On σ-Filtered Boolean Algebras

Dissertation

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Chapter 0

Introduction

Freese and Nation ([13]) used a property of partial orders which is now called Freese-Nation property (FN) in order to characterize projective lattices. Projective Boolean algebras have this property. Heindorf ([23]) characterized the Boolean algebras with the FN as the rc-filtered Boolean algebras. These algebras are sometimes called openly generated. In the book by Heindorf and Shapiro ([23]) a generalization of the FN is considered, the weak Freese-Nation property (WFN). Heindorf ([23]) characterized the Boolean algebras with the WFN as being σ-filtered. Fuchino, Koppelberg, and Shelah ([16]) introduced a further generalization of the FN, the κ-Freese-Nation property (κ-FN), for any regular cardinal κ. Their approach is more set-theoretic than Heindorf's, but implicitly they proved that for all partial orders the κ-FN is equivalent to what would be called κ-filteredness. rc-filteredness is $\aleph_0$-filteredness and σ-filteredness is $\aleph_1$-filteredness. Roughly speaking, a partial order is κ-filtered iff it has many nicely embedded suborders. How nice these embeddings are, depends on κ. The smaller κ, the nicer the embeddings. A partial order $(P, \leq)$ has the κ-Freese-Nation property iff there is a function $f : P \to [P]^{<\kappa}$ such that for all $a, b \in P$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ with $a \leq c \leq b$. Every partial order of size $\leq \kappa$ has the κ-FN. FN is $\aleph_0$-FN and WFN is $\aleph_1$-FN. For a partial order P let WFN(P) denote the statement 'P has the WFN'. The study of the κ-FN, especially for $\kappa = \aleph_1$, was continued by Fuchino, Koppelberg, Shelah, and Soukup in [17] and [19].

Koppelberg ([28]) introduced and studied the notion of tight σ-filteredness of a Boolean algebra, which generalizes projectivity. Using this notion, she gave uniform proofs of several mostly known results about the existence of certain homorphisms into countably complete Boolean algebras. Tight σ-filteredness is a strengthening of the WFN, in the same way as projectivity strengthens the FN. Every Boolean algebra of size $\leq \aleph_1$ which has the FN is projective. Similarly, every Boolean algebra of size $\leq \aleph_2$ which has the WFN is tighly σ-filtered.

My research concerning tight σ-filtrations was initiated by a list of questions addressed by Fuchino. The first task was to give a usable characterization of tight σ-filteredness. The relation between tight σ-filteredness and σ-filteredness is very similar to the relation between projectivity and rc-filteredness. However, while projective Boolean algebras are precisely the retracts of free Boolean algebras, a similar characterization of tightly σ-filtered Boolean algebras does not seem to be available. But as it turns out, tightly σ-filtered Boolean algebras can be characterized in a similar way as projective Boolean algebras have been characterized by Ščepin, Haydon, and Koppelberg. (See [23] or [29].) This characterization of tight σ-filteredness can be used to get some results on the Stone spaces of tightly σ-filtered Boolean algebras. The parallel results for projective Boolean algebras were used by Koppelberg ([29]) to show that for every uncountable regular cardinal λ there are only $2^{<\lambda}$ isomorphism types of projective Boolean algebras of size λ. This does not hold for tightly σ-filtered Boolean algebras. For every infinite cardinal λ there are 2^λ pairwise non-isomorphic tightly σ-filtered Boolean algebras of size λ.

One of the main reasons why the WFN and tight σ-filteredness are interesting is that in some models of set theory infinite complete Boolean algebras can have these properties. This is not the case with projectivity or FN. It was shown by Fuchino, Koppelberg, and Shelah ([16]) that adding a small number of Cohen reals to a model of CH results in a model of WFN($\mathfrak{P}(\omega)$). Fuchino and Soukup ([19]) later extended this result showing that adding any number of Cohen reals to a model of CH+$\neg 0^\sharp$ yields a model of WFN($\mathfrak{P}(\omega)$). $\mathfrak{P}(\omega)$ plays an important role considering questions about the WFN of complete Boolean algebras since it is a retract of every infinite complete Boolean algebra and the WFN is hereditary with respect to retracts. In short, if any infinite Boolean algebra has the WFN, then so does $\mathfrak{P}(\omega)$. Using the characterization mentioned above, it turns out that the same is true for tight σ-filteredness. Fuchino, Koppelberg, and Shelah ([16]) observed that $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies that the unboundness number $\mathfrak{b}$ is $\aleph_1$. It follows that the question whether there are any infinite complete Boolean algebras with the WFN cannot be answered in ZFC alone.

One of Fuchino's questions about tight σ-filteredness was whether it is consistent that $\mathfrak{P}(\omega)$ is tightly σ-filtered while the continuum is $\geq \aleph_3$. The only reason for $\mathfrak{P}(\omega)$ being tightly σ-filtered known so far is $\mathrm{WFN}(\mathfrak{P}(\omega))$ together with $2^{\aleph_0} \leq \aleph_2$. Investigating whether $\mathfrak{P}(\omega)$ is tightly σ-filtered in certain models of set theory, I noticed that it is even difficult to get models of $\neg\,\mathrm{CH} + \mathrm{WFN}(\mathfrak{P}(\omega))$, apart from starting with a model of CH and extending the continuum by adding Cohen reals. This led to a systematic study of $\mathrm{WFN}(\mathfrak{P}(\omega))$ in various models of set theory. Together with Fuchino and Soukup, I found that if $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds, then, as far as the reals are concerned, the universe behaves very similar to a model of set theory that was obtained by adding Cohen reals to a model of CH.

While it is quite easy to see that $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies $\mathrm{WFN}(\mathfrak{P}(\omega)/fin)$ and $\mathrm{WFN}(\mathbb{C}(\omega))$, where $\mathbb{C}(\omega)$ is the Cohen algebra, i.e. the completion of the countably generated free Boolean algebra, it is not so clear whether $\mathrm{WFN}(\mathfrak{P}(\omega))$ also implies $\mathrm{WFN}(\mathbb{R}(\omega))$, where $\mathbb{R}(\omega)$ is the measure algebra of the Cantor space. It does, however. If the universe is not too bad, that is, if $0^\sharp$ does not exist, then $\mathrm{WFN}(\mathfrak{P}(\omega))$ even implies that all measure algebras have the WFN and the class of complete Boolean algebras with the WFN has nice closure properties. The argument used here is similar to an argument used by Fuchino and Soukup ([19]) in order to get their result about $\mathrm{WFN}(\mathfrak{P}(\omega))$ in Cohen extensions and to obtain a nice characterization of partial orders with the WFN. It was shown in [16] that all complete Boolean algebras A with $\mathrm{WFN}(A)$ satisfy the c.c.c. In [19] it was proved that if $0^\sharp$ does not exist and CH holds, then $\mathrm{WFN}(A)$ holds for all complete c.c.c. Boolean algebras A. Moreover, under CH, for all complete c.c.c. Boolean algebras A of size $< \aleph_\omega$, $\mathrm{WFN}(A)$ holds. This together with the fact that under $\neg 0^\sharp$ the class of complete Boolean algebras with the WFN has nice closure properties contrasts with some recent results of Soukup. He proved that if the existence of a supercompact cardinal is consistent with ZFC, then it is also consistent that GCH holds, but there is a complete c.c.c. Boolean algebra without the WFN. Using a similar argument, he also proved that it is consistent with ZFC that WFN($\mathfrak{P}(\omega)$) holds, but there is a complete c.c.c. Boolean algebra of size $\aleph_2$ not having the WFN.

0.1 Overview

In the first chapter I introduce the basic notions for this thesis such as tight κ-filteredness and κ-FN and recall the known results. At some places I give straightforward generalizations of known results. Tight κ-filteredness is a generalization of Koppelberg's tight σ-filteredness. Tight σ-filteredness is tight $\aleph_1$-filteredness.

The second chapter deals with tightly κ-filtered Boolean algebras. κ-FN and tight κ-filteredness are equivalent for Boolean algebras of size $\leq \kappa^+$. Any tightly κ-filtered Boolean algebra has the κ-FN.

I give a characterization of tightly κ-filtered Boolean algebras which is similar to the characterization of projective Boolean algebras developed by to Haydon, Koppelberg, and Ščepin. (See [23] or [29].) I show that for every infinite cardinal κ the number of tightly σ-filtered Boolean algebras of size κ is precisely 2^κ, contrasting the result of Koppelberg ([29]) that there are only $2^{<\kappa}$ projective Boolean algebras of size κ for every regular $\kappa > \aleph_0$.

For every infinite regular cardinal κ, I construct (in ZFC) a Boolean algebra which has the FN but is not tightly κ-filtered. This construction is a generalization of one of Ščepin's constructions of a Boolean algebra which is rc-filtered but not projective. (See [23].)

I show that adding ω_3 Cohen reals to a model of CH yields a model of ZFC where $\mathfrak{P}(\omega)$ is not tightly σ-filtered, even though WFN($\mathfrak{P}(\omega)$) holds. A very similar proof shows (in ZFC) that the Cohen algebra over $(2^{\aleph_0})^{++}$ generators, i.e. the completion of the free Boolean algebra over $(2^{\aleph_0})^{++}$ generators, is not tightly σ-filtered. It follows that no complete Boolean algebra of size $\geq (2^{\aleph_0})^{++}$ is tightly σ-filtered.

The third chapter deals with the WFN, mostly for complete Boolean algebras. I characterize those proper notions of forcing P for which $\mathfrak{P}(\omega)$ of the ground model M is σ-embedded in $\mathfrak{P}(\omega)$ in $M[G]$ for every P-generic G. I observe that many forcing notions fail to have this property. (In fact, all forcing notions I have considered that are generated by a name for a real and do not collapse cardinals, except for Cohen forcing.) It follows that in many iterated forcing extensions WFN($\mathfrak{P}(\omega)$) fails. For example, adding ω_2 random reals to a model of CH yields a model of $\neg$WFN($\mathfrak{P}(\omega)$). I show that adding a Hechler real over ω_2 Cohen reals to a model of CH also gives a model of $\neg$WFN($\mathfrak{P}(\omega)$). This shows that even adding one real by some σ-centered forcing can destroy WFN($\mathfrak{P}(\omega)$).

It turns out that WFN($\mathfrak{P}(\omega)$) implies that the covering number of the ideal of meager subsets of ${}^{\omega}2$ is large, by a joint result with Soukup. I prove that the groupwise density number $\mathfrak{g}$ is $\aleph_1$ under WFN($\mathfrak{P}(\omega)$). I show that under the assumption $\neg 0^\sharp$, WFN($\mathfrak{P}(\omega)$) implies the WFN of many complete c.c.c. Boolean algebras, among them all measure algebras. Without $\neg 0^\sharp$, my argument only works for algebras which are completely generated by less than $\aleph_\omega$ elements.

0.2 Sources

The first chapter mainly surveys the known results about κ-embeddings, κ-FN, and tight σ-filteredness from [23], [28], [29], [19], [16] and [17]. The second chapter is quite algebraic, although set-theoretic methods are used in several places. The methods and notions used in this chapter are mainly taken from the books by Heindorf and Shapiro ([23]) and Eklof and Mekler ([11]) and from Koppelberg's articles ([28], [29]). The set theory that is used here can be found in the books by Kunen ([32]) and Jech ([24]) and the reference for Boolean algebras is the first volume of the Handbook of Boolean Algebras ([30]). Everything that is needed about general topology is contained in Engelking's book ([12]). The third chapter heavily uses forcing. I basically rely on the books by Kunen ([32]) and Jech ([24]), but I also use several facts from more modern texts ([1], [21]). For cardinal invariants of the continuum, everything necessary is provided by Blass' article ([4]).

0.3 Acknowledgements

First I would like to thank Sabine Koppelberg for supervising this thesis and for teaching me the basics of modern set theory and the theory of Boolean algebras.

It was Sakaé Fuchino who brought my attention to tightly σ-filtered Boolean algebras. I want to thank him for making it possible for me to stay in Japan for one year and to work with him and Lajos Soukup, which resulted in two papers ([14], [15]), so far. Thanks are due to all the aforementioned persons for many fruitful discussions on the topics of this thesis. Especially Lajos Soukup pushed me to prove certain theorems.

Lutz Heindorf gave me a copy of [23] and Sabine Koppelberg gave me a copy of [30]. Both books have been very useful. Sakaé Fuchino gave me a copy of his japanese translation ([27]) of Kanamori's book ([26]), which has not been extremely useful yet, for the obvious reason. But I hope sometime I will be able to read it. The LaTeX-document class used for typesetting this thesis is due to Carsten Schultz.

Chapter 1

Preparation

Throughout this chapter let κ be an infinite regular cardinal.

1.1 κ-embeddings

1.1.1. Definition. A partially ordered structure $(P, \leq)$ is an algebraic structure P together with a partial order $\leq$. Of course, P itself may have no functions or relations. In this case $(P, \leq)$ is just a partial order. Typically, I will omit $\leq$ and write P instead of $(P, \leq)$. Likewise, I will rarely distinguish between an algebraic structure and its underlying set. As a subset X of a partially ordered structure P is cofinal in P iff it contains an upper bound of every element of P, X is coinitial in P iff it contains a lower bound of every element of P. The cofinality of P is the minimal cardinality of a cofinal subset of P and is denoted by $\mathrm{cf}(P)$. Similarly, the coinitiality of P is the minimal size of a coinitial subset of P and is denoted by $\mathrm{ci}(P)$.

Let P and Q be partially ordered structures such that $P \leq Q$, i.e. such that P is a substructure of Q. Then for $x \in Q$ the initial segment $\{a \in P : a \leq x\}$ is denoted by $P \downarrow x$ and the final segment $\{a \in P : a \geq x\}$ by $P \uparrow x$. P is called a *κ-substructure* of Q iff for each $x \in Q$ the initial segment $P \downarrow x$ and the final segment $P \uparrow x$ have cofinality respectively coinitiality $< \kappa$. In this case I write $P \leq_\kappa Q$. $P \leq_\sigma Q$ means $P \leq_{\aleph_1} Q$. The word 'substructure' can be replaced by 'suborder' or 'subalgebra', depending on the type of objects I am dealing with. An isomorphism between a partially ordered structure P and a κ-substructure P' of a partially ordered structure Q is called a κ-embedding.

Now let A and B be Boolean algebras such that A is a subalgebra of B and let $x \in B$. I write $A \upharpoonright x$ instead of $A \downarrow x$. $A \upharpoonright x$ can be regarded as an ideal of A or, if x is an element of A, as a Boolean algebra, namely the relative algebra of A with respect to x. The intended meaning will always be clear from the context. A is a *relatively complete* subalgebra of B iff $A \leq_{\aleph_0} B$. In this case I write $A \leq_{\mathrm{rc}} B$. Note that $A \leq_{\mathrm{rc}} B$ iff for every $x \in B$ the ideal $A \upharpoonright x$ is generated by a single element. The mapping lpr_A^B assigning to $x \in B$ the generator of $A \upharpoonright x$ is called the *lower projection* from B to A. □

In the following the letters A, B, and C will refer to Boolean algebras unless stated differently. Thus $A \leq B$ means that A is a subalgebra of B. Note that $A \leq_\kappa B$ iff $A \leq B$ and for every $x \in B$ the ideal $A \upharpoonright x$ has cofinality $< \kappa$. Also note that $A \leq_\kappa B$ iff $A \leq B$ and for every ideal I of B which has cofinality $< \kappa$ the ideal $I \cap A$ also has cofinality $< \kappa$.

The following two lemmas collect some frequently used facts on $\leq_\kappa$.

1.1.2. Lemma. *Let A and B be Boolean algebras such that $A \leq B$ and $x \in B$. Then $A \leq_\kappa A(x)$ iff $A \upharpoonright x$ and $A \upharpoonright -x$ both have cofinality $< \kappa$.*

Proof. The direction from the left to the right is trivial. For the other direction let $E \subseteq A$ and $F \subseteq A$ be sets of size $< \kappa$ which are cofinal in $A \upharpoonright x$ and $A \upharpoonright -x$ respectively. Suppose $y \in A(x)$. Then there are $v, w \in A$ such that $y = (v + x) \cdot (w + (-x))$. Let $z \in A$ be such that $z \leq y$. Then $z - v \leq x$ and $z - w \leq -x$. Hence $z - v \leq a$ and $z - w \leq b$ for some $a \in E$ and some $b \in F$. It follows that $z \leq (v + a) \cdot (w + b)$. Clearly, $(v + a) \cdot (w + b) \leq y$ for every $a \in E$ and every $b \in F$. Hence $\{(v + a) \cdot (w + b) : a \in E \wedge b \in F\}$ is cofinal in $A \upharpoonright y$. □

1.1.3. Lemma. *Let P, Q, and R be partial orders.*

a) $P \leq_\kappa Q \leq_\kappa R \Rightarrow P \leq_\kappa R$.

b) If Q is the union of a family $\mathcal{Q}$ of suborders of Q and $P \leq_\kappa Q'$ for every $Q' \in \mathcal{Q}$, then $P \leq_\kappa Q$.

c) If $(P_\alpha)_{\alpha<\lambda}$ is an ascending chain of κ-suborders of Q and $\mathrm{cf}(\lambda) < \kappa$, then $\bigcup_{\alpha\leq\lambda} P_\alpha \leq_\kappa Q$.

Now let A, B, and C be Boolean algebras.

d) $A \leq_\kappa B$, $X \in [B]^{<\kappa} \Rightarrow A(X) \leq_\kappa B$.

e) $A \leq_{\mathrm{rc}} B$, $C \leq B$, *and* $\mathrm{lpr}^B_A[C] \subseteq C \Rightarrow A \cap C \leq_{\mathrm{rc}} C$.

