1}$ by n-ary functions

For the convenience of the reader and since we have further use for one part of the proof, namely Lemma 2.3, we include a proof of Theorem 1.1.

We first show that κ functions are indeed sufficient to cover an $n+1$-dimensional cube of size κ^{+n}.

Lemma 2.1. *For every $n \in \omega$ and every infinite cardinal κ, $(\kappa^{+n})^{n+1}$ can be covered by κ n-ary functions.*

Proof. We use an induction on n. For $n = 0$ the statement is trivial. (Recall that for $n = 0$ an n-ary function is a constant.) Suppose $n = m + 1$ and $(\kappa^{+m})^{m+1}$ can be covered by κ m-ary functions.

Then for every $\alpha < \kappa^{+n}$ there is a family $\mathcal{F}_\alpha = \{f^\beta_\alpha : \beta < \kappa\}$ of m-ary functions on $\alpha + 1$ which covers $(\alpha + 1)^{m+1}$.

Let $\beta < \kappa$ and $\alpha_0, \dots, \alpha_m < \kappa^{+n}$. Choose a permutation σ of $m + 1$ such that $\alpha_{\sigma(m)} \geq \alpha_0, \dots, \alpha_m$. Put

$$g_\beta(\alpha_0, \dots, \alpha_m) := f^\beta_{\alpha_{\sigma(m)}}(\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(m-1)}).$$

Claim 2.2. $\{g_\beta : \beta < \kappa\}$ covers $(\kappa^{+n})^{n+1}$.

For the claim let $\alpha_0, \dots, \alpha_n < \kappa$. We have to find $\beta < \kappa$ such that $(\alpha_0, \dots, \alpha_n)$ is covered by g_β. We may assume that $\alpha_n \geq \alpha_0, \dots, \alpha_m$.

By the choice of $\mathcal{F}_{\alpha_n}$, there is some $\beta < \kappa$ such that $(\alpha_0, \dots, \alpha_m)$ is covered by $f^\beta_{\alpha_n}$. It follows that $(\alpha_0, \dots, \alpha_n)$ is covered by g_β. $\square$

The fact that $(\kappa^{+n})^{n+1}$ cannot be covered by less that κ n-ary functions follows by induction on n from the following lemma, which gives a bit more information.

For a set X, $n \in \omega$, and a class $\mathcal{C}$ of functions let $I_{\mathcal{C},n}(X)$ denote the σ-ideal on X^{n+1} generated by the sets

$$\{(x_0, \dots, x_n) \in X^{n+1} : (x_0, \dots, x_n) \text{ is covered by } f\}$$

where $f \in \mathcal{C}$ is an n-ary function on X.

The *covering number* $\operatorname{cov}(I)$ of some ideal I on a set X is the least number of sets from the ideal needed to cover the underlying set X. (Provided, of course, the whole ideal covers the space. It makes sense to define $\operatorname{cov}(I) := \infty$, otherwise.)

Lemma 2.3. *Let X be an infinite set, $\mathcal{C}$ a class of functions, and $n \in \omega$. Suppose, for every $f : X^{n+1} \to X$ with $f \in \mathcal{C}$ and every $x \in X$, that the function $f_x : X^n \to X; (x_1, \dots, x_n) \mapsto f(x, x_1, \dots, x_n)$ is an element of $\mathcal{C}$. Then* $\operatorname{cov}(I_{\mathcal{C},n}(X)) \leq \operatorname{cov}(I_{\mathcal{C},n+1}(X))^+$.

Proof. We may assume $\operatorname{cov}(I_{\mathcal{C},n+1}(X))^+ < \infty$. Let $\mathcal{F} \subseteq \mathcal{C}$ be a family of size $\kappa := \operatorname{cov}(I_{\mathcal{C},n+1}(X))$ of $n+1$-ary functions on X covering X^{n+2}. For simplicity assume that $\mathcal{F}$ is closed under permutation of the arguments, i.e., for all $f \in \mathcal{F}$ and every permutation σ of $n+1$, the function that maps $(x_0, \dots, x_n)$ to $f(x_{\sigma(0)}, \dots, x_{\sigma(n)})$ is an element of $\mathcal{F}$.

Let M be an elementary submodel of H_χ for some sufficiently large χ such that $\mathcal{F} \subseteq M$, $X \in M$, and $|M|=|M \cap X|= \kappa^+$. Suppose $\kappa^+ < \operatorname{cov}(I_{\mathcal{C},n}(X))$. Then there is $(x_0, \dots, x_n) \in X^{n+1}$ which is not covered by $\{f_x : x \in X \cap M \wedge f \in \mathcal{F}\}$. Let N be an elementary submodel of H_χ such that $\mathcal{F} \subseteq N$, $(x_0, \dots, x_n), X \in N$, and $|N|= \kappa$.

Choose $x \in (X \cap M) \setminus N$. By the choice of $(x_0, \dots, x_n)$, there is no $f \in \mathcal{F}$ such that $(x_0, \dots, x_n)$ is covered by f_x. On the other hand, for $f \in \mathcal{F}$ we have $f(x_0, \dots, x_n) \neq x$ since f and $(x_0, \dots, x_n)$ are elements of N but x is not. Since $\mathcal{F}$ is closed under permutations of the arguments, it follows that $(x, x_0, \dots, x_n)$ is not covered by $\mathcal{F}$. A contradiction. $\square$

3. Adding continuous functions

Let $n > 0$ be a natural number and let f be a function from a subset of $\mathbb{R}^n$ to $\mathbb{R}$. We define a forcing notion adding a countable family $\mathcal{F}$ of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ covering f (in the usual sense, i.e., $f \subseteq \bigcup \mathcal{F}$). Here we do not assume that f is a Borel function. If we talk about f in the generic extension, we mean the same set of pairs as in the ground model. Of course, our forcing notion adds new reals. Thus, even if f is a total function in the ground model, it is only a partial function in the extension. The functions in $\mathcal{F}$ are total functions in the extension.