Proof. a), b), and e) are easy. For c) let $R := \bigcup_{\alpha<\lambda} P_\alpha$. Fix a cofinal set $X \subseteq \lambda$ of size $< \kappa$. For $q \in Q$ and $\alpha \in X$ let Y^q_α be a cofinal subset of $P_\alpha \downarrow q$ of size $< \kappa$. Then $\bigcup_{\alpha\in X} Y^q_\alpha$ is cofinal in $R \downarrow q$ and has size $< \kappa$ by regularity of κ. By the same argument, $R \uparrow q$ has coinitiality $< \kappa$.

d) was shown by Koppelberg for $\kappa \leq \aleph_1$ ([29]). The proof for the general case is the same. Let C be the subalgebra of B generated by X. Suppose $b \in B$. For each $c \in C$ fix a set $Y_c \in [A]^{<\kappa}$ which is cofinal in $A \restriction -c + b$. I *claim* that the algebra $D \leq A(X)$ generated by $C \cup \bigcup_{c\in C} Y_c$ contains a cofinal subset of $A(X) \restriction b$.

Let $a \in A(X) \restriction b$. There are $n \in \omega$, $a_0, \dots, a_{n+1} \in A$, and $c_0, \dots, c_{n-1} \in C$ such that $a = \sum_{i<n} a_i c_i$. Since $a \leq b$, $a_i \leq -c_i + b$ for each $i < n$. Hence, for each $i < n$ there is $a'_i \in Y_{c_i}$ such that $a_i \leq a'_i \leq -c_i + b$. Now $a \leq \sum_{i<n} a'_i c_i \leq b$ and $\sum_{i<n} a'_i c_i \in D$. This proves the claim. By regularity of κ, $|D| < \kappa$. □

1.2 κ-filtrations

A partial order is κ-filtered iff it has many κ-suborders. In order to give a precise formulation of 'many', I introduce various notions of skeletons.

1.2.1. Definition. Let $\mathcal{S}$ be a family of suborders of a partial order P. $\mathcal{S}$ is called a $<\kappa$-*skeleton* of P iff the following conditions hold:

(i) $\mathcal{S}$ is closed under unions of subchains.

(ii) For every suborder Q of P there are $\mu < \kappa$ and $R \in \mathcal{S}$ such that $Q \subseteq R$ and $|R| \leq |Q| + \mu$.

$\mathcal{S}$ is called a κ-*skeleton* of P iff $\mathcal{S}$ satisfies (i) as above and instead of (ii) the following holds:

(ii)′ Every suborder Q of P is included in a member R of $\mathcal{S}$ such that $|R|{=}|Q|+\kappa$.

$\mathcal{S}$ is called a *skeleton* iff it is an $\aleph_0$-skeleton. □

The exact definition of κ-filteredness is the following:

1.2.2. Definition. A partial order P is κ-filtered iff it has a κ-skeleton $\mathcal{S}$ consisting of κ-suborders. P is *σ-filtered* iff it is $\aleph_1$-filtered. A Boolean algebra A is *rc-filtered* iff it is $\aleph_0$-filtered. □

Note that if $\mathcal{S}$ is a κ-skeleton of a Boolean algebra A, then it includes a κ-skeleton $\mathcal{S}'$ of A consisting of subalgebras of A. Thus a Boolean algebra A is κ-filtered iff it has a κ-skeleton consisting of κ-subalgebras. If κ is uncountable, then every $<\kappa$-skeleton of a Boolean algebra A contains a $<\kappa$-skeleton consisting of subalgebras of A. However, the latter is not true for $\kappa = \aleph_0$ since every infinite Boolean algebra A has a $<\aleph_0$-skeleton which contains no finite subalgebra of A.

The other notion, apart from κ-filteredness, that will be investigated in this thesis is tight κ-filteredness. At least using the definition given below, this notion only makes sense for Boolean algebras. While κ-filteredness and tight κ-filteredness seem to be unrelated at first sight, it will turn out later that tight κ-filteredness is stronger than κ-filteredness.

1.2.3. Definition. Let A be a Boolean algebra and δ an ordinal. A continuous ascending chain $(A_\alpha)_{\alpha<\delta}$ of subalgebras of A is called a (wellordered) *filtration* of A.

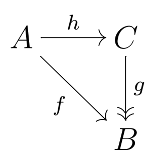
A filtration $(A_\alpha)_{\alpha<\delta}$ is called *tight* iff $A_0 = 2$ and there is a sequence $(x_\alpha)_{\alpha<\delta}$ in A such that $A_{\alpha+1} = A_\alpha(x_\alpha)$ holds for all $\alpha < \delta$.

A filtration $(A_\alpha)_{\alpha<\delta}$ is called a *κ-filtration* (*rc-filtration*, *σ-filtration*) iff $A_\alpha \leq_\kappa A_{\alpha+1}$ ($A_\alpha \leq_{\mathrm{rc}} A_{\alpha+1}$, $A_\alpha \leq_\sigma A_{\alpha+1}$) holds for all $\alpha < \delta$. A is *tightly κ-filtered* iff it has a tight κ-filtration. □

1.3 Universal properties

This section will not really be needed for the rest of this thesis, but it provides some motivation for studying tight κ-filteredness. Tightly κ-filtered Boolean algebras have properties similar to projectivity. While no infinite complete Boolean algebra is projective, in some models of set theory interesting complete Boolean algebras are for example tightly σ-filtered. This has nice applications concerning the existence of certain homomorphisms.

1.3.1. Definition. A Boolean algebra A is *projective* iff for any two Boolean algebras B and C, every epimorphism $g : C \to B$, and every homomorphism $f : A \to B$ there is a homomorphism $h : A \to C$ such that



commutes. □

While this definition works in every category, the following characterization provides more insight into the structure of projective Boolean algebras.

1.3.2. Definition and Lemma. *A is a* retract *of B iff there are homomorphisms $e : A \to B$ and $p : B \to A$ such that $p \circ e = \mathrm{id}_A$. A Boolean algebra A is projective iff it is a retract of a free Boolean algebra.*

Proof. Abstract nonsense. □

This lemma is true in every category with sufficiently many free objects. However, there are categories in which this lemma does not hold since there are non-trivial projective objects, but no non-trivial free objects. (See [20] for an example.)

By theorems by Haydon, Koppelberg, and Ščepin, the tightly rc-filtered Boolean algebras are exactly the projective Boolean algebras. (See [29] or [23].) The following theorem generalizes one direction of this to tightly κ-filtered Boolean algebras and was proved by Koppelberg ([28]) for $\kappa = \aleph_1$.

Her proof works for uncountable κ as well. Let me introduce some additional notions first.

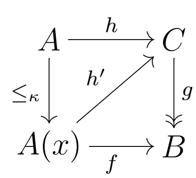
1.3.3. Definition. A Boolean algebra A has the *κ-separation property* (κ-s.p. for short) iff for any two subsets S and T of A of size $< \kappa$ with $S \cdot T := \{s \cdot t : s \in S \wedge t \in T\} = \{0\}$ there is $a \in A$ such that $s \leq a$ for all $s \in S$ and $t \leq -a$ for all $t \in T$. An ideal I of a Boolean algebra A is *κ-directed* iff every subset of I of size $< \kappa$ has an upper bound in I. □

In particular, every κ-complete Boolean algebra has the κ-s.p. Similarly, every κ-ideal, i.e. every ideal which is closed under sums of less than κ elements, is κ-directed.

1.3.4. Theorem. *Let A be a tightly κ-filtered Boolean algebra. If B and C are Boolean algebras, C has the κ-s.p., $g : C \to B$ is an epimorphism such that the kernel of g is κ-directed, and $f : A \to B$ is a homomorphism, then there is a homomorphism $h : A \to C$ such that $g \circ h = f$.* □

The proof needs

1.3.5. Lemma. *Let A and A' be Boolean algebras such that A' is a simple extension of A, i.e. $A' = A(x)$ for some $x \in A'$. Assume that $A \leq_\kappa A(x)$, B and C are Boolean algebras, C has the κ-s.p., $g : C \to B$ is an epimorphism with κ-directed kernel, $f : A' \to B$ is a homomorphism, and $h : A \to C$ is a homomorphism such that $g \circ h = f \restriction A$. Then there is an extension $h' : A' \to C$ of h such that $g \circ h' = f$, i.e.*



commutes.

Proof. Let $S, T \in [A]^{<\kappa}$ be such that S is cofinal in $A \restriction x$ and T is cofinal in $A \restriction -x$. Fix some $z \in C$ such that $g(z) = f(x)$. Since the kernel of g is κ-directed, there is $i \in g^{-1}(0)$ such that for all $s \in S$, $h(s) \leq z + i$ and for all $t \in T$, $h(t) \leq -z + i$. Note that $\{i, z - i, -z - i\}$ is a partition of unity in C. By the κ-s.p. of C, there is $j \in C$ such that $j \leq i$, $h(s) \cdot i \leq j$ for all $s \in S$, and $h(t) \cdot i \leq -j$ for all $t \in T$. Let $z' := (z - i) + j$. Now it is a straightforward consequence of Sikorski's extension theorem that there is an extension $h' : A' \to C$ of h such that $h'(x) = z'$. Since $A' = A(x)$, this extension is unique. It is easy to see that h' works for the lemma. □

Proof of the theorem. Fix a tight κ-filtration of A and construct h by transfinite induction along this filtration, using Lemma 1.3.5 at the successor stages. □

In particular, this theorem gives that if A has the κ-s.p., $f : A \to B$ is an epimorphism with κ-directed kernel, and B is tightly κ-filtered, then there is an homomorphism $h : B \to A$ such that $f \circ h = \mathrm{id}_B$. h is called a *lifting* for f. Note that h is injective.

1.3.6. Definition. Let $\mathcal{M}$ be the ideal of meager subsets of the Cantor space ${}^{\omega}2$ and let $\mathcal{N}$ be the ideal of subsets of ${}^{\omega}2$ of measure zero. Here the measure on ${}^{\omega}2$ is just the product measure induced by the measure on 2 mapping the singletons to $\frac{1}{2}$. Let $\mathrm{Bor}({}^{\omega}2)$ be the σ-algebra of Borel subsets of ${}^{\omega}2$ and let $\mathbb{C}(\omega) := \mathrm{Bor}({}^{\omega}2)/\mathcal{M}$ and $\mathbb{R}(\omega) := \mathrm{Bor}({}^{\omega}2)/\mathcal{N}$. $\mathbb{C}(\omega)$ is the *Cohen algebra* or *category algebra* and $\mathbb{R}(\omega)$ is the *measure algebra* or *random algebra*. Let $p : \mathrm{Bor}({}^{\omega}2) \to \mathbb{R}(\omega)$ and $q : \mathrm{Bor}({}^{\omega}2) \to \mathbb{C}(\omega)$ be the quotient mappings. A lifting for p is a *Borel lifting for measure* and a lifting for q is a *Borel lifting for category*. □

Using her version of Theorem 1.3.4, Koppelberg gave uniform proofs of several mostly known results about the existence of certain homomorphism into Boolean algebras with the countable separation property. Among other things, she observed that under CH and after adding $\aleph_2$ Cohen reals to a model of CH, $\mathbb{C}(\omega)$ and $\mathbb{R}(\omega)$ are tightly σ-filtered. This implies the existence of Borel liftings for measure and category in the respective models. (See [28].) Originally, the results on Borel liftings in these models were obtained by von Neumann, Stone, Carlson, Frankiewicz, and Zbierski.

One may ask whether the existence of a Borel lifting implies the existence of a tight σ-filtration of the respective algebra. At least for measure, this it not the case. According to Burke ([9]), Veličkovič has shown that after adding $\aleph_2$ random reals to a model of CH, there is a Borel lifting for measure. It will turn out later that in that model $\mathbb{R}$ is not tightly σ-filtered.

I do not know whether tight κ-filteredness can be characterized by some property like the one in Theorem 1.3.4. However, there will be several internal characterizations of tight κ-filteredness in the second chapter.

1.4 The κ-Freese-Nation property

1.4.1. Definition. A partial order $(P, \leq)$ has the *κ-Freese-Nation property* (κ-FN for short) iff there is a function $f : P \to [P]^{<\kappa}$ such that for all $a, b \in P$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ such that $a \leq c \leq b$. f is called a *κ-FN-function* for P. The $\aleph_0$-FN is the original Freese-Nation property (FN), which was used by Freese and Nation to characterize projective lattices ([13]). The $\aleph_1$-FN is called *weak* Freese-Nation property (WFN for short) and was introduced by Heindorf and Shapiro ([23]). WFN(P) denotes the statement 'P has the WFN'. □

It is easily seen that small partial orders have the κ-FN.

1.4.2. Lemma. *([16]) Every partial order P of size $\leq \kappa$ has the κ-FN.* □

By a result by Heindorf ([23]), a Boolean algebra is rc-filtered iff it has the FN. Similarly, in [23] it is proved that for Boolean algebras the WFN is the same as σ-filteredness. Fuchino, Koppelberg, and Shelah ([16]) have shown that for all regular infinite κ a partial order P has the κ-FN iff it is κ-filtered. However, they formulated κ-filteredness in terms of elementary submodels of some H_χ rather than in terms of skeletons. But these two formulation are easily seen to be equivalent.

When dealing with elementary submodels of some H_χ, I will usually assume that χ is 'large enough' or 'sufficiently large'. This simply means that χ is chosen so large that all the objects I am going to consider are contained in H_χ and all the properties of these objects I am going to use are absolute over H_χ. In the context of forcing sometimes a class M is considered which is a model of a 'sufficiently large fragment of ZFC'. This means that M satisfies enough of ZFC to carry out the arguments I am going to use. The problem is that typically, one cannot get suitable set models for all of ZFC. See [32] for these questions. I use ZFC* to abbreviate 'sufficiently large fragment of ZFC'.

The basic observations in order to get the desired characterization of partial orders with the κ-FN are the following:

1.4.3. Lemma. *a)([16]) If f is a κ-FN-function for a partial order P and $Q \subseteq P$ is closed under f, then $Q \leq_\kappa P$.*

b) If Q is a κ-suborder of a partial order P and P has the κ-FN, then Q has the κ-FN, too.

c)([16]) Let δ be a limit ordinal and let $(P_\alpha)_{\alpha\leq\delta}$ be an increasing continuous chain of partial orders such that $P_\alpha \leq_\kappa P_\delta$ for every $\alpha < \delta$. If P_α has the κ-FN for every $\alpha < \delta$, then P_δ has the κ-FN as well.

Proof. Only b) has not been proved in [16]. Let f be a κ-FN-function for P. For each $p \in P$ fix $X_p \in [Q]^{<\kappa}$ such that X_p is cofinal in $Q \downarrow p$. For each $q \in Q$ let $g(q) := \bigcup_{p\in f(q)} X_p$. g is a κ-FN-function for Q: By regularity of κ, $|g(q)| < \kappa$ for every $q \in Q$. Let $q, r \in Q$ be such that $q \leq r$. Now there is $p \in f(q) \cap f(r)$ such that $q \leq p \leq r$. Let $p' \in X_p$ be such that $q \leq p' \leq p$. Now $q \leq p' \leq r$ and $p' \in g(q) \cap g(r)$. □

From Lemma 1.4.3 one can obtain the following charaterization of partial orders with the κ-FN:

1.4.4. Theorem. *(Implicitly in [16]) Let $(P, \leq)$ be a partial order and χ large enough. The following are equivalent:*

(i) P has the κ-FN.

(ii) For every elementary submodel M of H_χ such that $(P, \leq), \kappa \in M$ and $\kappa \subseteq M$, $P \cap M \leq_\kappa P$ holds.

(iii) P is κ-filtered.

Proof. (i)⇒(ii) is proved in [16] for elementary submodels of size κ, but the same argument works here as well. Since M knows that P has the κ-FN, there is $f \in M$ such that $f : P \to [P]^{<\kappa}$ is a κ-FN-function for P. For each $p \in P \cap M$, $f(p) \in M$. Since $|f(p)| < \kappa$ and $\kappa \subseteq M$, $f(p) \subseteq M$. It follows that $P \cap M$ is closed under f. By Lemma 1.4.3, $P \cap M \leq_\kappa P$.

Now assume (ii). Fix a wellorder $\trianglelefteq$ of H_χ. (iii) is then witnessed by

$$\mathcal{S} := \{P \cap M : M \precsim (H_\chi, \trianglelefteq) \wedge (P, \leq), \kappa \in M \wedge \kappa \subseteq M\} :$$

Clearly, every subset X of P is included in some $Q \in \mathcal{S}$ such that $|Q| \leq |X| + \kappa$. By (ii), every $Q \in \mathcal{S}$ is a κ-suborder of P. Let $\mathcal{T} \subseteq \mathcal{S}$ be a chain. Since $\trianglelefteq$ is a wellordering of H_χ, $(H_\chi, \trianglelefteq)$ has definable Skolem functions. For each $Q \in \mathcal{T}$ let M_Q be the Skolem hull of Q in H_χ. By definition, every $Q \in \mathcal{T}$ has the form $P \cap M$ for some elementary submodel M of $(H_\chi, \trianglelefteq)$. Therefore $M_Q \cap P = Q$. It follows that $\{M_Q : Q \in \mathcal{T}\}$ is a chain of elementary submodels of H_χ. Thus $N := \bigcup_{Q \in \mathcal{T}} M_Q \precsim H_\chi$. Therefore $\bigcup \mathcal{T} = P \cap N \in \mathcal{S}$.

For (iii)⇒(i) let $\mathcal{S}$ be a κ-skeleton of P consisting of κ-suborders. Clearly $P \in \mathcal{S}$. Assume that P does not have the κ-FN. Let $Q \in \mathcal{S}$ be of minimal size such that Q does not have the κ-FN. By Lemma 1.4.2, $|Q| > \kappa$. By the properties of $\mathcal{S}$, there is a strictly increasing continuous chain $(Q_\alpha)_{\alpha<|Q|}$ in $\mathcal{S} \cap [P]^{<|Q|}$ such that $Q \subseteq \bigcup_{\alpha<|Q|} Q_\alpha$. By the choice of Q, every Q_α has the κ-FN. By part c) of Lemma 1.4.3, $\bigcup_{\alpha<\lambda} Q_\alpha$ has the κ-FN. This contradicts part b) of Lemma 1.4.3. □

A more advanced version of this theorem has been found by Fuchino and Soukup. In this theorem only very nice submodels of H_χ have to be considered.

1.4.5. Definition. Let χ be a cardinal such that $\kappa < \chi$. $M \precsim H_\chi$ is *V_κ-like* iff $M = \bigcup_{\alpha<\kappa} M_\alpha$ for a continuously increasing chain $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of M of size $< \kappa$ such that for each $\alpha < \kappa$, $(M_\beta)_{\beta\leq\alpha} \in M_{\alpha+1}$. □

It is easy to see that every subset of H_χ of size κ is a subset of some V_κ-like elementary submodel of H_χ. Fuchino and Soukup proved the following:

1.4.6. Theorem. *([19]) Let P be a partial order and let χ be large enough.*

a) If $0^\sharp$ does not exist, then P has the κ-FN iff for every V_κ-like elementary submodel M of H_χ such that $P \in M$, $P \cap M \leq_\kappa P$.

b) If $|P| < \aleph_\omega$, then P has the κ-FN iff for every V_κ-like elementary submodel M of H_χ such that $P \in M$, $P \cap M \leq_\kappa P$. □

Unfortunately, part a) of this theorem really needs some assumptions on the non-existence of certain large cardinals, as was also shown by Fuchino and Soukup ([19]). In the proof of this theorem, as well as in the proofs of of similar theorems that will be stated later, $\neg 0^\sharp$ is used in the following way:

The proof uses some transfinite induction on cardinals. There occurs a problem at uncountable cardinals of countable cofinality. In order to proceed with the induction at a stage λ with $\mathrm{cf}(\lambda) = \aleph_0$, some weak form of the $\Box$-principle as well as some assumption like $\mathrm{cf}([\lambda]^{\aleph_0}) = \lambda^+$ is needed.

The following lemma comes in handy when one wants to find out whether or not certain complete Boolean algebras have the WFN. The κ-FN does not reflect to suborders in general, but to suborders which are retracts.

1.4.7. Definition. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partial orders. A mapping $e : P \to Q$ is an *order embedding* iff for all $a, b \in P$, $a \leq_P b$ iff $e(a) \leq_Q e(b)$. P is an *order retract* of Q iff there are monotone mappings $e : P \to Q$ and $p : Q \to P$ such that $p \circ e = \mathrm{id}_P$. □

1.4.8. Lemma. *([16]) Let P and Q be partial orders. If P is an order retract of Q and Q has the κ-FN, then P has the κ-FN.* □

If P is order embeddable into Q and sufficiently complete, then P is an order retract of Q.

1.4.9. Corollary. *Let P and Q be partial orders and let $e : P \to Q$ be an order embedding. If Q has the κ-FN and in P every subset has a least upper bound, then P has the κ-FN.*

Proof. For each $q \in Q$ let $p(q) := \sup\{p \in P : e(p) \leq q\}$. $p : Q \to P$ is monotone and $p \circ e = \mathrm{id}_P$. Thus P is an order retract of Q and Lemma 1.4.8 applies. □

Since $\mathfrak{P}(\omega)$ embeds into every infinite complete Boolean algebra, $\mathfrak{P}(\omega)$ has the κ-FN iff any infinite complete Boolean algebra does. The most interesting case seems to be $\kappa = \aleph_1$. Fuchino, Koppelberg, and Shelah ([16]) noticed that $\mathfrak{P}(\aleph_1)$ does not have the WFN, i.e. $\neg\,\mathrm{WFN}(\mathfrak{P}(\aleph_1))$ is provable from ZFC. Therefore, again by the corollary above, no complete Boolean algebra without c.c.c. has the WFN. As mentioned earlier, for every partial order P of size $\aleph_1$, WFN(P) holds. Thus CH implies WFN($\mathfrak{P}(\omega)$). It is possible to enlarge the continuum by adding Cohen reals without destroying WFN($\mathfrak{P}(\omega)$). Here adding κ Cohen reals means forcing with $\mathrm{Fn}(\kappa, 2)$. In [16] and [19] the following facts about WFN($\mathfrak{P}(\omega)$) were established:

1.4.10. Theorem. *a)([16]) Adding less than $\aleph_\omega$ Cohen reals to a model of CH gives a model of* WFN($\mathfrak{P}(\omega)$).

b)([19]) Adding any number of Cohen reals to a model of CH+$\neg 0^\sharp$ gives a model of WFN($\mathfrak{P}(\omega)$).

c)([16]) WFN($\mathfrak{P}(\omega)$) *implies that the unboundedness number $\mathfrak{b}$ is $\aleph_1$.* □

It follows that the question whether there are any infinite complete Boolean algebras having the WFN cannot be settled in ZFC. It will turn out that the universe must be quite similar to a model obtained by adding Cohen reals to a model of CH if WFN($\mathfrak{P}(\omega)$) holds, at least as far as the reals are concerned. Note that the Cohen algebra $\mathbb{C}$ and $\mathfrak{P}(\omega)$ both are retracts of each other. Therefore one of them has WFN iff the other one does. This was noticed by Koppelberg ([28]).

The usual ways of refuting the κ-FN of some partial order P are either showing that P has an order retract which does not have the κ-FN or giving a counter-example to part a) of Lemma 1.4.3. Concerning WFN($\mathfrak{P}(\omega)$), I will only use the second method. The following lemma has probably never been stated explicitly, but it should be well-known.

1.4.11. Lemma. *Suppose either that M and N are transitive models of* ZFC* *and $M \subseteq N$ such that M is a definable class in N, or that $N = V$ and M is an elementary submodel of some H_χ, where χ is a sufficiently large* *cardinal. Then for all* $P, Q \in \{({}^\omega\omega, \leq), ({}^\omega\omega, \leq^*), \mathfrak{P}(\omega), \mathfrak{P}(\omega)/fin\}$,

$$N \models (P \cap M \leq_\sigma P \Leftrightarrow Q \cap M \leq_\sigma Q).$$

Proof. I argue in N. The equivalence for $\mathfrak{P}(\omega)$ and $\mathfrak{P}(\omega)/fin$ follows easily from the fact that fin is a countable subset of M. Similarly, the equivalence holds for $({}^\omega\omega, \leq)$ and $({}^\omega\omega, \leq^*)$ since for each $f : \omega \to \omega$ the set $\{g \in {}^\omega\omega : g =^* f\}$ is a countable subset of M if $f \in M$.

Mapping each $x \subseteq \omega$ to its characteristic function gives an order embedding from $\mathfrak{P}(\omega)$ into $({}^\omega\omega, \leq)$. Since $\mathfrak{P}(\omega)$ is complete, it is an order retract of $({}^\omega\omega, \leq)$. The mappings proving this are elements of M if M is an elementary submodel of H_χ for some large χ. If M is a definable class, then the restrictions of these mappings to M are in M. It is easy to see that this implies

$$({}^\omega\omega \cap M, \leq) \leq_\sigma ({}^\omega\omega, \leq) \Rightarrow \mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega).$$

Now suppose $\mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$ and let $f \in {}^\omega\omega$. Let $x := \{(n, m) \in \omega \times \omega : m \leq f(n)\}$ and let C be an at most countable cofinal subset of $(\mathfrak{P}(\omega \times \omega) \restriction x) \cap M$. For each $c \in C$ and each $n \in \omega$ let $f_c(n) := \max\{m \in \omega : (n, m) \in c\}$. Now for each $c \in C$, $f_c \in M$. $\{f_c : c \in C\}$ is cofinal in $({}^\omega\omega \cap M, \leq) \downarrow f$.

Assume $({}^\omega\omega \cap M, \leq) \uparrow f$ is non-empty. Let D be a countable cofinal subset of $(\mathfrak{P}(\omega \times \omega) \restriction \omega \times \omega \setminus x) \cap M$. I may assume that for all $d \in D$ and all $n \in \omega$ there is some $m \in \omega$ such that $(n, m) \in d$ since there is $g \in {}^\omega\omega \cap M$ such that $f \leq g$ by assumption. For each $d \in D$ and each $n \in \omega$ let $g_d(n) := \min\{m \in \omega : (n, m) \in d\}$. Now for each $d \in D$, $g_d \in M$. $\{g_d : d \in D\}$ is coinitial in $({}^\omega\omega \cap M, \leq) \uparrow f$. □

From this lemma together with Lemma 1.4.4 it follows that WFN($\mathfrak{P}(\omega)$), WFN($\mathfrak{P}(\omega)/fin$), WFN($({}^\omega\omega, \leq)$), and WFN($({}^\omega\omega, \leq^*)$) are equivalent. This was partially observed by Koppelberg in [28].

Chapter 2

On Tightly κ-Filtered Boolean Algebras

Again, in this chapter I assume that κ is regular and infinite.

2.1 The number of tightly σ-filtered Boolean algebras

By a result by Koppelberg ([29]), there are only $2^{<\lambda}$ pairwise non-isomorphic projective Boolean algebras of size λ for every regular uncountable cardinal λ and there are 2^λ pairwise non-isomorphic projective Boolean algebras of size λ for every singular infinite cardinal λ. However, a similar statement does not hold for tightly σ-filtered Boolean algebras.

2.1.1. Theorem. *For every infinite cardinal λ there are 2^λ pairwise non-isomorphic tightly σ-filtered Boolean algebras of size λ satisfying the c.c.c.* $\square$

The proof of the theorem uses the following lemma, which says that stationary sets consisting of ordinals of countable cofinality can be coded by tightly σ-filtered Boolean algebras.

2.1.2. Lemma. *Let λ be an uncountable regular cardinal and let S be a subset of λ consisting of ordinals of cofinality $\aleph_0$. Then there are a Boolean algebra A of size λ and a tight σ-filtration $(A_\alpha)_{\alpha<\lambda}$ of A such that the following hold:*

a) $A_\alpha \not\leq_{\mathrm{rc}} A$ *for all* $\alpha \in S$

b) $A_\alpha \leq_{\mathrm{rc}} A$ *for all* $\alpha \in \lambda \setminus S$.

Proof. For every $\alpha \in S$ let $(\delta^\alpha_n)_{n\in\omega}$ be a strictly increasing sequence of ordinals with least upper bound α and $S \cap \{\delta^\alpha_n : n \in \omega\} = \emptyset$. I will construct $(A_\alpha)_{\alpha<\lambda}$ together with a sequence $(x_\alpha)_{\alpha<\lambda}$ such that

(i) $A_0 = 2$,

(ii) $A_{\alpha+1} = A_\alpha(x_\alpha)$ for all $\alpha < \lambda$,

(iii) x_α is independent over A_α whenever $\alpha \notin S$,

(iv) $A_\alpha \upharpoonright x_\alpha$ is generated by $\{x_{\delta^\alpha_n} : n \in \omega\}$ and $A_\alpha \upharpoonright -x_\alpha = \{0\}$ whenever $\alpha \in S$,

(v) $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ holds for all limit ordinals $\beta < \lambda$.

Clearly, the construction can be done and is uniquely determined. I have to show that a) and b) of the lemma hold for $(A_\alpha)_{\alpha<\lambda}$.

For a) let $\alpha \in S$. Then $A_\alpha \upharpoonright x_\alpha$ is non-principal. For suppose $a \in A_\alpha$ is such that $a \leq x_\alpha$. Since $(\delta^\alpha_n)_{n\in\omega}$ is cofinal in α, there is $n \in \omega$ such that $a \in A_{\delta^\alpha_n}$. Since $\delta^\alpha_n \notin S$, $x_{\delta^\alpha_n}$ is independent over $A_{\delta^\alpha_n}$ by construction. Hence $a + x_{\delta^\alpha_n}$ is strictly larger than a, but still smaller than x_α. So a does not generate $A_\alpha \upharpoonright x_\alpha$.

For b) let $\alpha \notin S$. By induction on $\gamma < \lambda$, I show that $A_\alpha \leq_{\mathrm{rc}} A_\gamma$ holds for every $\gamma \geq \alpha$. $A_\alpha \leq_{\mathrm{rc}} A_\alpha$ holds trivially. Suppose γ is a limit ordinal and $A_\alpha \geq_{\mathrm{rc}} A_\beta$ holds for all $\beta < \gamma$ such that $\alpha \leq \beta$. Then $A_\alpha \leq_{\mathrm{rc}} A_\gamma$ follows from Lemma 1.1.3. Now suppose $\gamma = \beta + 1$ for some $\beta \geq \alpha$. There are two cases:

I. $\beta \notin S$. In this case $A_\beta \leq_{\mathrm{rc}} A_\gamma$ by construction. By hypothesis, $A_\alpha \leq_{\mathrm{rc}} A_\beta$. By Lemma 1.1.3, this implies $A_\alpha \leq_{\mathrm{rc}} A_\gamma$.

II. $\beta \in S$. This is the non-trivial case. I *claim* that $A_\delta \leq_{\mathrm{rc}} A_\delta(x_\beta)$ holds for every $\delta < \beta$. This can be seen as follows: By Lemma 1.1.2, it is sufficient to show that both $A_\delta \upharpoonright x_\beta$ and $A_\delta \upharpoonright -x_\beta$ are principal. But $A_\delta \upharpoonright -x_\beta \subseteq A_\beta \upharpoonright -x_\beta = \{0\}$ by construction. Let $a \in A_\delta$ be such that

$a \leq x_\beta$. Let $m := \{n \in \omega : x_{\delta_n^\beta} \in A_\delta\}$. Clearly $m \in \omega$. Let $T \in [\omega]^{<\omega}$ be such that $a \leq \sum\{x_{\delta_n^\beta} : n \in T\}$. Then

$$a \leq \sum\{x_{\delta_n^\beta} : n \in T \cap m\} + \sum\{x_{\delta_n^\beta} : n \in T \setminus m\}.$$

Since $\sum\{x_{\delta_n^\beta} : n \in T \setminus m\}$ is independent over A_δ by construction,

$$a \leq \sum\{x_{\delta_n^\beta} : n \in T \cap m\} \leq \sum\{x_{\delta_n^\beta} : n < m\} \leq x_\beta.$$

This shows that $A_\delta \restriction x_\beta$ is generated by $\sum\{x_{\delta_n^\beta} : n < m\}$ and the claim holds. Now $A_\gamma = A_\beta(x_\beta) = \bigcup_{\alpha \leq \delta < \beta} A_\delta(x_\beta)$. Hence $A_\alpha \leq_{\mathrm{rc}} A_\gamma$ follows from the claim together with Lemma 1.1.3.

This shows b). □

In order to show that the Boolean algebra A constructed in the lemma above satisfies the c.c.c., I use an argument which was used by Soukup ([15]) to prove that, modulo the consistency of the existence of a supercompact cardinal, it is consistent with ZFC+GCH that there is a complete c.c.c. Boolean algebra without the WFN.

2.1.3. Lemma. *The Boolean algebra A constructed in the proof of Lemma 2.1.2 satisfies the c.c.c.*

Proof. Assume A does not satisfy the c.c.c. Let $C \subset A$ be an uncountable antichain. Let $X := \{x_\alpha : \alpha < \lambda\}$. For $x \in X$ let $x^0 := x$ and $x^1 := -x$. I may assume that each $a \in C$ is an elementary product of elements of X, i.e. there is $X_a \in [X]^{<\aleph_0}$ and $f_a : X_a \to 2$ such that $a = \prod_{x \in X_a} x^{f_a(x)}$. After thinning out C if necessary, I may assume that $\{X_a : a \in C\}$ is a Δ-system with root R, there is $f : R \to 2$ such that $f_a \restriction R = f$ for all $a \in C$, and all X_a are of the same size, say n.

Claim. Let $Y \in [X]^{<\omega}$ and $g : Y \to 2$ be such that $\prod_{x \in Y} x^{g(x)} = 0$. Then there are $\alpha \in S$ and $i \in \omega$ with $x_\alpha, x_{\delta_i^\alpha} \in Y$ such that $g(x_\alpha) = 1$ and $g(x_{\delta_i^\alpha}) = 0$.

First note that for $y, z \in X$, $y^{g(y)} \cdot z^{g(z)} = 0$ holds iff there are $\alpha \in S$ and $i \in \omega$ with $\{y, z\} = \{x_\alpha, x_{\delta_i^\alpha}\}$ such that $g(x_\alpha) = 1$ and $g(x_{\delta_i^\alpha}) = 0$. Now I show the claim by in induction on $\max\{\alpha < \lambda : x_\alpha \in Y\}$. The case $|Y| < 3$ is trivial.

Assume the claim has been proved for $\max\{\alpha < \lambda : x_\alpha \in Y\} < \beta$. Suppose $\max\{\alpha < \lambda : x_\alpha \in Y\} = \beta$ and for no two elements $y, z \in Y$, $y^{g(y)} \cdot z^{g(z)} = 0$. For $\beta \notin S$ the argument is easy. By assumption, $b := \prod_{x \in Y \setminus \{x_\beta\}} x^{g(x)} \neq 0$. By construction, x_β and b are independent. Thus $\prod_{x \in Y} x^{g(x)} \neq 0$.

Now suppose $\beta \in S$ and $\prod_{x \in Y} x^{g(x)} = 0$. By construction, $A_\beta \restriction -x_\beta = \{0\}$. Thus $b := \prod_{x \in Y \setminus \{x_\beta\}} x^{g(x)} \not\leq -x_\beta$. Therefore $g(x_\beta) = 1$ and $b \leq x_\beta$. By construction, there is $m \in \omega$ such that $b \leq \sum_{i<m} x_{\delta_i^\beta}$. It follows from the inductive hypothesis that $b \cdot \prod_{i<m} -x_{\delta_i^\beta} \neq 0$. This contradicts the choice of m and the claim is proved.