For technical reasons we prefer to work over a compact space. The unit interval $I := [0, 1]$ is homeomorphic to the two-point compactification of $\mathbb{R}$. By transforming a given function f from a subset of $\mathbb{R}^n$ to $\mathbb{R}$ into a function from a subset of I^n to I, then adding countably many continuous n-ary functions on I covering the transform of f, and finally transforming the restrictions of the new continuous functions to $(0, 1)^n$ back to n-ary functions on $\mathbb{R}^n$, we obtain a countable family of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ which covers the original f.

Consider the set $C(I^n, I)$ of continuous functions from I^n to I equipped with the topology of uniform convergence, i.e., the topology induced by the sup-norm $\|\cdot\|_\infty$ on $C(I^n, \mathbb{R})$. Clearly, the space $C(I^n, I)$ is separable. Choose a countable dense set $D_n \subseteq C(I^n, I)$.

Definition 3.1. Let $n > 0$ and let f be a function from a subset of I^n to I. Then $p = (f_p, F_p, \varepsilon_p)$ is a condition in $\mathbb{P}(I^n, f)$ if $f_p \in D_n$, F_p is a finite subset of $\mathrm{dom}(f)$, ε_p is a real number > 0, and for all $x \in F_p$, $|f(x) - f_p(x)| < \varepsilon_p$.

A condition $p \in \mathbb{P}(I^n, f)$ extends $q \in \mathbb{P}(I^n, f)$, i.e., $p \leq q$, if $F_p \supseteq F_q$, $\varepsilon_p \leq \varepsilon_q$, and $\| f_p - f_q \|_\infty \leq \varepsilon_q - \varepsilon_p$.

It is easily checked that $\leq$ is transitive on $\mathbb{P}(I^n, f)$. If $p, q \in \mathbb{P}(I^n, f)$ satisfy $f_p = f_q$, then p and q are compatible, the condition $(f_p, F_p \cup F_q, \min(\varepsilon_p, \varepsilon_q))$ being a common extension. By induction, every finite collection of conditions with the same first component has a common extension. Since there are only countably many possibilities for the first component f_p of a condition p, namely the elements of D_n, $\mathbb{P}(I^n, f)$ is σ-centered.

Let G be $\mathbb{P}(I^n, f)$-generic. Then for every $i \in \omega$ there is some $p \in G$ such that $\varepsilon_p < \frac{1}{i+1}$. Let $g_i := f_p$. Since G is a filter, the sequence $(g_i)_{i\in\omega}$ uniformly converges to some continuous function $f_G : I^n \to I$. Recall that even though officially the f_p are functions in the ground model, we identify them with the functions in the generic extension that have the same Borel definition. The functions in the generic extension are simply the unique continuous extensions of the old functions to the new reals. The function f_G is of course a function that exists only in the generic extension. If for some $x \in I^n$ there is $p \in G$ with $x \in F_p$, then $f_G(x) = f(x)$.

Now let $\mathbb{P}^*(I^n, f)$ denote the finite support product of countably many copies of $\mathbb{P}(I^n, f)$, say with index set ω. Suppose G is $\mathbb{P}^*(I^n, f)$-generic over the ground model. For $i \in \omega$ let G_i denote the projection of G to the i-th copy of $\mathbb{P}(I^n, f)$ in the product $\mathbb{P}^*(I^n, f)$. An easy density argument shows that for each $x \in \mathrm{dom}(f)$ there is $i \in \omega$ such that for some $p \in G_i$, $x \in F_p$. For this i we have $f(x) = f_{G_i}(x)$. It follows that f is covered by the functions f_{G_i}, $i \in \omega$.

Lemma 3.2. *For every $n > 0$ and every function f from a subset of $\mathbb{R}^n$ to $\mathbb{R}$ there is a σ-centered forcing notion $\mathbb{P}^*(\mathbb{R}^n, f)$ which adds a sequence $(f_i)_{i\in\omega}$ of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that $f \subseteq \bigcup_{i\in\omega} f_i$.*

Proof. Fix a homeomorphism $h : \mathbb{R} \to (0, 1)$. Let $b : \mathcal{P}(\mathbb{R}^n \times \mathbb{R}) \to \mathcal{P}((0, 1)^n \times (0, 1))$ be the bijection induced by h. The forcing notion $\mathbb{P}^*(\mathbb{R}^n, f) := \mathbb{P}^*(I^n, b(f))$ is σ-centered since it is a finite support product of countably many σ-centered forcing notions. Let G be $\mathbb{P}^*(I^n, b(f))$-generic and for $i \in \omega$ let G_i be the projection of G to the i-th coordinate in the product $\mathbb{P}^*(I^n, b(f))$. Let $f_i := b^{-1}(f_{G_i} \upharpoonright (0, 1)^n)$ where $f_{G_i} : I^n \to I$ is the generic function coded by G_i. It is easily checked that $(f_i)_{i\in\omega}$ has the desired properties. $\square$

Similarly, for every function f from a subset of 2^ω to 2^ω there is a σ-centered forcing notion $\mathbb{P}^*(2^\omega, f)$ adding countably many continuous functions from 2^ω to 2^ω covering f. Since 2^ω is homeomorphic to $(2^\omega)^n$ for every $n > 0$, the same forcing can be used to add continuous functions covering a function f from a subset of $(2^\omega)^n$ to 2^ω.