For each $a \in C$ let $X_a = \{x_{a,i} : i < n\}$. Clearly, I may assume that C has size $\aleph_1$. Let $\leq$ be a wellorder on C of ordertype ω_1. For each $\{a, b\} \in [C]^2$ choose a color $c(\{a, b\}) \in n^2$ such that

$$\forall (i, j) \in n^2 (c(\{a, b\}) = (i, j) \wedge a \leq b \Rightarrow x_{a,i}^{f_a(x_{a,i})} \cdot x_{b,j}^{f_b(x_{b,j})} = 0).$$

It follows from the claim that c can be defined. Clearly, for all $\{a, b\} \in [C]^2$, if $c(\{a, b\}) = (i, j)$ and $a \leq b$, then $x_{a,i}, x_{b,j} \notin R$. Baumgartner and Hajnal ([3]) established the following partition result:

$$\forall m \in \omega \forall \alpha < \omega_1 (\omega_1 \to (\alpha)^2_m).$$

In particular, $\omega_1 \to (\omega + 2)^2_{n^2}$ holds. That is, there are $(i, j) \in n^2$ and a subset C' of C of ordertype $\omega + 2$ such that for all $\{a, b\} \in [C']^2$, $c(\{a, b\}) = (i, j)$. Let a and b be the last two elements of C'. Assume $x_{a,j} = x_\alpha$ for some $\alpha \in S$. By construction of A, for all $c \in C' \setminus \{a, b\}$, $x_{c,i} = x_{\delta_k^\alpha}$ for some $k \in \omega$. By the Δ-system assumption, all the $x_{c,i}$'s are different. This implies $x_{a,j} = x_{b,j}$, contradicting the Δ-system assumption.

Now assume that for all $\alpha \in S$, $x_{a,j} \neq x_\alpha$. In this case, for all $c \in C' \setminus \{a, b\}$, $x_{c,i} = x_\alpha$ for some $\alpha \in S$. Let d and e be the first two elements of C'. Now for all $c \in C' \setminus \{d, e\}$, $x_{c,j} = x_{\delta_k^\alpha}$ for some $k \in \omega$. By the Δ-system assumption, all the $x_{c,j}$'s are different. This implies $x_{d,i} = x_{e,i}$, contradicting the Δ-system assumption. This finishes the proof of the lemma. □

Proof of the theorem. Let λ be an infinite cardinal. If $\lambda = \aleph_0$, then there are 2^λ pairwise non-isomorphic Boolean algebras of size λ and all of them are projective, hence tightly σ-filtered. Also, if λ is singular, then there are 2^λ pairwise non-isomorphic projective Boolean algebras by the result of Koppelberg mentioned before. Projective Boolean algebras satisfy the c.c.c.

For regular uncountable λ let $\mathcal{P}$ be a disjoint family of stationary subsets of $\{\alpha < \lambda : \mathrm{cf}(\alpha) = \aleph_0\}$ of size λ. Such a family exists by the wellknown results of Ulam and Solovay. For every subset $\mathcal{T}$ of $\mathcal{P}$ let $A^{\mathcal{T}}$ be the Boolean algebra which is constructed in the lemma from the set $S := \bigcup \mathcal{T}$ and let $(A^{\mathcal{T}}_\alpha)_{\alpha<\lambda}$ be its associated tight σ-filtration. Then for $\mathcal{T}, \mathcal{T}' \subseteq \mathcal{P}$ with $\mathcal{T} \neq \mathcal{T}'$ the Boolean algebras $A^{\mathcal{T}}$ and $A^{\mathcal{T}'}$ are non-isomorphic.

For suppose $h : A^{\mathcal{T}} \longrightarrow A^{\mathcal{T}'}$ is an isomorphism. W.l.o.g. I may assume that $\mathcal{T} \setminus \mathcal{T}'$ is nonempty. The set $\{\alpha < \lambda : h[A^{\mathcal{T}}_\alpha] = A^{\mathcal{T}'}_\alpha\}$ is club in λ. Since $\bigcup(\mathcal{T} \setminus \mathcal{T}')$ is stationary, there is $\alpha \in \bigcup(\mathcal{T} \setminus \mathcal{T}')$ such that $h[A^{\mathcal{T}}_\alpha] = A^{\mathcal{T}'}_\alpha$. But $A^{\mathcal{T}}_\alpha \not\leq_{\mathrm{rc}} A^{\mathcal{T}}$ and $A^{\mathcal{T}'}_\alpha \leq_{\mathrm{rc}} A^{\mathcal{T}'}$, a contradiction.

By Lemma 2.1.3, the Boolean algebras $A^{\mathcal{T}}$ satisfy the c.c.c. □

The two lemmas above give even more:

2.1.4. Theorem. *Let λ be an uncountable and regular cardinal. Then there is a family of size 2^λ of tightly σ-filtered c.c.c. Boolean algebras of size λ such that no member of this family is embeddable into another one as an rc-subalgebra.*

Proof. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are subsets of $\mathcal{P}$, where P is as in the proof of the theorem above. Assume there is an embedding $e : A^{\mathcal{T}} \to A^{\mathcal{T}'}$ such that $e[A^{\mathcal{T}}] \leq_{\mathrm{rc}} A^{\mathcal{T}'}$. Let $C \subseteq \lambda$ be a club such that $e[A^{\mathcal{T}}_\alpha] = A^{\mathcal{T}'}_\alpha \cap e[A^{\mathcal{T}}]$ and $\mathrm{lpr}^{A^{\mathcal{T}'}}_{e[A^{\mathcal{T}}]}[A^{\mathcal{T}'}_\alpha] \subseteq A^{\mathcal{T}'}_\alpha$ hold for every $\alpha \in C$. Let $\alpha \in C \cap \bigcup \mathcal{T}$. Then $e[A^{\mathcal{T}}_\alpha] \not\leq_{\mathrm{rc}} e[A^{\mathcal{T}}]$ and hence $e[A^{\mathcal{T}}_\alpha] \not\leq_{\mathrm{rc}} A^{\mathcal{T}'}$. Since $A^{\mathcal{T}'}_\alpha$ is closed under $\mathrm{lpr}^{A^{\mathcal{T}'}}_{e[A^{\mathcal{T}}]}$, $e[A^{\mathcal{T}}_\alpha] \leq_{\mathrm{rc}} A^{\mathcal{T}'}_\alpha$. Hence $A^{\mathcal{T}'}_\alpha \not\leq_{\mathrm{rc}} A^{\mathcal{T}'}$. Therefore $C \cap \bigcup \mathcal{T} \subseteq C \cap \bigcup \mathcal{T}'$. Thus, since $\mathcal{P}$ consists of stationary sets, $\mathcal{T} \subseteq \mathcal{T}'$. Now let I be an independent family of subsets of $\mathcal{P}$ of size 2^λ. In particular, the elements of I are pairwise

$\subseteq$-incomparable. Thus the family $\{A^{\mathcal{T}} : \mathcal{T} \in I\}$ consists of pairwise non-rc-embeddable tightly σ-filtered c.c.c. Boolean algebras of size λ. □

2.2 Characterizations of Tightly κ-Filtered Boolean Algebras

In this section I give characterizations of tightly κ-filtered Boolean algebras which are similar to the characterizations known for projective Boolean algebras. For these characterizations I have to assume that κ is uncountable, simply because some of the proofs given below do not work for $\kappa = \aleph_0$. However, some of the characterizations given below are parallel to those of projective Boolean algebras. The main difference to the projective case is that projective Boolean algebras are exactly the retracts of free Boolean algebras. A similar characterization of tightly κ-filtered Boolean algebras does not seem to be available. For the characterization of tightly κ-filtered Boolean algebras I will use the concept of commuting subalgebras of a Boolean algebra.

2.2.1. Definition. Let A and B be subalgebras of the Boolean algebra C. Then A and B *commute* iff for every $a \in A$ and every $b \in B$ such that $a \cdot b = 0$ there is $c \in A \cap B$ such that $a \leq c$ and $b \leq -c$.

A family $\mathcal{F}$ of subsets of a Boolean algebra A is called *commutative* iff it consists of pairwise commuting subalgebras. □

The connection between κ-subalgebras and commutative families is given by

2.2.2. Lemma. *Let $\mathcal{F}$ be a commutative family of subalgebras of A such that every $a \in A$ is contained in some $B \in \mathcal{F}$ of size $< \kappa$. Then $\mathcal{F}$ consists of κ-subalgebras of $\mathcal{F}$.*

Proof. Let $C \in \mathcal{F}$ and $a \in A$. Then there is $B \in \mathcal{F}$ such that $a \in B$. I *claim* that B contains a cofinal subset of $C \restriction a$. Let $c \in C \restriction a$. Now $-a \cdot c = 0$. Since B and C commute, there is $b \in B \cap C$ such that $c \leq b$ and $-a \leq -b$. But now $c \leq b \leq a$. □

This lemma is implicitly contained in the book by Heindorf and Shapiro ([23]) for the case $\kappa = \aleph_1$.

It turns out that additivity of skeletons is what separates tight κ-filteredness from κ-filteredness.

2.2.3. Definition. A $<\kappa$-skeleton (respectively κ-skeleton) $\mathcal{S}$ of a Boolean algebra A is called *additive* iff for every subset $T \subseteq \mathcal{S}$ the Boolean algebra $\langle \bigcup T \rangle$ generated in A by $\bigcup T$ is a member of $\mathcal{S}$. □

In order to make the similarities between projective Boolean algebras and tightly κ-filtered Boolean algebras apparent, I quote the following from Heindorf and Shapiro ([23]):

2.2.4. Theorem. *The following are equivalent for a Boolean algebra A:*

(i) A is projective.

(ii) For some ordinal δ, A is the union of a continuous chain $(A_\alpha)_{\alpha<\delta}$ consisting of rc-subalgebras such that $A_{\alpha+1}$ is countably generated over A_α for every $\alpha < \delta$ and A_0 is countable.

(iii) A has a tight rc-filtration.

(iv) A has an additive commutative skeleton.

(v) A has an additive skeleton consisting of rc-embedded subalgebras.

(vi) A is the union of a family $\mathcal{C}$ of countable subsets of A such that $\langle \bigcup S \rangle \leq_{\mathrm{rc}} A$ for every $S \subseteq \mathcal{C}$. □

The characterization of tightly κ-filtered Boolean algebras is the following:

2.2.5. Theorem. *Let κ be an uncountable regular cardinal. The following are equivalent for a Boolean algebra A:*

(i) For some ordinal δ, A is the union of a chain $(A_\alpha)_{\alpha<\delta}$ of κ-subalgebras which is continous at limit ordinals of cofinality $\geq \kappa$ such that $A_{\alpha+1}$ is $\leq \kappa$-generated over A_α for every $\alpha < \delta$ and A_0 has size $\leq \kappa$.

(ii) A has a tight κ-filtration.

(iii) A has an additive commutative $< \kappa$-skeleton.

(iv) A has an additive $< \kappa$-skeleton consisting of κ-embedded subalgebras.

(v) A has an additive κ-skeleton consisting of κ-embedded subalgebras.

(vi) A is the union of a family $\mathcal{C}$ of subsets of size $< \kappa$ of A such that for all $S, T \subseteq \mathcal{C}$ the algebras $\langle \bigcup S \rangle$ and $\langle \bigcup T \rangle$ commute.

(vii) A is the union of a family $\mathcal{C}$ of subsets of size $< \kappa$ of A such that for every $S \subseteq \mathcal{C}$, $\langle \bigcup S \rangle \leq_\kappa A$.

(viii) A is the union of a family $\mathcal{C}$ of subsets of size $\leq \kappa$ of A such that for every $S \subseteq \mathcal{C}$, $\langle \bigcup S \rangle \leq_\kappa A$.

Proof. (i)⇒(ii) was proved by Koppelberg ([28]) for $\kappa = \aleph_1$. The proof for arbitrary regular κ is exactly the same. Let $(A_\alpha)_{\alpha<\delta}$ be a filtration of A as in (i). First make the sequence continuous by inserting the appropriate unions at those limit stages which lack continuity. Since this only happens at limits of cofinality $< \kappa$, the filtration remains a κ-filtration by part d) of Lemma 1.1.3. For $\alpha \leq \lambda$ let $X \in [A_{\alpha+1}]^{\leq\kappa}$ be such that $A_\alpha(X) = A_{\alpha+1}$. Let $X = \{x_\delta : \delta < \kappa\}$. Now insert $(A_\alpha(\{x_\gamma : \gamma < \beta\}))_{\beta<\kappa}$ between A_α and $A_{\alpha+1}$. Similarly, insert a continuous tight filtration of A_0 below A_0. The new filtration is a κ-filtration by part c) of Lemma 1.1.3 and it is tight by construction.

(iii)⇒(iv) follows from Lemma 2.2.2.

(iv)⇒(v) is trivial.

(iii)⇒(vi), (iv)⇒(vii), and (v)⇒(viii) can be seen using the same argument: Let the $\mathcal{C}$ consist of the elements of the $< \kappa$-skeleton (κ-skeleton) of size $< \kappa$ (of size $\leq \kappa$). Then additivity of the $< \kappa$-skeleton (κ-skeleton) yields the desired property of $\mathcal{C}$.

(vi)⇒(vii) follows from Lemma 2.2.2 applied to the family $\mathcal{F}$ of all subalgebras of A generated by a union of elements of $\mathcal{C}$.

(vii)⇒(i) and (viii)⇒(i) are easily seen using the following argument: Let $A = \{a_\alpha : \alpha <|A|\}$. For every $\alpha <|A|$ choose $B_\alpha \in \mathcal{C}$ such that $a_\alpha \in B_\alpha$. Let $A_\alpha := \langle\bigcup_{\beta<\alpha} B_\beta\rangle$ for every $\alpha <|A|$. $(A_\alpha)_{\alpha<|A|}$ works for (i).

(ii)⇒(iii) is the only part that requires some work. Let $(x_\alpha)_{\alpha<\delta} \in {}^\delta A$ be such that $(\langle\{x_\beta : \beta < \alpha\}\rangle)_{\alpha<\delta}$ is a tight κ-filtration of A. For every $S \subseteq \delta$ let $A_S := \langle\{x_\beta : \beta \in S\}\rangle$. With this notation the filtration is simply $(A_\alpha)_{\alpha<\delta}$. Choose $f : \delta \longrightarrow [\delta]^{<\kappa}$ such that for every $\alpha < \delta$ the ideals $A_\alpha \upharpoonright x_\alpha$ and $A_\alpha \upharpoonright -x_\alpha$ are generated by $(A_\alpha \upharpoonright x_\alpha) \cap A_{f(\alpha)}$ and $(A_\alpha \upharpoonright -x_\alpha) \cap A_{f(\alpha)}$ respectively and such that $f(\alpha) \subseteq \alpha$. Let $\mathcal{S} := \{A_T : T \subseteq \delta \wedge \bigcup f[T] \subseteq T\}$. $\mathcal{S}$ is an additive $< \kappa$-skeleton:

Clearly, every subset of A of size at least κ is included in a member of $\mathcal{S}$ of the same size. Moreover, any subset of A of size $< \kappa$ is included in an element of $\mathcal{S}$ of size $< \kappa$. Suppose $\mathcal{T} \subseteq \mathcal{S}$. Let $\mathcal{U} \subseteq \mathfrak{P}(\delta)$ be such that $\mathcal{T} = \{A_T : T \in \mathcal{U}\}$. Then $\langle\bigcup \mathcal{T}\rangle = A_{\bigcup\mathcal{U}} \in \mathcal{S}$ since $\bigcup\mathcal{U}$ is closed under f. In particular, $\mathcal{S}$ is closed under unions of subchains.

It remains to show that $\mathcal{S}$ is commutative.

Suppose $S, T \subset \kappa$ are closed under f. It is sufficient to show that $A_{S\cap\alpha}$ and $A_{T\cap\alpha}$ commute for every $\alpha < \delta$. I will do so by induction on α. The limit stages of the induction are trivial. Suppose $\alpha = \beta+1$. W.l.o.g. I may assume $\beta \in S$. Let $u \in A_{S\cap\alpha}$ and $v \in A_{T\cap\alpha}$ be such that $u \cdot v = 0$. W.l.o.g. I may assume that u is of the form $a \cdot x_\beta$ for some $a \in A_{S\cap\beta}$. The case $u = a - x_\beta$ is completely analogous. Only the following cases are interesting:

I. $v = b - x_\beta$ for some $b \in A_{T\cap\beta}$ and $\beta \in T$. Then $x_\beta \in A_S \cap A_T$, $u \leq x_\beta$ and $v \leq -x_\beta$.

II. $v = b \cdot x_\beta$ for some $b \in A_{T\cap\beta}$ and $\beta \in T$. Then $a \cdot b \cdot x_\beta = 0$. Hence $a \cdot b \leq -x_\beta$. Take $c \in A_{f(\beta)}$ such that $a \cdot b \leq c \leq -x_\beta$. Then $(a - c) \cdot (b - c) = 0$, $a \cdot x_\beta \leq a - c$ and $b \cdot x_\beta \leq b - c$. Now $a - c \in A_{S\cap\beta}$ and $b - c \in A_{T\cap\beta}$. By hypothesis, there is $r \in A_{T\cap\beta} \cap A_{S\cap\beta}$ such that $a - c \leq r$ and $b - c \leq -r$. r is as required.

III. $v \in A_{T\cap\beta}$. Then $a \cdot v \leq -x_\beta$. Choose $c \in A_{f(\beta)}$ such that $a \cdot v \leq c \leq -x_\beta$. Then $a \cdot v - c = 0$ and $u = a \cdot x_\beta \leq a - c$. Since $a - c \in A_{S\cap\beta}$, there is $r \in A_{S\cap\beta} \cap A_{T\cap\beta}$ such that $a - c \leq r$ and $v \leq -r$.

This completes the induction and (ii)⇒(iii) of the theorem follows. □

2.2.6. Remark. It follows from the proof of this theorem that A is tightly κ-filtered iff it has a tight κ-filtration indexed by $|A|$. □

The assumption $\kappa > \aleph_0$ was only needed for this theorem. From now on I only assume κ to be regular and infinite. The following corollary is very useful when one wants to show that some Boolean algebra is not tightly κ-filtered.

2.2.7. Corollary. *Let κ be an infinite regular cardinal. If a Boolean algebra A is tightly κ-filtered, then there is a function $f : A \to [A]^{<\kappa}$ such that for any two sets $X, Y \subseteq A$ which are closed under f, $\langle X \cup Y \rangle \leq_\kappa A$.*

Proof. By Theorem 2.2.5 respectively Theorem 2.2.4, there is a subset $\mathcal{C}$ of $[A]^{<\kappa}$ such that $A = \bigcup \mathcal{C}$ and for each $\mathcal{S} \subseteq \mathcal{C}$, $\langle \bigcup S \rangle \leq_\kappa A$. For each $a \in A$ choose $f(a) \in \mathcal{C}$ such that $a \in f(a)$. f works for the corollary. □

The characterization of tight κ-filteredness also gives

2.2.8. Corollary. *a) Every Boolean algebra A of size κ is tightly κ-filtered.*

b) Every Boolean algebra of size κ^+ which has the κ-FN is tightly κ-filtered.

c) Every tightly κ-filtered Boolean algebra has the κ-FN.

d) If a Boolean algebra A is a retract of a tightly κ-filtered Boolean algebra B, then A is tightly κ-filtered, too.