4. Covering $\mathbb{R}^{n+1}$ by a small number of continuous functions

Let $\mathcal{C}ont$ denote the class of continuous functions between topological spaces. Following the notation in Section 2, $I_{\mathcal{C}ont,n}(\mathbb{R})$ is the σ-ideal generated by the sets of the form

$$\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : (x_0, \ldots, x_n) \text{ is covered by } f\}$$

where f is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$.

Using Lemma 3.2 and Theorem 1.1 it is easy to construct models of set theory where $\mathbb{R}^{n+1}$ is covered by a small number of continuous n-ary functions. The only restrictions are $\mathrm{cov}(I_{\mathcal{C}ont,n}(\mathbb{R})) \geq \aleph_1$ and $\mathrm{cov}(I_{\mathcal{C}ont,n}(\mathbb{R}))^{+n} \geq 2^{\aleph_0}$. The first restriction follows from the fact that graphs of continuous functions are meager and powers of $\mathbb{R}$ cannot be covered by countably many meager sets. The second restriction follows from Theorem 1.1.

Theorem 4.1. *Let $n \in \omega$. Let $\kappa \geq \aleph_1$ be a cardinal and suppose the universe V satisfies $2^{\aleph_0} \leq \kappa^{+n}$. Then there is a generic extension $V[G]$ in which $\mathbb{R}^{n+1}$ can be covered by κ continuous n-ary functions and which has the same cardinals and the same size of the continuum as V.*

Proof. The case $n = 0$ is trivial. So assume $n > 0$. For a family $\mathcal{E}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}$ let $\mathbb{P}_{\mathcal{E}}$ be the finite support product of the forcings $\mathbb{P}^*(\mathbb{R}^n, f)$, $f \in \mathcal{E}$. $\mathbb{P}_{\mathcal{E}}$ is c.c.c. and does not change the size of the continuum as long as $|\mathcal{E}| \leq 2^{\aleph_0}$. Let $(\mathbb{Q}_\alpha)_{\alpha \leq \omega_1}$ be a finite support iteration such that for all $\alpha < \omega_1$, $\mathbb{Q}_{\alpha+1} = \mathbb{Q}_\alpha * \mathbb{P}_{\dot{\mathcal{E}}_\alpha}$ where

$$\mathbb{Q}_\alpha \Vdash \dot{\mathcal{E}}_\alpha \subseteq {}^{\mathbb{R}^n}\mathbb{R} \wedge |\dot{\mathcal{E}}_\alpha| = \kappa \wedge \dot{\mathcal{E}}_\alpha \text{ covers } \mathbb{R}^{n+1}.$$

Let G be $\mathbb{Q}_{\omega_1}$-generic over the universe. We argue in $V[G]$. For every $\mathbb{Q}_{\omega_1}$-name $\dot{x}$ let $\dot{x}[G]$ denote its evaluation with respect to G.

For each $\alpha < \omega_1$ let $\mathcal{F}_\alpha$ be the union of the countable sets of continuous functions added by the factors of $\mathbb{P}_{\dot{\mathcal{E}}_\alpha}[G]$. $\mathcal{F}_\alpha$ is of size κ. Put $\mathcal{F} := \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$. Now for each point $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ there is $\alpha < \omega_1$ such that $x_0, \dots, x_n$ have been added before stage α of the iteration. The point $(x_0, \dots, x_n)$ is covered by some function $g \in \dot{\mathcal{E}}_\alpha[G]$. g is covered by countably many functions from $\mathcal{F}_\alpha$. In particular, $(x_0, \dots, x_n)$ is covered by some $f \in \mathcal{F}_\alpha \subseteq \mathcal{F}$.

It follows that $\mathcal{F}$ is a family of size $\leq \kappa$ of continuous n-ary real functions covering $\mathbb{R}^{n+1}$. $V[G]$ has the same cardinals as V since it is a c.c.c. extension of V. Also, $2^{\aleph_0}$ is the same in $V[G]$ as in V. $\square$

Corollary 4.2. *The following is consistent with ZFC:*

$$2^{\aleph_0} = \aleph_{27} \wedge \mathrm{cov}(I_{\mathcal{C}ont,7}(\mathbb{R})) = \aleph_{20}$$

It is worth noting that by Lemma 2.3, $\mathrm{cov}(I_{\mathcal{C}ont,n}(\mathbb{R})) \leq \mathrm{cov}(I_{\mathcal{C}ont,n+1}(\mathbb{R}))^+$. It follows that if $\mathrm{cov}(I_{\mathcal{C}ont,n}(\mathbb{R}))^{+n} = 2^{\aleph_0}$ (as in the model in Corollary 4.2 for $\kappa = \aleph_{20}$ and $n = 7$), then the covering numbers of the ideals $I_{\mathcal{C}ont,i}(\mathbb{R})$ are pairwise different for $i \leq n$.

Theorem 4.1 says that it is consistent for $\mathrm{cov}(I_{\mathcal{C}ont,n}(\mathbb{R}))$ to be small. The dual of the covering number of an ideal I is the *uniformity* $\mathrm{non}(I)$ of I, the smallest size of a subset of the underlying set of I which is not in the ideal. This is where it becomes important that we defined $I_{\mathcal{C},n}(X)$ to be a σ-ideal. Using Ramsey's theorem, it can be shown that for every infinite space X and every infinite $A \subseteq X$, A^{n+1} cannot be covered by finitely many functions from X^n to X. But since we are looking at σ-ideals, every set that can be covered by countably many functions of the considered class is still in the ideal. Using the idea of the proof of Theorem 4.1 we can show

Theorem 4.3. *Assume MA(σ-centered). Then for every $n \in \omega$*

$$\mathrm{non}(I_{\mathcal{C}ont,n}(\mathbb{R})) = \min(\aleph_1^{+n}, 2^{\aleph_0}).$$

Proof. Since the ideal $I_{\mathcal{C}ont,n}(\mathbb{R})$ consists of meager subsets of $\mathbb{R}^{n+1}$, we have $\mathrm{non}(I_{\mathcal{C}ont,n}(\mathbb{R})) \leq 2^{\aleph_0}$. By Theorem 1.1, $\mathrm{non}(I_{\mathcal{C}ont,n}(\mathbb{R})) \leq \aleph_1^{+n}$.

Now suppose $Y \subseteq \mathbb{R}^{n+1}$ is of size $< \min(\aleph_1^{+n}, 2^{\aleph_0})$. We have to show that Y can be covered by countably many continuous n-ary functions.

We may assume that Y is infinite. By passing to a larger set of the same size, we may assume that Y is of the form Z^{n+1} for some $Z \subseteq \mathbb{R}$. By Theorem 1.1 there is a countable family $\mathcal{F}$ of functions from Z^n to Z covering $Y = Z^{n+1}$. Note that the functions in $\mathcal{F}$ are of size $|Z| < 2^{\aleph_0}$.

MA(σ-centered) implies that every function from a subset of $\mathbb{R}^n$ of size $< 2^{\aleph_0}$ to $\mathbb{R}$ is covered by countably many continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. This is easily seen using the forcing notions $\mathbb{P}^*(\mathbb{R}^n, f)$. It follows that for every $f \in \mathcal{F}$ there is a countable set $\mathcal{G}_f$ of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that $f \subseteq \bigcup \mathcal{G}_f$. The set $\bigcup_{f \in \mathcal{F}} \mathcal{G}_f$ is countable and covers Y. $\square$

5. Covering powers of 2^ω and ω^ω

It is natural to ask about the relation between the covering number of the ideal $I_{\mathcal{C}ont,n}(\mathbb{R})$ and the covering numbers of $I_{\mathcal{C}ont,n}(X)$ for other Polish spaces X. In [2] it was observed that if X is the disjoint union of $\mathbb{R}$ and 2^ω, then $2^{\aleph_0}$ continuous functions from X to X are necessary to cover X^2. However, it was also shown that the covering numbers of the ideals $I_{\mathcal{C}ont,1}(X)$ are the same for $X = \mathbb{R}$, $X = 2^\omega$, and $X = \omega^\omega$. We generalize this to all finite dimensions and show

Theorem 5.1. *For all $n \in \omega$,*

$$\mathrm{cov}(I_{\mathcal{C}ont,n}(\mathbb{R})) = \mathrm{cov}(I_{\mathcal{C}ont,n}(2^\omega)) = \mathrm{cov}(I_{\mathcal{C}ont,n}(\omega^\omega)).$$

The proof of this theorem needs some preparation. We mainly have to show that $\mathrm{cov}(I_{\mathcal{C}ont,n}(2^\omega))$ is not smaller than the *dominating number* $\mathfrak{d}$, the least number of copies of 2^ω needed to cover ω^ω. For every $x \in 2^\omega$ which has infinitely often the value 1 let $e_x : \omega \to \omega$ be the increasing enumeration of $x^{-1}(1)$ and let $d_x : \omega \to \omega$ be defined by $d_x(0) := e_x(0)$ and $d_x(n) := e_x(n) - e_x(n-1)$ for every $n > 0$.

Let λ be a sufficiently big cardinal and consider the structure $(H_\lambda, \in)$ with Skolem functions. For an elementary submodel M of H_λ and sets $x_1, \dots, x_n \in H_\lambda$ let $M[x_1, \dots, x_n]$ denote the Skolem hull of $M \cup \{x_1, \dots, x_n\}$. $f : \omega \to \omega$ is *unbounded over M* if for every $g \in \omega^\omega \cap M$ there are infinitely many $n \in \omega$ such that $f(n) > g(n)$. A function $g : \omega \to \omega$ for which $\{n \in \omega : f(n) > g(n)\}$ is finite is a *bound of f*.

The crucial fact, which was implicitly used in [2], is the following:

Lemma 5.2. *Let $M \preccurlyeq H_\lambda$. Suppose $x, y \in 2^\omega$ are such that $x \notin M$ and d_y is unbounded over $M[x]$. Then there is no continuous function $f : 2^\omega \to 2^\omega$ such that $f \in M$ and $f(y) = x$.*

Proof. Fix M, x, and y as above. Let $f : 2^\omega \to 2^\omega$ be a continuous function with $f \in M$. Then for all $z \in 2^\omega \cap M$, $f(z) \neq x$. In other words, $f^{-1}(x) \cap M = \emptyset$. The function $d_z : \omega \to \omega$ is defined for all $z \in 2^\omega$ that are not eventually constant. In particular, d_z is defined for all $z \in f^{-1}(x)$. Since $f^{-1}(x)$ is compact and $d : z \mapsto d_z$ is continuous, the set $Z := \{d_z : z \in f^{-1}(x)\} \subseteq \omega^\omega$ is compact and thus bounded. Since $Z \in M[x]$, $d_y \notin Z$. It follows that $f(y) \neq x$. $\square$

We need a generalisation of Lemma 5.2 to n-ary functions, which we derive from Lemma 5.2 itself. We first observe that the unboundedness of d_x is equivalent to the unboundedness of e_x.