Proof. a) follows immediately from (i) in Theorem 2.2.5 respectively from (ii) in Theorem 2.2.4.

For b) let A be a Boolean algebra of size κ^+ which has the κ-FN. By Lemma 1.4.4, A is κ-filtered. Let $\mathcal{S}$ be a κ-skeleton of A consisting of κ-subalgebras. In $\mathcal{S}$ choose a strictly increasing sequence $(A_\alpha)_{\alpha<\kappa^+}$ such that $A = \bigcup_{\alpha<\kappa^+} A_\alpha$ and for all $\alpha < \kappa^+$, $|A_\alpha| = \kappa$. By (i) of Theorem 2.2.5 respectively (ii) of Theorem 2.2.4, A is tightly κ-filtered.

c) follows easily from (v) of Theorem 2.2.5 respectively (v) of Theorem 2.2.4.

For d) let $p : B \to A$ and $e : A \to B$ be homomorphisms such that $p \circ e = \mathrm{id}_A$. By Theorem 2.2.5 respectively Theorem 2.2.4, B has an additive κ-skeleton $\mathcal{T}$ consisting of κ-subalgebras. Let $\mathcal{T}'$ be the set of those elements of $\mathcal{T}$ which are closed under $e \circ p$. It is easy to see that $\mathcal{T}'$ is an additive κ-skeleton for B as well. Now let

$$\mathcal{S} := \{p[C] : C \in \mathcal{T}'\}.$$

Again, it is easy to see that $\mathcal{S}$ is an additive κ-skeleton for A. I *claim* that $\mathcal{S}$ consists of κ-subalgebras of A.

Let $C \in \mathcal{T}'$ and $a \in A$. Let Y be a cofinal subset of $C \upharpoonright e(a)$ of size $< \kappa$. Then $p[Y]$ is a cofinal subset of $p[C] \upharpoonright a$ of size $< \kappa$. This proves the claim.

By Theorem 2.2.5 respectively Theorem 2.2.4, A is tightly κ-filtered. $\square$

2.3 Stone spaces of tightly κ-filtered Boolean algebras

The implication (i)⇒(viii) and the proof of (viii)⇒(i) of Theorem 2.2.5 show that for a tightly κ-filtered Boolean algebra there is a lot of freedom in the choice of a tight κ-filtration of A. This fact allows it to generalize certain results by Koppelberg ([29]) on Stone spaces of projective Boolean algebras to Stone spaces of tightly κ-filtered Boolean algebras. Let A be a tightly κ-filtered Boolean algebra of size λ and X be its Stone space. I am interested in the subspace of X of points of small character.

2.3.1. Definition. Let M_λ be the subspace of X that consists of the ultrafilters of A which have character $< \lambda$. For Boolean algebras $B \leq C$ an ultrafilter p of B *splits* in C iff there are distinct ultrafilters q and r of C both extending p. $\square$

Note that p splits in C iff there is $c \in C$ such that $p \cup \{c\}$ and $p \cup \{-c\}$ both have the finite intersection property.

2.3.2. Theorem. *Let A be a tightly κ-filtered Boolean algebra of size λ, where $\kappa < \lambda$, λ is regular, and $|\delta|^{<\kappa} < \lambda$ holds for every $\delta < \lambda$. Let X and M_λ be as above. Then M_λ is an intersection of subsets of X which are unions* *of less than κ clopen sets and is determined by a subalgebra B of A of size $< \lambda$, i.e. there is $B \leq A$ such that $|B| < \lambda$ and $p \cap B$ does not split in A for any $p \in M_\lambda$.*

Proof. For the first assertion it is enough to show that for every point p in the complement of M_λ, there is a set $a_p \subseteq X \setminus M_\lambda$ such that $p \in a_p$ and a_p is the intersection of less than κ clopen subsets of X.

Let $p \in X \setminus M_\lambda$. Then there is a κ-filtration $(A'_\alpha)_{\alpha<\lambda}$ pf A such that the following hold for all $\alpha < \lambda$:

a) $p \cap A'_\alpha$ splits in $A'_{\alpha+1}$

b) $A'_{\alpha+1}$ is κ-generated, but not $< \kappa$-generated over A'_α.

This filtration can be constructed as in the proof of (viii)⇒(i) of Theorem 2.2.5 using the fact $\chi(p) = \lambda$ to get a) together with some extra care to get b). Now this filtration can easily be refined to a tight κ-filtration $(A_\alpha)_{\alpha<\lambda}$ such that $p \cap A_\alpha$ splits in $A_{\alpha+1}$ for every ordinal $\alpha < \lambda$ of cofinality $\geq \kappa$.

A moment's reflection shows that for all $\alpha < \lambda$ the set a_α of ultrafilters of A_α which split in $A_{\alpha+1}$ is an intersection of less than κ clopen sets in the Stone space of A_α. More exactly: Let $x \in A_{\alpha+1}$ be such that $A_\alpha(x) = A_{\alpha+1}$. An ultrafilter q of A_α splits in $A_{\alpha+1}$ iff $q \cup \{x\}$ and $q \cup \{-x\}$ both are centered. Let I_x and I_{-x} be cofinal subsets of size $< \kappa$ of $A_\alpha \restriction x$ and $A_\alpha \restriction -x$ respectively. Now $q \cup \{x\}$ and $q \cup \{-x\}$ both are centered iff q is disjoint from $I_x \cup I_{-x}$. But this holds iff the point q in the Stone space of A_α is contained in the intersection of the clopen sets corresponding to complements of elements of $I_x \cup I_{-x}$.

For every $\alpha < \lambda$ let I_α be a subset of A_α of size $< \kappa$ which generates the filter corresponding to a_α.

W.l.o.g. I may assume that the underlying set of A is λ. Let

$$S := \{\alpha < \lambda : \alpha \text{ is a limit ordinal of cofinality } \geq \kappa \\ \text{and the underlying set of } A_\alpha \text{ is } \alpha\}.$$

Since λ is a regular cardinal larger than κ, S is a stationary subset of λ. Let $f : \lambda \longrightarrow \lambda$ be the mapping which assigns to each $\alpha < \lambda$ the least upper bound of I_α. Then f is regressive on S. Hence there is a stationary subset T of S such that f is constant on T. Let δ be the value of f on T. Since δ has less than λ subsets of size $< \kappa$, there is a stationary subset U of T such that the mapping $F : \alpha \longmapsto I_\alpha$ is constant on U. Let I be the value of F on U and let a_p be the corresponding closed subset of X which is an intersection of less than κ clopen sets. For every ultrafilter $q \in a_p$ and every $\alpha \in U$, $q \cap A_\alpha$ splits in $A_{\alpha+1}$. Therefore each $q \in a_p$ has character λ. Hence $a_p \subseteq X \setminus M_\lambda$. Finally, $p \in a_p$ by construction. This proves the first assertion of the theorem.

For the second assertion suppose that M_λ is not determined by a subalgebra of A of size less than λ. By a similar argument as above, get a tight κ-filtration $(A_\alpha)_{\alpha<\lambda}$ such that for every ordinal $\alpha < \lambda$ of cofinality $\geq \kappa$ there is an ultrafilter $p \in M_\lambda$ such that $p \cap A_\alpha$ splits in $A_{\alpha+1}$. As above, there is a stationary subset U of λ consisting of ordinals of cofinality $\geq \kappa$ and a subset I of A of size $< \kappa$ such that for every $\alpha \in U$ the filter generated by I in A_α corresponds to the closed subset of the Stone space of A_α of those ultrafilters which split in $A_{\alpha+1}$. Let a be the closed subset of X corresponding to I. a is an intersection of less than κ clopen sets. By construction, $a \cap M_\lambda$ is non-empty. But all points in M_λ have character less than λ and all points in a have character λ because λ is regular. Thus M_λ and a are disjoint. This contradicts the choice of the filtration. □

2.4 Boolean algebras that are rc-filtered, but not tightly κ-filtered

In this section the arguments will be mainly topological. Let me collect some topological characterizations of the Stonean duals of κ-embeddings.

2.4.1. Lemma. *Let A be a subalgebra of the Boolean algebra B. Let X and Y be the Stone spaces of A and B respectively. Let $\phi : Y \to X$ be the Stonean dual of the inclusion of A into B. The following statements are equivalent:*

(i) $A \leq_\kappa B$

(ii) For each clopen set $b \subseteq Y$, $\chi(\phi[b], X) < \kappa$.

(iii) For each closed set $b \subseteq Y$ such that $\chi(b, Y) < \kappa$, $\chi(\phi[b], X) < \kappa$.

Proof. Stone duality. □

Recall that for a closed subset a of topological space X the pseudo-character of a is the minimal size of an open family $\mathcal{F}$ in X such that $\bigcap \mathcal{F} = a$. For a Boolean space it sufficient to consider clopen families $\mathcal{F}$. The pseudo-character of a equals the character of a if X is compact.

The concept of a symmetric power of a topological space was used by Ščepin in order to get an openly generated space that is not Dugundji or, in terms of Boolean algebras, to get a Boolean algebra that is rc-filtered but not projective. I will give a slight generalizion of his result.

2.4.2. Definition. Let X be a topological space. Let $\sim_X$ be the equivalence relation on X^2 that identifies (x, y) and (y, x) for all $x, y \in X$. Let $\mathrm{SP}^2(X) := X^2/\sim$. If X is the Stone space of the Boolean algebra A, then $\mathrm{SP}^2(X)$ is also a Boolean space and the algebra of clopen subsets of $\mathrm{SP}^2(X)$ corresponds to the subalgebra $\mathrm{SP}^2(A)$ of $A \oplus A$ consisting of those elements which are fixed by the automorphism of $A \oplus A$ that interchanges the two copies of A. □

2.4.3. Lemma. *(Ščepin, see [23]) SP^2 is a covariant functor from the category of Boolean algebras into itself where the definition of SP^2 on homomorphisms is the natural one. Let A be a Boolean algebra. Then the embedding $\mathrm{SP}^2(A) \to A \oplus A$ is relatively complete. SP^2 is continuous, i.e. if $(A_\alpha)_{\alpha<\lambda}$ is an ascending chain of subalgebras of A, then*

$$\mathrm{SP}^2(\bigcup_{\alpha<\lambda} A_\alpha) = \bigcup_{\alpha<\lambda} \mathrm{SP}^2(A_\alpha).$$

SP^2 preserves cardinalities, i.e. if A is infinite, then $|A|=|\mathrm{SP}^2(A)|$. $\mathrm{SP}^2(A)$ is rc-filtered provided that A is. □

It turns out that $\mathrm{SP}^2(\mathrm{Fr}(\lambda))$ is not tightly κ-filtered if λ is large enough. This will follow easily from

2.4.4. Lemma. *Let A, B, and C be infinite Boolean algebras such that the Stone space of A has character $\geq \kappa$.*

Then

$$\langle \mathrm{SP}^2(A \oplus B) \cup \mathrm{SP}^2(A \oplus C)\rangle \not\leq_\kappa \mathrm{SP}^2(A \oplus B \oplus C).$$

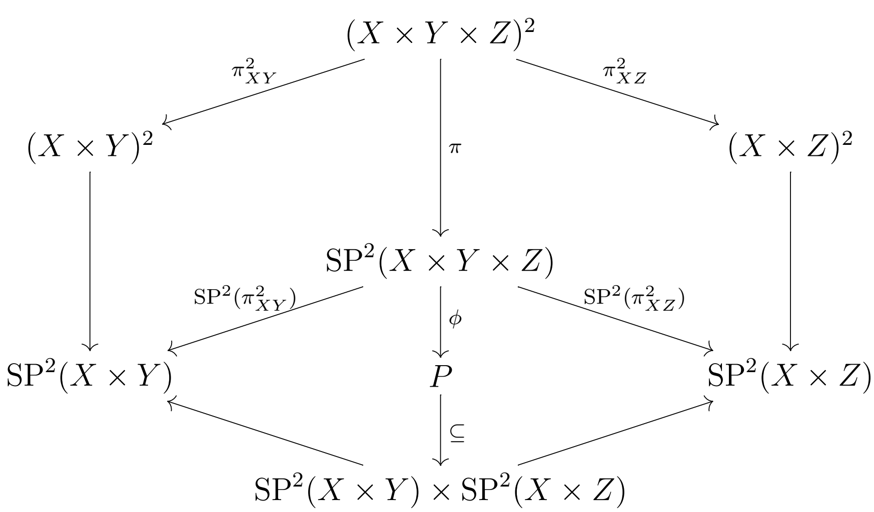
Proof. I prove the topological dual. Let X, Y, and Z be the Stone spaces of A, B, and C respectively. To commence I introduce names for several mappings. Let π^2_{XY} and π^2_{XZ} denote the projections of $(X \times Y \times Z)^2$ onto $(X \times Y)^2$ and $(X \times Z)^2$ respectively. Let π denote the quotient map from $(X \times Y \times Z)^2$ onto $\mathrm{SP}^2(X \times Y \times Z)$. It follows from Lemma 2.4.3 that π is open. Let π_{XY} and π_{XZ} denote the projections of $X \times Y \times Z$ onto $X \times Y$ and $X \times Z$ respectively. Now $\mathrm{SP}^2(\pi_{XY})$ and $\mathrm{SP}^2(\pi_{XZ})$ are also defined. Let

$$\phi : \mathrm{SP}^2(X \times Y \times Z) \to \mathrm{SP}^2(X \times Y) \times \mathrm{SP}^2(X \times Z);$$
$$p \mapsto (\mathrm{SP}^2(\pi_{XY})(p), \mathrm{SP}^2(\pi_{XZ})(p))$$

and $P := \operatorname{Im}\phi$. Note that ϕ is the Stonean dual of the inclusion from

$$\langle \mathrm{SP}^2(A \oplus B) \cup \mathrm{SP}^2(A \oplus C)\rangle$$

into $\mathrm{SP}^2(A \oplus B \oplus C)$. The picture looks like this:



Here the mappings that are not labeled are the natural ones.

Now let $U_1, U_2 \subseteq Y$ and $V_1, V_2 \subseteq Z$ be non-empty, clopen, and disjoint.

Claim 1: $\pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$ is clopen in $\mathrm{SP}^2(X \times Y \times Z)$ but $(\phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$ has character $\geq \kappa$ in P.

This claim together with Lemma 2.4.1 proves the lemma. For its proof I need

Claim 2:

$$\begin{aligned} W &:= (\phi^{-1} \circ \phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2] \\ &= \pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2] \cup \bigcup_{x \in X} \pi[\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1]. \end{aligned}$$

Proof of Claim 2: Let $(a_1, b_1, c_1, a_2, b_2, c_2)$ be such that $\pi(a_1, b_1, c_1, a_2, b_2, c_2)$ is contained in W but not in $\pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$. Then there is $(a_1', b_1', c_1', a_2', b_2', c_2') \in X \times U_1 \times V_1 \times X \times U_2 \times V_2$ s.t.

$$(\phi \circ \pi)(a_1, b_1, c_1, a_2, b_2, c_2) = (\phi \circ \pi)(a_1', b_1', c_1', a_2', b_2', c_2').$$

I may assume $a_1 = a_1'$ and $a_2 = a_2'$. Now the following holds: $\{b_1, b_2\} = \{b_1', b_2'\}$, $\{c_1, c_2\} = \{c_1', c_2'\}$, $b_1' \neq b_2'$, $c_1' \neq c_2'$, and hence $c_1 \neq c_2$ and $b_1 \neq b_2$.

Suppose $a_1 \neq a_2$. In this case

$$((a_1, b_1), (a_2, b_2)) \sim_{X \times Y} ((a_1', b_1'), (a_2', b_2'))$$

and

$$((a_1, c_1), (a_2, c_2)) \sim_{X \times Z} ((a_1', c_1'), (a_2', c_2')).$$

Moreover, $b_i = b_i'$ and $c_i = c_i'$ for $i = 1, 2$, and hence

$$\pi(a_1, b_1, c_1, a_2, b_2, c_2) \in \pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2],$$

a contradiction. Thus, $a_1 = a_2$. Since $\{b_1, b_2\} = \{b_1', b_2'\}$ and $\{c_1, c_2\} = \{c_1', c_2'\}$,

$$(a_1, b_1, c_1, a_2, b_2, c_2) \sim_{X \times Y \times Z} (a_1', b_1', c_2', a_2', b_2', c_1').$$

Therefore

$$(a_1, b_1, c_1, a_2, b_2, c_2) \in \bigcup_{x \in X} \pi[\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1].$$

Conversely, let $a \in X$, $b_i \in U_i$, and $c_i \in V_i$ for $i = 1, 2$. Now

$$\begin{aligned}(\phi \circ \pi)(a, b_1, c_2, a, b_2, c_1) &= (\phi \circ \pi)(a, b_1, c_1, a, b_2, c_2) \\ &\in (\phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2].\end{aligned}$$

This finishes the proof of Claim 2.

Proof of Claim 1: $\pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$ is clopen in $\mathrm{SP}^2(X \times Y \times Z)$ since

$$\begin{aligned}&(\pi^{-1} \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2] \\ &\quad = (X \times U_1 \times V_1 \times X \times U_2 \times V_2) \cup (X \times U_2 \times V_2 \times X \times U_1 \times V_1)\end{aligned}$$

is clopen in $(X \times Y \times Z)^2$.