Lemma 5.3. *Let $M \preccurlyeq H_\lambda$. For every $x \in 2^\omega$, d_x is unbounded over M iff e_x is unbounded over M.*

Proof. It is clear that for every $n \in \omega$, $d_x(n) \leq e_x(n)$. Therefore e_x is unbounded if d_x is. Now suppose that d_x is bounded by some function $b \in \omega^\omega \cap M$. We may assume that b is never 0. Let $y \subseteq \omega$ be the unique infinite set such that $d_y = b$. Since $b \in M$, also $y \in M$ and $e_y \in M$. Obviously e_y is a bound of e_x. $\square$

Lemma 5.4. *Let $M \preccurlyeq H_\lambda$. Suppose there are $x_0, \dots, x_n \in 2^\omega$ such that $x_0 \notin M$ and for all $i < n$, $d_{x_{i+1}}$ is unbounded over $M[x_0, \dots, x_i]$. Then there is no continuous function $f : (2^\omega)^n \to 2^\omega$ such that $f \in M$ and f covers $(x_0, \dots, x_n)$.*

Proof. For a contradiction assume that $f : (2^\omega)^n \to 2^\omega$ is a continuous function in M which covers $(x_0, \dots, x_n)$. Let σ be a permutation of $n+1$ such that $x_{\sigma(0)} = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Let $m := \sigma(0)$. Since $f, x_0, \dots, x_{n-1} \in M[x_0, \dots, x_{n-1}]$ and $x_n \notin M[x_0, \dots, x_{n-1}]$, $m \neq n$.

In $M[x_0, \dots, x_{m-1}]$ we have an $n-m$-ary continuous function on 2^ω which covers $(x_m, \dots, x_n)$. This function is obtained by plugging the parameters $x_0, \dots, x_{m-1}$ into f at the right places. It follows that we may assume, by varying n if necessary, that $m = 0$ and $x_0 = f(x_1, \dots, x_n)$.

For $x, y \in 2^\omega$ let $x \otimes y := (x(0), y(0), x(1), y(1), \dots)$. f gives rise to a continuous function $g : 2^\omega \to 2^\omega$ such that $x_0 = g((\dots(x_1 \otimes x_2) \otimes \dots) \otimes x_n)$. Using Lemma 5.2 we arrive at a contradiction, once we can show that $d_{(\dots(x_1 \otimes x_2)\otimes\dots)\otimes x_n}$ is unbounded over $M[x_0]$. But this follows by induction from

Claim 5.5. Let $N \preccurlyeq H_\lambda$. Suppose $y, z \in 2^\omega$, d_y is unbounded over M, and d_z is unbounded over $M[y]$. Then $d_{y\otimes z}$ is unbounded over M.

By Lemma 5.3, it is sufficient to prove the claim with d_y, d_z, and $d_{y\otimes z}$ replaced by e_y, e_z, and $e_{y\otimes z}$, respectively.

Suppose that e_y is unbounded over M and let $b \in \omega^\omega \cap M$. Consider b as a candidate for a bound of $e_{y\otimes z}$. We may assume that b is strictly increasing. Let $\overline{0} \in 2^\omega$ be the function that is constantly 0. By the unboundedness of e_y, $e_{y\otimes\overline{0}}$ is not bounded by the function $\{(n, b(2n)) : n \in \omega\} \in M$. Therefore in $M[y]$ we can find $a \in 2^\omega$ such that $a^{-1}(1)$ is infinite and whenever $e_{y\otimes a}(n)$ is odd, then $e_{y\otimes a}(n) > b(n)$.

Since e_z is unbounded over $M[y]$, $\{n \in \omega : e_z(n) > e_a(n)\}$ is infinite. Let $n \in \omega$ be such that $e_z(n) > e_a(n)$. Let $m \in \omega$ be such that $e_{y\otimes a}(m) = 2e_a(n) + 1$. Then $e_{y\otimes z}(m) > e_{y\otimes a}(m) > b(m)$. It follows that b is not a bound of $e_{y\otimes z}$. □

Lemma 5.6. *For all $n \in \omega$,* $\mathrm{cov}(I_{\mathcal{C}ont,n}(2^\omega)) \geq \mathfrak{d}$.

Proof. Let $\mathcal{F}$ be a family of continuous n-ary functions on 2^ω. Suppose $|\mathcal{F}| < \mathfrak{d}$. Let M be an elementary submodel of H_λ of size $|\mathcal{F}| + \aleph_0$ such that $\mathcal{F} \subseteq M$. Since $|M| < \mathfrak{d}$, there are $x_0, \dots, x_n \in 2^\omega$ such that $x_0 \notin M$ and for all $i < n$, $d_{x_{i+1}}$ is unbounded over $M[x_0, \dots, x_i]$. By Lemma 5.4, $(x_0, \dots, x_n)$ is not covered by a continuous n-ary function in M. In particular, $\mathcal{F}$ does not cover $(2^\omega)^{n+1}$. This shows the lemma. □

We are now ready to give the

Proof of Theorem 5.1. The proof is essentially the same as the proof of

$$\mathrm{cov}(I_{\mathcal{C}ont,1}(\mathbb{R})) = \mathrm{cov}(I_{\mathcal{C}ont,1}(2^\omega)) = \mathrm{cov}(I_{\mathcal{C}ont,1}(\omega^\omega))$$

given in [2]. Therefore we will just sketch the argument.

Let X be either $\mathbb{R}$ or ω^ω. Then 2^ω is homeomorphic to a subspace of X. Every family $\mathcal{F}$ of continuous functions from X^n to X which covers X^{n+1} gives rise to a family $\mathcal{F}'$ of no greater size of continuous partial functions from $(2^\omega)^n$ to 2^ω which covers $(2^\omega)^{n+1}$. $\mathcal{F}'$ is obtained by intersecting every $f \in \mathcal{F}$ with the copy of $(2^\omega)^{n+1}$ inside X^{n+1}. The functions from $\mathcal{F}'$ are defined on compact subsets of $(2^\omega)^n$ and therefore can be extended continuously to all of $(2^\omega)^n$ using the parallel of the Tietze-Urysohn Theorem for 2^ω. This shows

$$\mathrm{cov}(I_{\mathcal{C}ont,n}(2^\omega)) \leq \mathrm{cov}(I_{\mathcal{C}ont,n}(\omega^\omega)), \mathrm{cov}(I_{\mathcal{C}ont,n}(\mathbb{R})).$$

Now let $\mathcal{F}$ be a family of n-ary functions on 2^ω which covers $(2^\omega)^{n+1}$ and let X be one of the spaces $\mathbb{R}$ and ω^ω. By Lemma 5.6, $|\mathcal{F}|\geq \mathfrak{d}$. Recall that ω^ω can be covered by $\mathfrak{d}$ copies of 2^ω. Since ω^ω is homeomorphic to a co-countable subspace of $\mathbb{R}$, namely the set of irrational numbers, $\mathbb{R}$ can also be covered by $\mathfrak{d}$ copies of 2^ω.

It follows that X^{n+1} can be covered by $\mathfrak{d}$ sets that are products of $n+1$ copies of 2^ω. On each of these sets we have a copy of the family $\mathcal{F}$. The union of these copies of $\mathcal{F}$ is a family $\mathcal{F}'$ of partial continuous functions from X^n to X which covers X^{n+1}. The functions from $\mathcal{F}'$ are defined on compact subsets of X^n and therefore, by the Tietze-Urysohn Theorem, respectively by the corresponding theorem for ω^ω, they can be extended continuously to all of X^n. Thus we have family of not more than $\mathfrak{d}\cdot |\mathcal{F}|=|\mathcal{F}|$ continuous n-ary functions on X which covers X^{n+1}. This finishes the proof of the theorem. $\square$

Using almost the same proof as for Lemma 5.6, we see that the dual of Lemma 5.6 is also true. The *unboundedness number* $\mathfrak{b}$ is the least size of a subset of ω^ω that cannot be covered by countably many copies of 2^ω.

Lemma 5.7. *For all $n \in \omega$,* $\mathrm{non}(I_{\mathcal{C}ont,n}(2^\omega)) \leq \mathfrak{b}$.

Proof. Let $A \subseteq \omega^\omega$ be such that A cannot be covered by countably many copies of 2^ω. Let $\mathcal{F}$ be a countable family of continuous n-ary functions on 2^ω and let M be a countable elementary submodel of H_λ such that $\mathcal{F} \subseteq M$. Since every co-countable subset of A contains a function that is unbounded over M, there are $x_0, \dots, x_n \in 2^\omega$ such that $x_0 \notin M$ and for all $i < n$, $d_{x_{i+1}} \in A$ and $d_{x_{i+1}}$ is unbounded over $M[x_0, \dots, x_i]$. By Lemma 5.4, $(x_0, \dots, x_n)$ is not covered by a continuous n-ary function in M. In particular, $\mathcal{F}$ does not cover $B := \{(x_0, \dots, x_n) : \forall i < n+1(d_{x_i} \in A)\}$. This shows $B \notin I_{\mathcal{C}ont,n}(2^\omega)$. Clearly, $|B|\leq|A|$. $\square$

Lemma 5.7 enables us to dualize the proof of Theorem 5.1 and we get

Theorem 5.8. *For all $n \in \omega$,*

$$\mathrm{non}(I_{\mathcal{C}ont,n}(\mathbb{R})) = \mathrm{non}(I_{\mathcal{C}ont,n}(2^\omega)) = \mathrm{non}(I_{\mathcal{C}ont,n}(\omega^\omega)).$$

Proof. Since 2^ω embeds into $\mathbb{R}$ and into ω^ω, every set $A \subseteq (2^\omega)^{n+1}$ which is not contained in $I_{\mathcal{C}ont,n}(2^\omega)$ gives rise to subsets of $(\omega^\omega)^{n+1}$ and $\mathbb{R}^{n+1}$ of the same size that are not elements of the respective ideals on $(\omega^\omega)^{n+1}$ and $\mathbb{R}^{n+1}$. This uses an argument on extending partial continuous functions on 2^ω as in the proof of Theorem 5.1. We obtain

$$\mathrm{non}(I_{\mathcal{C}ont,n}(2^\omega)) \geq \mathrm{non}(I_{\mathcal{C}ont,n}(\mathbb{R})), \mathrm{non}(I_{\mathcal{C}ont,n}(\omega^\omega)).$$