For the character part of Claim 1 let $\Delta^2[X]$ be the diagonal $\{(x, x) : x \in X\}$ of X^2. Now

$$\begin{aligned}&\chi((\phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2], P) \\ &\quad \geq \chi\left(\bigcup_{x \in X} \pi[\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1], \mathrm{SP}^2(X \times Y \times Z)\right) \\ &\quad \geq \chi\Bigg(\bigcup_{x \in X}(\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1) \\ &\qquad \cup \bigcup_{x \in X}(\{x\} \times U_2 \times V_1 \times \{x\} \times U_1 \times V_2), (X \times Y \times Z)^2\Bigg) \\ &\quad \geq \chi(\Delta^2[X], X^2) \geq \chi(X).\end{aligned}$$

Here the last inequality can be seen as follows. Let $\mu := \chi(\Delta^2[X], X^2)$ and let $\{U^\alpha : \alpha < \mu\}$ be a local base at $\Delta^2[X]$. For each $x \in X$ and each $\alpha < \mu$ pick an open set $U^\alpha_x \subseteq X$ containing x such that $(U^\alpha_x)^2 \subseteq U^\alpha$. Now

$(\bigcap_{\alpha<\mu} U_x^\alpha)^2 = \bigcap_{\alpha<\mu}(U_x^\alpha)^2 \subseteq \Delta^2[X]$. Hence $\bigcap_{\alpha<\mu} U_x^\alpha = \{x\}$. Thus x has pseudo-character $\leq \mu$. Since X is compact, x has character $\leq \mu$. □

Now I am ready to prove a theorem which yields the promised examples of rc-filtered Boolean algebras which are not tightly κ-filtered.

2.4.5. Theorem. *Let κ and λ be regular.* $\mathrm{SP}^2(\mathrm{Fr}\,\lambda)$ *is tightly κ-filtered iff $\lambda \leq \kappa^+$.*

Proof. $A := \mathrm{SP}^2(\mathrm{Fr}\,\lambda)$ is rc-filtered by Lemma 2.4.3. In particular, A is κ-filtered for every regular cardinal κ. For $\lambda \leq \kappa^+$, $|A| \leq \kappa^+$. Hence, by the characterization of tightly κ-filtered Boolean algebras, A is tightly κ-filtered. This proves the easy implication of the theorem.

Now let $\lambda > \kappa^+$. Suppose A is tightly κ-filtered. Then there is a function $f : A \to [A]^{<\kappa}$ as in Corollary 2.2.7. For $S \subseteq \lambda$ let $\mathrm{SP}(S) := \mathrm{SP}^2(\mathrm{Fr}\,S)$ and consider this algebra as a subalgebra of A in the obvious way. Since SP^2 is continuous and cardinal preserving, there are disjoint sets $S, T \in [\lambda]^{\kappa^+}$ such that $\mathrm{SP}(S)$ and $\mathrm{SP}(S \cup T)$ are closed under f. Choose $S' \subseteq S \cup T$ such that $\mathrm{SP}(S')$ is closed under f and $|S' \cap S| = |S' \cap T| = \kappa$. Let $S_0 := S' \cap S$ and $T_0 := S' \cap T$. Finally, choose $S_1 \in [S]^\kappa$ disjoint from S_0 such that $\mathrm{SP}(S_0 \cup S_1)$ is closed under f. Since $\mathrm{SP}(S_0 \cup S_1)$ and $\mathrm{SP}(S_0 \cup T_0)$ are closed under f and by the choice of f,

$$\langle \mathrm{SP}(S_0 \cup S_1) \cup \mathrm{SP}(S_0 \cup T_0) \rangle \leq_\kappa A.$$

This contradicts Lemma 2.4.4. □

Clearly, this theorem implies

2.4.6. Corollary. *For each regular cardinal κ there is a Boolean algebra A such that A is rc-filtered but not tightly κ-filtered.* □

2.5 Complete Boolean algebras and tight σ-filtrations

Fuchino and Soukup ([19]) have shown that there may be arbitrarily large complete Boolean algebras which are σ-filtered. More exactly, if CH holds and $0^\sharp$ does not exist, then all complete c.c.c. Boolean algebras are σ-filtered. In this section, I look at the stronger property of having a tight σ-filtration. It turns out that no infinite complete Boolean algebra of size larger than $(2^{\aleph_0})^+$ is tightly σ-filtered. It is sufficient to prove that the completion of the free Boolean algebra over $(2^{\aleph_0})^{++}$ generators has no tight σ-filtration, since the Balcar-Franěk Theorem implies that this algebra is a retract of every complete Boolean algebra of size larger than $(2^{\aleph_0})^+$. A similar argument will show that adding $\aleph_3$ Cohen reals to a model of CH yields a model where $\mathfrak{P}(\omega)$ is not tightly σ-filtered but still has the WFN.

2.5.1. Definition. For a set X let the *Cohen algebra* $\mathbb{C}(X)$ over X be the completion of the free Boolean algebra $\mathrm{Fr}(X)$ over X. For $X \subseteq Y$, $\mathbb{C}(X)$ will be regarded as a complete subalgebra of $\mathbb{C}(Y)$ in the obvious way. □

A technical lemma

Both results mentioned above depend heavily on the next lemma or rather on its more convenient second version, but neither one uses the full strength of the lemma. However, this seems to be approximately the weakest lemma that works for both proofs. It roughly says that the left-hand-side of the inequality $(*)$ only badly approximates the right-hand-side.

2.5.2. Lemma. *Let A, B, and C be Boolean algebras, $n \in \omega$, and $(a_i)_{i\leq n} \in A^{n+1}$ and $(b_i)_{i\leq n} \in B^{n+1}$ antichains with $a_i, b_i \neq 0$ for all $i \in n+1$. For each $k < n$ and each $i < n+1$ let $u_i^k, v_i^k \in C$ be such that*

$$(*) \quad \sum_{k<n} \left(\sum_{i,j<n+1} a_i u_i^k v_j^k b_j \right) \leq \sum_{i<n+1} a_i b_i$$

holds in $A \oplus B \oplus C$. Then for each $c \in C^+$ there are $d \in (C \restriction c)^+$ and $i < n+1$ such that

$$a_i b_i d \cdot \sum_{k<n} \left(\sum_{i,j<n+1} a_i u_i^k v_j^k b_j \right) = 0.$$

Proof. Since $(a_i)_{i<n+1}$ and $(b_i)_{i<n+1}$ are antichains without zero elements, by $(*)$, $u_i^k v_j^k = 0$ whenever $i \neq j$. Hence

$$\sum_{k<n} \left(\sum_{i,j<n+1} a_i u_i^k v_j^k b_j \right) = \sum_{k<n} \sum_{i<n+1} a_i u_i^k v_i^k b_i$$

$$= \sum_{i<n+1} a_i b_i \left(\sum_{k<n} u_i^k v_i^k \right).$$

Let $c \in C$. Let $P \subseteq C$ be the set of all atoms of the subalgebra of C that is generated by c together with the elements $u_i^k v_i^k$ for $k < n$ and $i < n+1$. Choose $d \in P$ such that $d \leq c$. Define the 2-valued matrix $(d_{ik})_{i<n+1,k<n}$ by letting $d_{ik} := 0$ iff $d u_i^k v_i^k = 0$ and $d_{ik} := 1$ iff $d \leq u_i^k v_i^k$. This matrix is well defined since d was taken from P. For each $k \leq n$, $(u_i^k v_i^k)_{i<n+1}$ is an antichain. Therefore each column of $(d_{ik})_{i<n+1,k<n}$ contains at most one 1. Hence there is $i < n+1$ such that the i'th row contains no 1. i and d work for the lemma. $\square$

The following version of this lemma will be more convenient for the intended application. For a Boolean algebra A let $\overline{A} := \mathrm{ro}(A)$ and consider A as a subalgebra of $\overline{A}$ in the usual way.

2.5.3. Lemma. *Let A, B, and C be Boolean algebras, $n \in \omega$, and $(a_i)_{i\leq n} \in A^{n+1}$ and $(b_i)_{i\leq n} \in B^{n+1}$ antichains with $a_i, b_i \neq 0$ for all $i \in n+1$. Suppose $\{x_k : k < n\} \subseteq \overline{A \oplus C}$ and $\{y_k : k < n\} \subseteq \overline{B \oplus C}$ are s.t.*

$$\sum_{k<n} x_k y_k \leq \sum_{i<n+1} a_i b_i$$

in $\overline{A \oplus B \oplus C}$. Then for each $c \in C^+$ there are $d \in (C \upharpoonright c)^+$ and $i < n+1$ such that $a_i b_i d \cdot \sum_{k<n} x_k y_k = 0$.

Proof. Let $(S^k)_{k<n}$ and $(T^k)_{k<n}$ be disjoint families of sets and for every $n < k$, $s \in S^k$, and $t \in T^k$ let $a_s \in A^+$, $v_s, w_t \in C^+$, and $b_t \in B^+$ such that $x_k = \sum_{s \in S^k} a_s v_s$ and $y_k = \sum_{t \in T^k} b_t w_t$. For $i < n+1$ and $k < n$ let $S_i^k := \{s \in S^k : a_s \leq a_i\}$ and $T_i^k := \{t \in T^k : b_t \leq b_i\}$. Then $(S_i^k)_{i<n+1}$ and

$(T_i^k)_{i<n+1}$ are partitions of S^k and T^k respectively. Moreover, if $i \neq j$, then for all $k < n$ and for all $s \in S_i^k$ and $t \in T_j^k$, $s \cdot t = 0$. Now

$$\begin{aligned}\sum_{k<n} x_k y_k &= \sum_{k<n} \left(\left(\sum_{s \in S^k} a_s v_s \right) \left(\sum_{s \in T^k} b_t w_t \right) \right) \\ &= \sum_{k<n} \sum_{s \in S^k, t \in T^k} a_s v_s w_t b_t = \sum_{k<n} \sum_{i<n+1} \sum_{s \in S_i^k, t \in T_i^k} a_s v_s w_t b_t \\ &\leq \sum_{k<n} \sum_{i<n+1} \sum_{s \in S_i^k, t \in T_i^k} a_i v_s w_t b_i = \sum_{k<n} \sum_{i<n+1} \left(a_i b_i \left(\sum_{s \in S_i^k, t \in T_i^k} v_s w_t \right) \right) \\ &\leq \sum_{i<n+1} a_i b_i.\end{aligned}$$

For each $k < n$ and $i < n+1$ let $v_i^k := \sum_{s \in S_i^k} v_s$ and $w_i^k := \sum_{t \in T_i^k} w_t$. Then $\sum_{s \in S_i^k, t \in T_i^k} v_s w_t = v_i^k w_i^k$ and thus

$$\sum_{k<n} x_k y_k \leq \sum_{k<n} \sum_{i<n+1} a_i v_i^k w_i^k b_i \leq \sum_{i<n+1} a_i b_i.$$

Now for each $c \in C^+$ suitable d and i exist by Lemma 2.5.2. □

Complete Boolean algebras of size $\geq (2^{\aleph_0})^{++}$ have no tight σ-filtration

The essential observation in order to get the theorem for $\mathbb{C}((2^{\aleph_0})^{++})$ is

2.5.4. Lemma. *Let X, Y, and Z be disjoint infinite sets. Let $C_0 := \mathbb{C}(X \cup Z)$, $C_1 := \mathbb{C}(Y \cup Z)$, and $C := \mathbb{C}(X \cup Y \cup Z)$. Then $\langle C_0 \cup C_1 \rangle \not\leq_\sigma C$.*

Proof. Let $X_0 \subset X$ and $Y_0 \subset X$ be countably infinite. Let $g : \omega \to \mathrm{Fr}(X_0 \cup Y_0)$ be a surjection, $f : \omega \times \omega \to \omega$ a bijection, and $f_0, f_1 : \omega \to \omega$ such that $f^{-1} = (f_0(\cdot), f_1(\cdot))$. Let $(c_i)_{i \in \omega}$ be an antichain in $\mathrm{Fr}(Z)$ without zero elements and put $x := \sum_{i \in \omega} c_i (g \circ f_0)(i)$. I *claim* that $\langle C_0 \cup C_1 \rangle \restriction x$ is not countably generated.

Proof of the claim: Let $\{x_n : n \in \omega\} \subseteq \langle C_0 \cup C_1 \rangle \restriction x$ be closed under finite joins. Let $n \in \omega$. Then there is $k \in \omega$ such that for each $i \in \omega$ there are $p^i_j \in C_0$ and $q^i_j \in C_1$, $j < k$, s.t.

$$c_i \cdot x_n = p^i_0 q^i_0 + \dots + p^i_{k-1} q^i_{k-1}.$$

Now there is $m \in \omega$ such that $(g \circ f_0) f(m,n) = \sum_{l<k+1} a_l b_l$ for two antichains $(a_l)_{l<k+1}$ and $(b_l)_{l<k+1}$ without zero elements in $\mathrm{Fr}(X_0)$ and $\mathrm{Fr}(Y_0)$ respectively. Since $c_{f(m,n)} \cdot x_n \leq \sum_{l<k+1} a_l b_l$, one can use Lemma 2.5.3 to get $l < k$ and $d \in \mathrm{Fr}(Z)^+$ such that $d \leq c_{f(m,n)}$ and $a_l b_l d x_{f(m,n)} = 0$. Let $y_{f(m,n)} := a_l b_l d$ and let $y_{f(m',n)} := 0$ for $m' \neq m$. Finally let $y := \sum_{i\in\omega} y_i$. Note that for suitable $(a'_i)_{i\in\omega} \in {}^\omega C_0$ and $(b'_i)_{i\in\omega} \in {}^\omega C_1$, $y = \left(\sum_{i\in\omega} a'_i c_i\right) \cdot \left(\sum_{i\in\omega} b'_i c_i\right)$. Therefore $y \in \langle C_0 \cup C_1 \rangle \restriction x$. However, $y \not\leq x_n$ for any $n \in \omega$. This proves the claim and hence finishes the proof of the lemma. □

Now I am ready to prove

2.5.5. Theorem. $\mathbb{C}((2^{\aleph_0})^{++})$ *is not tightly* σ*-filtered.* □

But before embarking the proof of this theorem, let me deduce from it

2.5.6. Corollary. *No complete Boolean algebra of size strictly larger than* $(2^{\aleph_0})^+$ *has a tight* σ*-filtration.*

Proof. Suppose A is a complete Boolean algebra of size $\geq (2^{\aleph_0})^{++}$. By the well-known Balcar-Franěk Theorem, $\mathrm{Fr}((2^{\aleph_0})^{++})$ embeds into A. By completeness of A, this embedding extends to $\mathbb{C}((2^{\aleph_0})^{++})$. Since the free algebra is dense in the Cohen algebra, this extension is an embedding as well. By completeness of $\mathbb{C}((2^{\aleph_0})^{++})$, $\mathbb{C}((2^{\aleph_0})^{++})$ is a retract of A. Since being tightly σ-filtered is hereditary with respect to retracts (Corollary 2.2.8) and by the theorem above, A is not tightly σ-filtered. □

Proof of the theorem. Suppose $A := \mathbb{C}((2^{\aleph_0})^{++})$ has a tight σ-filtration. Let $f : A \to [A]^{<\kappa}$ be a function as in Corollary 2.2.7. Since A satisfies c.c.c., every subalgebra of A of size $2^{\aleph_0}$ or $(2^{\aleph_0})^+$ is contained in a complete subalgebra of the same size. Hence, using the argument in the proof of Theorem 2.4.5, I can find non-empty disjoint sets $S_0, S_1, T_0 \subseteq \lambda$ of size $2^{\aleph_0}$ such that

$\mathbb{C}(S_0 \cup S_1)$ and $\mathbb{C}(S_0 \cup T_0)$ are closed under f. By the preceding lemma,

$$\langle \mathbb{C}(S_0 \cup S_1) \cup \mathbb{C}(S_0 \cup T_0) \rangle \not\leq_\sigma \mathbb{C}(S_0 \cup S_1 \cup T_0).$$

A contradiction. □

After adding many Cohen reals, $\mathfrak{P}(\omega)$ is not tightly σ-filtered

The proof of this theorem is very similar to the proof of Theorem 2.5.5. The parallel of Lemma 2.5.4 is

2.5.7. Lemma. *Let A and B be complete Boolean algebras both adding Cohen reals such that any countable set of ordinals in a generic extension by $A \oplus B$ of the ground model M is contained in a countable set in M. Let G be $(A \oplus B)$-generic over M. Let $P_0 := \mathfrak{P}(\omega)^{M[G\cap A]}$ and $P_1 := \mathfrak{P}(\omega)^{M[G\cap B]}$. Then $\langle P_0 \cup P_1 \rangle \not\leq_\sigma P := \mathfrak{P}(\omega)^{M[G]}$.*

Proof. Since A and B both add Cohen reals, there are countable atomless regular subalgebras A_0 and B_0 of A and B respectively. Let $g : \omega \to A_0 \oplus B_0$ be onto, $f : \omega \times \omega \to \omega$ a bijection, and $f_0, f_1 : \omega \to \omega$ such that $f^{-1} = (f_0(\cdot), f_1(\cdot))$, like in the proof of Lemma 2.5.4. Let $\sigma := g \circ f_0$. Consider σ as an $\overline{A \oplus B}$-name for a subset of ω. I will show that $\langle P_0 \cup P_1 \rangle \restriction \sigma_G$ is not countably generated. Suppose $S \in M[G]$ is a countable subset of this ideal which is closed under finite joins. For every $a \in S$ there is a name $\tau^a : \omega \to \overline{A \oplus B}$ such that $a = \tau^a_G$. Let $T := \{\tau^a : a \in S\}$. Since $S \subseteq \langle P_0 \cup P_1 \rangle$, I may assume that

(∗) for each $\tau \in T$ there is $k_\tau \in \omega$ such that for all $m \in \omega$ there are $p^m_0, \ldots, p^m_{k_\tau - 1} \in A$ and $q^m_0, \ldots, q^m_{k_\tau - 1} \in B$ such that $\tau(m) = \sum_{i<k_\tau} p^m_i q^m_i$.

Here the exact reasoning is like this: Each a in S is some Boolean combination of elements from P_0 and P_1. Hence, if τ is a name for a, i.e. if $\tau_G = a$, then there are a condition r in G and $k_\tau \in \omega$ such that

$$r \Vdash \exists p_0, \ldots, p_{k_\tau - 1} \in P_0 \exists q_0, \ldots, q_{k_\tau - 1} \in P_1 \left(\tau = \sum_{i<k_\tau} p_i q_i \right).$$

By the maximal principle, there are names $\{(m, p_i^m) : m \in \omega\}$ and $\{(m, q_i^m) : m \in \omega\}$ for the p_i and q_i respectively. From these names I can construct a name τ for a which works for $(*)$.

Now for each $\tau \in T$ choose $p_\tau \in G$ such that $p_\tau \Vdash \tau \subseteq \sigma$. Note that $p \Vdash \tau \subseteq \sigma$ iff $\tau(m) \leq -p + \sigma(m)$ for all $m \in \omega$. This is equivalent to $\tau(m) \cdot p \leq \sigma(m)$ for all $m \in \omega$. Let $\tau \in T$. From $p_\tau \in G$ it follows that $a = \tau_G^a = (\tau^a \cdot p_{\tau^a})_G$, where $\tau \cdot p$ is the function that maps every $m \in \omega$ to $\tau(m) \cdot p$. Since $\{p \cdot q : p \in A, q \in B\}$ is dense in $\overline{A \oplus B}$, I may assume $p_\tau = p^\tau \cdot q^\tau$ for some $p^\tau \in A$ and $q^\tau \in B$ for each $\tau \in T$. This is handy, since replacing each $\tau \in T$ by $\tau \cdot p^\tau \cdot q^\tau$ preserves property $(*)$.

Therefore I may assume that $(*)$ holds and for every $\tau \in T$, $\tau \leq \sigma$, i.e. for all $m \in \omega$ the inequality $\tau(m) \leq \sigma(m)$ holds. By assumption, T is contained in a countable set T' of names in the ground model. Since only those names $\tau \in T'$ that do not spoil $(*)$ and for which $\tau \leq \sigma$ holds are relevant and since these properties are definable in the ground model, I may assume that $(*)$ holds for T' and $\tau \leq \sigma$ holds for every $\tau \in T'$. Moreover, I may assume that T' is closed under finite joins, in the sense that for all $\tau, \tau' \in T'$ the name $\{(m, \tau(m) + \tau'(m)) : m \in \omega\}$ is also an element of T'. Let $(\tau_n)_{n\in\omega} \in M$ be an enumeration of T'. Since $A_0 \oplus B_0$ is a regular subalgebra of $\overline{A \oplus B}$, I will be done if I can prove the following

Claim. There is a name $\rho : \omega \to \overline{A \oplus B}$ for an element of $\langle P_0 \cup P_1\rangle$ such that for every $n \in \omega$ and every $r \in A_0 \oplus B_0$ there is $s \leq r$, $s \in A_0 \oplus B_0$, such that $s \Vdash \rho \not\leq \tau_n$.

Proof of the claim: Construct ρ as follows: For each $n \in \omega$ choose $k_n \in \omega$ and sequences $(p_{i,n}^m)_{i<k_n, m\in\omega}$ in A and $(q_{i,n}^m)_{i<k_n, m\in\omega}$ in B as promised in $(*)$ for τ_n. For $m, n \in \omega$ such that $\sigma(f(m,n)) = \sum_{i<k_n+1} a_i b_i$ for some antichains $(a_i)_{i<k_n}$ and $(b_i)_{i<k_n}$ in A_0^+ and B_0^+ respectively let $i < k_n + 1$ be such that $a_i b_i \tau_n(f(m,n)) = 0$. This is possible by Lemma 2.5.3. Note that in this case the algebra C mentioned in the lemma is trivial. Let $\rho(f(m,n)) := a_i b_i$. Now $a_i b_i \Vdash \rho \not\subseteq \tau_n$. In any other case let $\rho(f(m,n)) := 0$. Clearly, ρ is a name for an element of $\langle P_0 \cup P_1\rangle$.

ρ works for the claim: Let $n \in \omega$ and $r \in \overline{A_0 \oplus B_0}$. W.l.o.g. I may assume $r = a \cdot b$ for some $a \in A_0$ and $b \in B_0$. Let $m \in \omega$ such that

$\sigma(f(m,n)) = \sum_{i<k_n+1} a_i b_i \leq a \cdot b$ for some antichains $(a_i)_{i \leq k_n+1}$ and $(b_i)_{i \leq k_n+1}$ in A_0^+ and B_0^+ respectively. Note that the a_i and b_i are uniquely determined by $\sigma(f(m,n))$, up to permutation of the common index set. This is not really important here, but it makes the argument somewhat shorter. Now $\rho(f(m,n)) = a_i b_i$ for some $i < k_n + 1$ and $\tau_n(f(m,n)) \cdot a_i b_i = 0$. Hence $s := a_i b_i \Vdash \rho \not\subseteq \tau_n$ and $s \leq r$. This finishes the proof of the claim and hence the proof of the lemma. □

With this lemma at hand, I can prove the announced result on Cohen forcing. In fact, I will prove a slightly more general theorem.

2.5.8. Theorem. *Let λ be a cardinal such that $\lambda^{\aleph_0} = \lambda$ in the ground model M. Let $(A_\alpha)_{\alpha<\lambda^{++}}$ be a sequence Boolean algebras in the ground model, each adding at most λ new reals, such that*

$$A := \bigoplus_{\alpha<\lambda^{++}} A_\alpha$$

satisfies c.c.c. Let G be A-generic over M. Then

$$M[G] \models \mathfrak{P}(\omega) \text{ has no tight } \sigma\text{-filtration.}$$

In particular, adding $\aleph_3$ Cohen reals to a model of CH gives a model in which $\mathfrak{P}(\omega)$ fails to be tightly σ-filtered, though $\mathrm{WFN}(\mathfrak{P}(\omega))$ *still holds.*

Proof. For $S \subseteq \lambda^{++}$ let $A_S := \bigoplus_{\alpha \in S} A_\alpha$, $G_S := G \cap A_S$, and $P_S := \mathfrak{P}(\omega)^{M[G_S]}$. Suppose $\mathfrak{P}(\omega)$ has a tight σ-filtration in $M[G]$. I may assume that this is already forced by 1_A. In $M[G]$ let $f : \mathfrak{P}(\omega) \to [\mathfrak{P}(\omega)]^{\aleph_0}$ be a function as in Corollary 2.2.7. Let $\phi \in M$ be an $A_{\lambda^{++}}$-name for such a function. Using c.c.c., one can construct a function $g : {}^\omega A_{\lambda^{++}} \to [\lambda^{++}]^{\aleph_0}$ such that for every name $\tau : \omega \to A_{\lambda^{++}}$, $\Vdash \phi(\tau) \subseteq P_{g(\tau)}$. Call a subset S of λ^{++} *good* iff $\bigcup g[{}^\omega A_S] \subseteq S$. Let S and T be disjoint subsets of λ^{++} of size λ^+ such that S and $S \cup T$ are good. This is possible since $(\lambda^+)^{\aleph_0} = \lambda^+$. Now let $S_0, S_1 \subseteq S$ and $T_0 \subseteq T$ be disjoint sets of size λ such that $S_0 \cup S_1$ and $S_0 \cup T_0$ are good. Applying the last lemma to the algebras $\overline{A}_{S_1}$ and $\overline{A}_{T_0}$ with $M[G_{S_0}]$ as the ground model, it follows that $\langle P_{S_0 \cup S_1} \cup P_{S_0 \cup T_0} \rangle \not\leq_\sigma P_{S_0 \cup S_1 \cup T_0}$

in $M[G_{S_0}][G_{S_1\cup T_1}]$. By c.c.c., $\langle P_{S_0\cup S_1}\cup P_{S_0\cup T_0}\rangle \not\leq_\sigma P_{\lambda^{++}}$ holds in $M[G]$. This is a contradiction since by the choice of g, the algebras $P_{S_0\cup S_1}$ and $P_{S_0\cup T_0}$ are closed under ϕ_G. □

The pseudo product of Cohen forcings

While so far the only known way to obtain a model of $\neg\,\mathrm{CH}+\mathrm{WFN}(\mathfrak{P}(\omega))$ is to add Cohen reals to a model of CH, there is some freedom in the choice of the iteration used for adding the Cohen reals. In [18] Fuchino, Shelah, and Soukup introduced a new kind of side-by-side product of partial orders.

2.5.9. Definition. Let $(P_i)_{i\in X}$ be a family of partial orders where each P_i has a largest element 1_{P_i}. As usual, for $p\in\prod_{i\in X}P_i$ let $\mathrm{supp}(p):=\{i\in X: p(i)\neq 1_{P_i}\}$ be the *support* of p. Let $\prod^*_{i\in X}P_i:=\{p\in\prod_{i\in X}P_i:|\mathrm{supp}(p)|\leq\aleph_0\}$ be ordered such that for all $p,q\in\prod^*_{i\in X}P_i$,

$$p\leq q\Leftrightarrow\forall i\in X(p(i)\leq q(i))\wedge|\{i\in X:p(i)\neq q(i)\neq 1_{P_i}\}|<\aleph_0. \quad \square$$

Among other things, Fuchino, Shelah, and Soukup proved the following about this product:

2.5.10. Lemma. *Let $(P_i)_{i\in X}$ be as in the definition above.*

a) For every $Y\subseteq X$, $\prod^*_{i\in X}\cong\prod^*_{i\in Y}\times\prod^*_{i\in X\setminus Y}$.

b) Under CH, $\prod^*_{i\in X}\mathrm{Fn}(\omega,2)$ satisfies the $\aleph_2$-c.c. and is proper. □

I will show that $\mathfrak{P}(\omega)$ has the WFN after forcing with $\prod^*_{i\in X}\mathrm{Fn}(\omega,2)$ over a model of CH, provided $|X|$ is smaller than $\aleph_\omega$. I will use the well-known

2.5.11. Lemma. *Suppose the partial order P is a union of an increasing chain $(P_\alpha)_{\alpha<\lambda}$ of completely embedded suborders. Let G be P-generic over the ground model M and for each $\alpha<\lambda$ let $G_\alpha:=P_\alpha\cap G$. If λ has uncountable cofinality, then for every real $x\in M[G]$ there is $\alpha<\lambda$ such that $x\in M[G_\alpha]$.*

Proof. Let x be a real in $M[G]$. I may assume that x is a function from ω to 2. Let $\dot{x}$ be a P-name for x. For each $\alpha<\lambda$ let $\dot{x}_\alpha$ be a P_α-name for a function from ω to 2 such that

$$\forall n \in \omega \exists p \in P_\alpha \exists i \in 2((p \Vdash_P \dot{x}(n) = i) \Rightarrow (p \Vdash_{P_\alpha} \dot{x}_\alpha(n) = i)).$$

For each $n \in \omega$ let $\alpha_n < \lambda$ be such that there is $p \in G_\alpha$ deciding $\dot{x}(n)$. Let $\alpha := \sup_{n\in\omega} \alpha_n$. Now $\alpha < \lambda$ since λ has uncountable cofinality. Clearly, $(\dot{x}_\alpha)_{G_\alpha} = \dot{x}_G$. Thus $x \in M[G_\alpha]$. □

2.5.12. Theorem. *Let $\lambda < \aleph_\omega$ be an uncountable cardinal and suppose CH holds. Let $P := \prod^*_{\alpha<\lambda} \mathrm{Fn}(\omega, 2)$. Then*

$$\Vdash_P \mathrm{WFN}(\mathfrak{P}(\omega)) \text{ and } 2^{\aleph_0} = \lambda.$$

Proof. Let M be the ground model satisfying CH and let G be P-generic over M. It follows from Lemma 2.5.10 that P is cardinal preserving and that the continuum is λ in $M[G]$. Throughout this proof I will use Lemma 2.5.10 without referring to it anymore. For each $X \subseteq \lambda$ with $X \in M$ consider $P_X := \prod^*_{\alpha\in X} \mathrm{Fn}(\omega, 2)$ as a suborder of P in the obvious way and let $G_X := P_X \cap G$ and $\mathfrak{P}_X := (\mathfrak{P}(\omega))^{M[G_X]}$. $(\mathfrak{P}_\alpha)_{\alpha\leq\lambda}$ is continuous at limit ordinals of uncountable cofinality by Lemma 2.5.11.

Claim. In $M[G]$: For each $\alpha < \lambda$, $\mathfrak{P}_\alpha \leq_\sigma \mathfrak{P}(\omega)$.

Proof of the claim: I argue in $M[G]$. Let $\alpha < \lambda$. Let $x \in \mathfrak{P}(\omega)$. By $\aleph_2$-c.c. of P, in M there is a subset X of λ of size $< \aleph_2$ such that $x \in \mathfrak{P}_X$. By Lemma 2.5.11, in M there is a countable subset Y of $X \setminus \alpha$ such that $x \in M[G_\alpha][G_Y]$. The set $D := \{p \in P_Y : \mathrm{supp}(p) = Y\}$ is dense in P_Y. Thus there is $p \in G_Y \cap D$. It is easy to see that $P_Y \downarrow p$ is isomorphic to $\mathrm{Fn}(\omega, 2)$. Thus there is a Cohen real r over $M[G_\alpha]$ in $M[G]$ such that $x \in M[G_\alpha][r]$. It was shown in [16] that

$$M[G_\alpha][r] \models (\mathfrak{P}(\omega) \cap M[G_\alpha]) \restriction x \text{ has countable cofinality.}$$

(This also follows from Theorem 3.1.4 in the next chapter.) By properness of P, $\mathfrak{P}_\alpha \restriction x$ really has countable cofinality. This finishes the proof of the claim.

Now it follows by induction on the size of λ that $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds in

$M[G]$. The induction uses Lemma 1.4.3 and the fact that $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds under CH. □

Using the same argument as in the the proof of theorem 2.5.8, one can show that $\mathfrak{P}(\omega)$ is not tightly σ-filtered after forcing with $\prod^*_{\alpha<\omega_3} \mathrm{Fn}(\omega, 2)$ over a model of CH.

2.5.13. Theorem. *Assume CH and let $\lambda \geq \aleph_3$. Let $P := \prod^*_{\alpha<\lambda} \mathrm{Fn}(\omega, 2)$. Then*

$$\Vdash_P \mathfrak{P}(\omega) \text{ is not tightly } \sigma\text{-filtered.}$$

Proof. Again, in this proof I will use Lemma 2.5.10 without referring to it explicitly. Let G be P-generic over the ground model M. I argue in $M[G]$. Suppose that $\mathfrak{P}(\omega)$ is tightly σ-filtered. Let f be a function as in Corollary 2.2.7. For $X \subseteq \lambda$ with $X \in M$ let P_X, G_X, and $\mathfrak{P}_X$ be defined as in the proof of Theorem 2.5.12. Let $S \subseteq T \subseteq \lambda$ with $S, T \in M$ be such that $|S|=|T|=|T \setminus S|= \aleph_2$ and $\mathfrak{P}_S$ and $\mathfrak{P}_T$ are closed under f. This is possible by Lemma 2.5.11. In M choose disjoint sets $S_0, S_1 \subseteq S$ and a set $T_0 \subseteq T \setminus S$ such that $\mathfrak{P}_{S_0 \cup S_1}$ and $\mathfrak{P}_{S_0 \cup T_1}$ are closed under f. By Lemma 2.5.7,

$$M[G_{S_0 \cup S_1 \cup T_0}] \models \langle \mathfrak{P}(\omega) \cap (M[G_{S_0 \cup S_1}] \cup M[G_{S_0 \cup T_0}]) \rangle \not\leq_\sigma \mathfrak{P}(\omega).$$

Since $P_{\lambda \setminus (S_0 \cup S_1 \cup T_0)}$ is proper and $M[G] = M[G_{S_0 \cup S_1 \cup T_0}][G_{\lambda \setminus (S_0 \cup S_1 \cup T_0)}]$,

$$\langle \mathfrak{P}_{S_0 \cup S_1} \cup \mathfrak{P}_{S_0 \cup T_0} \rangle \not\leq_\sigma \mathfrak{P}(\omega).$$

This contradicts the choice of f. □

Chapter 3

The Weak Freese-Nation Property

While in the last chapter I considered tightly κ-filtered Boolean algebras, in this chapter I will drop 'tightly'. Moreover, I will only consider σ-filtered partial orders, mostly complete Boolean algebras. This chapter will be much more set-theoretic than the last one.

3.1 WFN($\mathfrak{P}(\omega)$) in forcing extensions

In this section I will show that WFN($\mathfrak{P}(\omega)$) is very fragile in the sense that in typical forcing extensions which have a larger continuum than the ground model and which are not Cohen extensions, WFN($\mathfrak{P}(\omega)$) fails. The following notions are crucial.

3.1.1. Definition. A notion of forcing P *yields a σ-extension of* $\mathfrak{P}(\omega)$ iff $\Vdash_P \mathfrak{P}(\omega) \cap \check{V} \leq_\sigma \mathfrak{P}(\omega)$. Similarly, P *yields a non-σ-extension of* $\mathfrak{P}(\omega)$ iff $\Vdash_P \mathfrak{P}(\omega) \cap \check{V} \not\leq_\sigma \mathfrak{P}(\omega)$. □

Typically, when enlarging the continuum by forcing, one uses some kind of long iteration of rather small forcings which add new reals. Very popular examples are countable support iterations of length ω_2 of proper forcings of size $\aleph_1$ over a model of CH or finite support iterations of length $> 2^{\aleph_0}$ of forcings satisfying c.c.c. These examples can be treated using

3.1.2. Lemma. *Let A be a partial order such that for every A-generic filter G over the ground model M every countable set of ordinals in $M[G]$ is covered* *by some countable set in M. Assume that forcing with A does not collapse cardinals. Suppose $\lambda :=|A|$ is regular. Let $S \subseteq \lambda$ be a stationary set of ordinals of uncountable cofinality. Suppose A is the union of an increasing chain $(A_\alpha)_{\alpha<\lambda}$ of completely embedded suborders which is continuous at limit ordinals in S. Assume for each $\alpha < \lambda$, $\Vdash_{A_\alpha} 2^{\aleph_0} < \lambda$. Suppose for each $\alpha \in S$, $\Vdash_{A_{\alpha+1}} \mathfrak{P}(\omega) \cap \check{V}[\dot{G} \cap \check{A}_\alpha] \not\leq_\sigma \mathfrak{P}(\omega)$. Then $\Vdash_A \neg\,\mathrm{WFN}(\mathfrak{P}(\omega))$.*

Proof. Let G be A generic over M. I argue in $M[G]$. For $\alpha < \lambda$ let $P_\alpha := \mathfrak{P}(\omega) \cap M[G \cap A_\alpha]$ and let $P_\lambda := \mathfrak{P}(\omega)$. For convenience, let $A_\lambda := A$.