To show

$$\mathrm{non}(I_{\mathcal{C}ont,n}(2^\omega)) \leq \mathrm{non}(I_{\mathcal{C}ont,n}(\mathbb{R})), \mathrm{non}(I_{\mathcal{C}ont,n}(\omega^\omega))$$

let X be one of the spaces ω^ω and $\mathbb{R}$ and let $A \subseteq X^{n+1}$ be a set such that $|A|< \mathrm{non}(I_{\mathcal{C}ont,n}(2^\omega))$. We may assume that A is of the form B^{n+1} for some $B \subseteq X$. By Lemma 5.7, B can be covered by a countable family $\mathcal{C}$ of copies 2^ω. Now for every $(C_0, \dots, C_n) \in \mathcal{C}^{n+1}$, $A \cap C_0 \times \dots \times C_n$ can be covered by countably many continuous functions from X^n to X, following the argument in the proof of Theorem 5.1. This implies $A \in I_{\mathcal{C}ont,n}(2^\omega)$. $\square$

We conclude this section with a remark on the other cardinal invariants of the ideals $I_{\mathcal{C}ont,1}(X)$. Let I be a σ-ideal on a set X. The *additivity* $\mathrm{add}(I)$ of I is the least size of a family $\mathcal{F} \subseteq I$ whose union is not in I. The *cofinality* $\mathrm{cof}(I)$ of I is the least size of a set which is cofinal in $(I, \subseteq)$.

Lemma 5.9. *Let X be a set of size $> \aleph_1$ and let $\mathcal{C}$ be a class of functions that includes all constant functions. Then* $\mathrm{add}(I_{\mathcal{C},1}(X)) = \aleph_1$ *and* $\mathrm{cof}(I_{\mathcal{C},1}(X)) =|X|$.

Proof. Let $\mathcal{F} \subseteq \mathcal{C}$ be a countable family of functions from X to X. For every $y \in X$ let $c_y : X \to X$ be the constant function with value y, i.e., the set $X \times \{y\}$.

Claim 5.10. For all but countably many $y \in X$, c_y is not covered by $\mathcal{F}$.

Let $A \subseteq X$ be a set of size $\aleph_1$. For each $y \in A$ let $A_y := \{f(y) : f \in \mathcal{F}\}$. Clearly, each A_y is countable. Let $B := X \setminus \bigcup_{y \in A} A_y$. Since $|A|<|X|$, B is nonempty.

Now for all $x \in B$, the set $C := \{f(x) : f \in \mathcal{F}\}$ is countable. For every $y \in A \setminus C$, c_y is not covered by $\mathcal{F}$. This shows the claim.

From the claim it follows that for every set $A \subseteq X$ of size $\aleph_1$ the set $\bigcup_{y \in X} c_y$ is not in $I_{\mathcal{C},1}(X)$. Similarly, if $\mathcal{A} \subseteq I_{\mathcal{C},1}(X)$ is a family of size $<|X|$, there is $y \in X$ such that c_y is not included in any member of $\mathcal{A}$. Thus, $\mathcal{A}$ is not cofinal in $I_{\mathcal{C},1}(X)$. □

Corollary 5.11. *a)* $\mathrm{add}(I_{\mathcal{C}ont,1}(\mathbb{R})) = \mathrm{add}(I_{\mathcal{C}ont,1}(\omega^\omega)) = \mathrm{add}(I_{\mathcal{C}ont,1}(2^\omega)) = \aleph_1$
b) $\mathrm{cof}(I_{\mathcal{C}ont,1}(\mathbb{R})) = \mathrm{cof}(I_{\mathcal{C}ont,1}(\omega^\omega)) = \mathrm{cof}(I_{\mathcal{C}ont,1}(2^\omega)) = 2^{\aleph_0}$

Proof. Since the ideals under consideration consist of meager subsets of their underlying spaces, the corollary holds under CH. If $2^{\aleph_0} > \aleph_1$, the corollary follows from Lemma 5.9. □

6. Linear orderings of Turing degrees

In this section we discuss the connection between coverings of $(2^\omega)^2$ by continuous functions and linear orderings of Turing degrees relative to a set of parameters.

Let $\mathcal{F}$ be a set of continuous functions on 2^ω. $\mathcal{F}$ induces a binary relation $\leq_{\mathcal{F}}$ on 2^ω as follows:

$$x \leq_{\mathcal{F}} y :\Leftrightarrow \exists f \in \mathcal{F}(f(y) = x)$$

It is easily checked that $\leq_{\mathcal{F}}$ is transitive if $\mathcal{F}$ is closed under composition of functions. It is obvious that $\leq_{\mathcal{F}}$ is reflexive if $\mathrm{id}_{2^\omega} \in \mathcal{F}$. A transitive and reflexive relation is a *quasi-ordering*. The relation $\leq_{\mathcal{F}}$ is linear, i.e., any two points are comparable, if and only if $\mathcal{F}$ covers $(2^\omega)^2$.

This shows

Lemma 6.1. $\mathrm{cov}(I_{\mathcal{C}ont,1}(2^\omega))$ *is the least size of a family $\mathcal{F}$ of continuous functions on 2^ω such that $\leq_{\mathcal{F}}$ is a linear quasi-ordering.*

It is well known that in the Sacks model (starting from the constructible universe L as the ground model) the constructible degrees of reals are wellordered of order type ω_2 (see [4]). If $\mathcal{F}$ is the set of constructible continuous functions from 2^ω to 2^ω, i.e., the set of continuous functions which have Borel codes in L, then $\leq_{\mathcal{F}}$ refines the quasi-ordering of constructible degrees. Therefore it is not surprising that we have $\mathrm{cov}(I_{\mathcal{C}ont,1}(2^\omega)) = \aleph_1$ in the Sacks model. In fact, $\leq_{\mathcal{F}}$ refines the quasi-ordering of Turing degrees relative to $\mathcal{F}$ as a set of parameters.

Definition 6.2. Let $x, y \in 2^\omega$ and let C be an oracle Turing machine. We say that x is Turing-reducible to y via C ($x \leq_C y$) if C equipped with the oracle y decides x. (Here we identify the elements of 2^ω with subsets of ω.) x is Turing-reducible to y via C relative to the parameter $z \in 2^\omega$ ($x \leq_{C,z} y$), if $x \leq_C y \otimes z$. (Here $\otimes$ should be considered as the Turing join.)

x is Turing-reducible to y relative to a parameter $z \in 2^\omega$ ($x \leq_{T,z} y$) if there is an oracle Turing machine C such that $x \leq_{C,z} y$. For $P \subseteq 2^\omega$ we say that x is Turing-reducible to y relative to P ($x \leq_{T,P} y$) if there is $z \in P$ such that $x \leq_{T,z} y$.

Let C be an oracle Turing machine and $z \in 2^\omega$. Consider the partial function $f_{C,z}$ on 2^ω that maps y to the unique x such that $x \leq_{C,z} y$ (if such an x exists). It may happen that C equipped with the oracle $y \otimes z$ does not halt on every input (the inputs being natural numbers). That is why $f_{C,z}$ can be partial. However, the domain of $f_{C,z}$ is a G_δ-set since for each natural number n the set of oracles on which C halts when it is asked to decide n is open because every computation is finite and thus only uses some finite part of the oracle. For the same reason (finiteness of computations), $f_{C,z}$ is continuous.

It follows that if $P \subseteq 2^\omega$ is such than $\leq_{T,P}$ is a linear quasi-ordering, then there is a family $\mathcal{F}$ of size $|P|$ (note that P must be infinite) of continuous functions defined on G_δ-subsets of 2^ω which covers $(2^\omega)^2$.

On the other hand, if $A \subseteq 2^\omega$ is G_δ and $f : A \to 2^\omega$ is continuous, then f can be coded in a reasonable way as a subset z of ω such that there is an oracle Turing machine C such that for all $x \in A$, $f(x) \leq_{C,z} x$. Thus, if $\mathcal{F}$ is a family of continuous partial functions defined on G_δ-subsets of 2^ω and $\mathcal{F}$ covers $(2^\omega)^2$, then for the set P of codes of the functions in $\mathcal{F}$, $\leq_{T,P}$ is linear. We have thus proved that he least size of a family of partial continuous functions defined on G_δ-subsets of 2^ω covering $(2^\omega)^2$ is equal to the smallest size of a set $P \subseteq 2^\omega$ such that $\leq_{T,P}$ is linear.

This proof easily relativizes to subsets of 2^ω and we obtain

Theorem 6.3. *Let X be an infinite subset of 2^ω and let κ be smallest size of a set $P \subseteq 2^\omega$ such that $\leq_{T,P}\restriction X$ is linear. Then the least size of a family of partial continuous functions defined on G_δ-subsets of 2^ω needed to cover X^2 is equal to $\kappa + \aleph_0$.*

Blass asked whether it is consistent with ZFC that for every set X of reals of size $\aleph_1$ there is a real p such that $\leq_{T,\{p\}}\restriction X$ is linear on X. This was answered positively by Zapletal [8]. His argument essentially showed the consistency of $\mathrm{non}(I_{\mathcal{C}ont,1}(2^\omega)) = \aleph_2$, which also follows from Theorem 4.3. Note that if $P \subseteq 2^\omega$ is countable, then all the parameters in P can be coded into a single parameter $r \in 2^\omega$ such that $\leq_{T,P} \subseteq \leq_{T,r}$. Now it follows from Theorem 6.3 that the least size of a set $X \subseteq 2^\omega$ such that for no $r \in 2^\omega$ the quasi-ordering $\leq_{T,r}\restriction X$ is linear is at least $\mathrm{non}(I_{\mathcal{C}ont,1}(2^\omega))$.

References

[1] K. Ciesielski, J. Pawlikowski, *Covering Property Axiom CPA*, preprint
[2] S. Geschke, M. Goldstern, M. Kojman, *Continuous pair-colorings on 2^ω and covering the square by functions*, submitted
[3] S. Geschke, M. Kojman, W. Kubiś, R. Schipperus, *Convex decompositions in the plane, meagre ideals and continuous pair colorings of the irrationals*, Israel Journal of Mathematics 131, 285–317 (2002)
[4] M. Groszek, *Applications of iterated perfect set forcing*, Ann. Pure Appl. Logic 39, No.1, 19-53 (1988)
[5] K. Kuratowski, *Sur une caractérisation des aleph*, Fundamenta Mathematicae 38, 14-17 (1951)
[6] B.J. van der Steeg, K.P. Hart, *A small transitive family of continuous functions on the Cantor set*, Topology and its Applications 123, 3, 409–420 (2002)
[7] J. Steprans *Decomposing Euclidean space with a small number of smooth sets*, Transactions of the American Mathematical Society 351, No. 4, 1461-1480 (1999)
[8] J. Zapletal, handwritten notes

(Abraham) Department of Mathematics, Ben Gurion University of the Negev, Beer Sheva, Israel

E-mail address: abraham@math.bgu.ac.il

(Geschke) Fachbereich Mathematik und Informatik, Freie Universität Berlin, Arnimallee 3, 14195 Berlin

E-mail address: geschke@math.fu-berlin.de