By Lemma 2.5.11, $(P_\alpha)_{\alpha\leq\lambda}$ is continuous at limit ordinals in S and at λ. Note that the continuum is at least λ since $|S|= \lambda$ and $P_{\alpha+1} \setminus P_\alpha \neq \emptyset$ for $\alpha \in S$. By assumption, $|P_\alpha|< \lambda$ for each $\alpha < \lambda$. Therefore $2^{\aleph_0} = \lambda$. For each $\alpha < \lambda$ let $Q_{\alpha+1} := P_{\alpha+1}$ and for every limit ordinal $\delta < \lambda$ let $Q_\delta := \bigcup_{\alpha<\delta} P_\alpha$. Now $(Q_\alpha)_{\alpha<\lambda}$ is continuously increasing and agrees with $(P_\alpha)_{\alpha<\lambda}$ on S. Let $f : \mathfrak{P}(\omega) \to [\mathfrak{P}(\omega)]^{\aleph_0}$ be any function. Since S is stationary, there is $\alpha \in S$ such that Q_α is closed under f. Now $Q_\alpha = P_\alpha$ and

$$M[G \cap A_{\alpha+1}] \models \mathfrak{P}(\omega) \cap M[G \cap A_\alpha] \not\leq_\sigma \mathfrak{P}(\omega).$$

Therefore there is $x \in P_{\alpha+1} \subseteq \mathfrak{P}(\omega)$ such that no in $M[G \cap A_{\alpha+1}]$ countable set includes a cofinal subset of $Q_\alpha \upharpoonright x$. Since $[2^{\aleph_0}]^{\aleph_0} \cap M$ is cofinal in $[2^{\aleph_0}]^{\aleph_0}$, $Q_\alpha \upharpoonright x$ really has uncountable cofinality. Thus f is not a WFN-function for $\mathfrak{P}(\omega)$. Since f was arbitrary, WFN($\mathfrak{P}(\omega)$) fails. □

A characterization of the forcings that yield σ-extensions of $\mathfrak{P}(\omega)$

The following lemma characterizes those proper notions of forcing which yield σ-extensions of $\mathfrak{P}(\omega)$. I am not going to introduce properness, since I will only use the following property of proper forcing extensions:

Every countable set of ordinals in the extension is included in a countable set in the ground model.

In particular, $\aleph_1$ of the ground model remains a cardinal in the extension. Note that all c.c.c. forcings as well as many other forcing notions, especially the standard forcing notions used for manipulating the cardinal invariants of the continuum, are proper.

It is convenient to introduce some additional notions first.

3.1.3. Definition. Let P be a partial order and C be a subset of P. Then $S(C)$ denotes the set of all greatest upper bounds of subsets of C that exist in P. A subset Q of P is *predense* below $p \in P$ iff for each $r \leq p$ there is $q \in Q$ such that $q \leq p$ and q is compatible with r. □

3.1.4. Lemma. *Let M be a transitive model of* ZFC*. *Let $A \in M$ be a complete Boolean algebra such that for each A-generic filter over M every countable set of ordinals in $M[G]$ is included in a countable set of ordinals which is an element of M. Then the following are equivalent:*

(i) For each A-generic filter G, $M[G] \models \mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$.

(ii) In M: For every countable subset C of A there is a dense set of $a \in A$ such that $(a \cdot S(C))^+$ has a countable subset B which is in A predense below each element of $(a \cdot S(C))^+$. Here $a \cdot S(C)$ means $\{a \cdot s : s \in S(C)\}$. □

The formulation in (ii) sounds exceedingly strange. The problem is that the algebra generated by $S(C)$ does not have to be a regular subalgebra of A. I wanted a formulation that does not use generic filters. The property of the countable set B can be described as follows: For any generic filter G containing $a \cdot c$ for some $c \in S(C)$ there is $b \in B$ such that $b \leq a \cdot c$ and $b \in G$. In particular, B is dense in $(a \cdot S(C))^+$.

Proof of the lemma. Suppose (i) holds for A. Let C be any countable subset of A and $b \in A$. I have to show that there is $a \in A^+$ with $a \leq b$ such that $(a \cdot S(C))^+$ has a countable subset which is in A predense below every element of $(a \cdot S(C))^+$.

Let $\sigma : \omega \to C$ be onto. I regard σ as a name for a subset of ω. Let G be an A-generic filter over M containing b. By (i), there is a countable set D of subsets of ω such that $I := (\mathfrak{P}(\omega) \cap M) \restriction \sigma_G$ is generated by a subset of D. By the properties of the extension, I may assume that D is an element of M. Now there is $a \in G$ such that a forces that I is generated by a subset of D. Since G is a filter, I can choose a below b. I may assume that for each $d \in D$ there is a non-zero $a_d \leq a$ such that $a_d \Vdash \check{d} \subseteq \sigma$ since only these d's are interesting. Let $[\![\check{d} \subseteq \sigma]\!]$ denote the truth value of the statement '$\check{d} \subseteq \sigma$'.

Claim 1. $[\![\check{d} \subseteq \sigma]\!] = \prod\{\sigma(n) : n \in d\}$.

This is easily seen. In particular, $[\![\check{d} \subseteq \sigma]\!] \in S(C)$. By enlarging the a_d's if necessary, I may assume that for each $d \in D$, a_d is the product of a and $[\![\check{d} \subseteq \sigma]\!]$. Now $a_d \in (a \cdot S(C))^+$ for every $d \in D$.

Claim 2. $\{a_d : d \in D\}$ is predense below each $c \in (a \cdot S(C))^+$.

Proof of Claim 2: Let C' be a subset of C such that $c := a \cdot \prod C' > 0$. Let $c' \leq c$ be such that $c' > 0$. Let $e := \{n \in \omega : \sigma(n) \geq c'\}$. Now c' forces that I is generated by a subset of D and $e \subseteq \sigma$. Let H be generic containing c'. Then there is $d \in D$ such that $e \subseteq d$ and $a_d \in H$. Since $a_d \leq \sigma(n)$ for each $n \in e$, $a_d \leq c$. Since a_d and c' both are elements of H, they are compatible. This proves Claim 2 and hence one direction of the equivalence.

For the other direction suppose (ii) holds. Let G be A-generic over M and let σ be a name for a subset of ω. I may assume that σ is a function from ω to A. Let $C := \operatorname{Im} \sigma$. By (ii), there is $a \in G$ such that $(a \cdot S(C))^+$ has a countable subset B which is predense below every non-zero element of $a \cdot S(C)$. For each $b \in B$ let $d_b := \{n \in \omega : \sigma(n) \geq b\}$. Let $D := \{d_b : b \in B\}$.

Claim 3. $I := (\mathfrak{P}(\omega) \cap M) \restriction \sigma_G$ is generated by a subset of D.

Proof of Claim 3: Let $e \in I$. Then there is $c \leq a$ with $c \in G$ such that $c \Vdash \check{e} \subseteq \sigma$. By Claim 1, $c \Vdash \check{e} \subseteq \sigma$ holds precisely if for each $n \in e$, $\sigma(n) \geq c$. Hence $a \cdot \prod\{\sigma(n) : n \in e\}$ also forces $\check{e} \subseteq \sigma$ and is an element of G. Since B is predense below $a \cdot \prod\{\sigma(n) : n \in e\}$, there is $b \in B \cap G$ such that $b \Vdash \check{e} \subseteq \sigma$. But now $e \subseteq d_b \subseteq \sigma_G$. This shows Claim 3 and therefore finishes the proof the lemma. □

This lemma is quite abstract and technical, but it has interesting consequences. For example, it follows that any proper ${}^\omega\omega$-bounding forcing notion which adds a new real gives a non-σ-extension of $\mathfrak{P}(\omega)$. This can be seen as follows: For complete Boolean algebras ${}^\omega\omega$-boundingness is the same as weak (ω, ω)-distributivity. Let A be a proper complete weakly (ω, ω)-distributive

Boolean algebra and let $C \leq A$ be countable such that an enumeration of C is a name for a new real. Suppose A yields a σ-extension of $\mathfrak{P}(\omega)$. Then by (ii) of the preceding lemma, there is a dense set of $a \in A$ such that $(S(C)\cdot a)^+$ has a countable dense subset D_a. I may assume that D_a contains $C \cdot a$. Let $B_a := \langle D_a \rangle_{A \restriction a}$. Then D_a is dense in B_a and B_a is atomless. Let $\{b_n : n \in \omega\}$ be a maximal antichain in D_a. Since D_a is dense in B_a, this antichain is a maximal antichain in B_a. For each $n \in \omega$ pick $\{c_n^m : m \in \omega\} \subseteq C$ such that $b_n = \prod\{c_n^m : m \in \omega\}$. Now

$$b := a - \sum\{b_n : n \in \omega\} = a \cdot \prod_{n\in\omega}\sum_{m\in\omega} -c_n^m$$

$$= a \cdot \sum_{f:\omega\to\omega}\prod_{n\in\omega}\sum_{m<f(n)} -c_n^m.$$

Since C is a subalgebra of A, $\prod_{n\in\omega}\sum_{m<f(n)} -c_n^m \in S(C)$ for every $f : \omega \to \omega$. Thus b is zero or else there is some element of D_a below b, contradicting the maximality of the antichain. Therefore B_a is a regular subalgebra of $A \restriction a$. But this contradicts weak (ω,ω)-distributivity.

However, later I will prove a much more general result. But the argument above is still useful, as it leads to

3.1.5. Remark. Let A be a weakly (ω,ω)-distributive complete c.c.c. Boolean algebra and let C be a subalgebra of A which completely generates A. Then $S(C)$ is dense in A.

Proof. First assume that C is countable. Let B be the subalgebra of A that is generated by $S(C)$ (using only finite operations). Since C is a subalgebra of A, $S(C)$ is dense in B. Let K be a maximal antichain in B. By c.c.c., K is countable. By the same argument as above, it follows that K is already maximal in A. Thus B is a regular subalgebra of A. This means that B is dense in the complete subalgebra of A generated by B. But B completely generates A and thus B is dense in A. Therefore $S(C)$ is dense in A.

Now let C be arbitrary. Let $a \in A^+$. I have to show that there is $b \in S(C)$ such that $0 < b \leq a$. By c.c.c., there is a countable subalgebra C' of C such that a is contained in the complete subalgebra of A generated by C'. By the first part of the proof, there is $b \in S(C')$ such that $0 < b \leq a$. Clearly, $b \in S(C)$. □

In order to show that certain forcing notions yield non-σ-extensions of $\mathfrak{P}(\omega)$, it is usually sufficient to apply the following version of the preceding lemma:

3.1.6. Lemma. *Let M be a transitive model of a sufficiently large part of ZFC. Let $A \in M$ be a complete atomless Boolean algebra such that for each A-generic filter G over M every countable set of ordinals in $M[G]$ is included in a countable set of ordinals which is an element of M. Suppose A has a countable subset C such that $S(C)$ is dense in A. Assume that for no $a \in A$ the algebra $A \restriction a$ has a countable dense subset. (Note that this holds in particular if forcing extensions obtained using A cannot be obtained by just adding one Cohen real.) Then for every A-generic filter G over M,*

$$M[G] \models \mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega).$$

Proof. W.l.o.g. I may assume that C is a subalgebra of A. It is easy to show that (ii) of the lemma above does not hold for A:

Suppose it does. Let a be such that $(a \cdot S(C))^+$ has a countable subset B which is predense below every non-zero element of $a \cdot S(C)$. In particular, such a set B is dense in the set $(a \cdot S(C))^+$. Since $S(C)$ is dense in A, $a \cdot S(C)$ is dense below a. Hence B is dense below a. But now $A \restriction a$ has a countable dense subset. A contradiction. □

Many examples

In this section I show that many forcing notions meet the conditions in Lemma 3.1.6. It follows that they yield non-σ-extensions of $\mathfrak{P}(\omega)$. For most of these forcings it will turn out later that it is not necessary to apply Lemma 3.1.6 to show that they yield non-σ-extensions of $\mathfrak{P}(\omega)$. In the section 3.2 I will collect some purely combinatorial criteria for when an extension N of some model M yields a non-σ-extension of $\mathfrak{P}(\omega)$. These criteria work even if the extensions are not obtained by forcing. However, for forcing extensions, this section provides a more uniform approach and some of the results of this section do not follow from the combinatorial criteria mentioned above.

3.1.7. Definition. A forcing notion P *meets the conditions in Lemma 3.1.6* iff $\mathrm{ro}(P)$ has a countable subset C such that $S(C)$ is dense in $\mathrm{ro}(P)$ and for every P-generic filter G over the ground model M every in $M[G]$ countable set of ordinals is covered by some set in M which is countable in M and there is no Cohen real $x \in M[G]$ over M such that $M[G] = M[x]$. □

I need some additional forcing theoretic notions.

3.1.8. Definition. For every partial order $(P, \leq)$ and all $p, q \in P$ let $p \leq^* q$ iff there is no $r \leq p$ such that $r \perp q$. A subset D of P is **dense* iff D is dense in P with respect to $\leq^*$. Similarly, $D \subseteq P$ is **dense below* $p \in P$ iff D is dense below p in P with respect to $\leq^*$.

Clearly, $\leq^*$ extends $\leq$. Note that $p \leq^* q$ iff the image of p in $\mathrm{ro}(P)$ under the canonical mapping is smaller or equal to the image of q under this mapping. Using $\leq^*$, I can argue in P itself rather than in $\mathrm{ro}(P)$.

3.1.9. Lemma. *Let $(P, \leq)$ be a partial order and let $e : P \to \mathrm{ro}(P)$ be the canonical mapping. Let $C \subseteq P$. If*

$$\forall p, q \in P(\forall c \in C(p \leq c \Rightarrow q \leq c) \Rightarrow q \leq^* p),$$

then $S(e[C])$ is dense in $\mathrm{ro}(P)$. In particular, if $S(C)$ is dense in P, then $S(e[C])$ is dense in $\mathrm{ro}(P)$.

Proof. Easy, using the fact that $e[P]$ is dense in $\mathrm{ro}(P)$. □

In the following, I will sometimes use this lemma without referring to it.

I first consider the measure algebra of the Cantor space and Sacks forcing.

3.1.10. Definition. *Random forcing* is the measure algebra $\mathbb{R}(\omega)$ of the Cantor space ${}^{\omega}2$ which already has been introduced. *Sacks forcing* is the partial order $\mathbb{S}$ consisting of all perfect subsets of the unit interval ordered by inclusion. □

Note that the generic objects for these forcings can be coded by single reals. A Sacks real is the unique element of the intersection of all elements of an $\mathbb{S}$-generic filter. The real coding an $\mathbb{R}(\omega)$-generic filter is obtained in a similar way and is called a random real.

3.1.11. Lemma. *$\mathbb{R}(\omega)$ and $\mathbb{S}$ meet the conditions in Lemma 3.1.6.*

Proof. Random forcing and Sacks forcing are both proper and ${}^{\omega}\omega$-bounding. As mentioned above, for complete Boolean algebras the latter property is equivalent to weak (ω, ω)-distributivity, which is hereditary with respect to regular subalgebras and relative algebras. Cohen forcing does not share this property. Hence random forcing and Sacks forcing both do not add Cohen reals. Let $C_{\mathbb{R}(\omega)}$ be the subset of $\mathbb{R}(\omega)$ that consists of equivalence classes of clopen sets. Then $S(C_{\mathbb{R}(\omega)})$ consists of the equivalence classes of closed subsets of ${}^{\omega}2$. Since every subset of ${}^{\omega}2$ of positive measure includes a closed set of positive measure, the set $S(C_{\mathbb{R}(\omega)})$ is dense in $\mathbb{R}(\omega)$.

For Sacks forcing let $C_{\mathbb{S}}$ be the set of finite unions of infinite closed intervals with rational endpoints. Clearly, this set is countable. Also, it is easy to see that $S(C_{\mathbb{S}})$ is dense. Hence random forcing and Sacks forcing both meet the conditions in Lemma 3.1.6. □

Similarly, Lemma 3.1.6 applies to amoeba forcing.

3.1.12. Definition. *Amoeba forcing* is the partial order $\mathbb{A}$ consisting of all open subsets of ${}^{\omega}2$ of measure $< \frac{1}{2}$ ordered by reverse inclusion. □

Amoeba forcing is σ-linked and thus proper. Another notion of forcing is connected with amoeba forcing, localization forcing $\mathbb{LOC}$.

3.1.13. Definition. *Localization forcing* is the partial order $\mathbb{LOC}$ consisting of all $s \in {}^{\omega}([\omega]^{<\aleph_0})$ such that $\forall n \in \omega(|s(n)| \leq n)$ and $\exists k \in \omega \forall^{\infty} n(|s(n)| \leq k)$. The order is componentwise inclusion. □

$\mathbb{LOC}$ is also proper and $\mathbb{A}$ completely embeds into $\mathbb{LOC}$. Both forcings are treated in [1]. Again the respective generic filters can be coded by a single real.

3.1.14. Lemma. *$\mathbb{A}$ and $\mathbb{LOC}$ both meet the conditions in Lemma 3.1.6.*

Proof. Consider $\mathbb{A}$ first. It is easy to see that for the set $C_{\mathbb{A}}$ of clopen subsets of the Cantor space with measure $< \frac{1}{2}$ the set $S(C_{\mathbb{A}})$ is dense in $\mathbb{A}$. Thus it remains to show that for no $p \in \mathbb{A}$ there is a countable set *dense below p. Let $p \in \mathbb{A}$ and suppose $D = \{d_n : n \in \omega\}$ is a countable set of conditions in $\mathbb{A}$. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{2} - \mu(p)$. For each $n \in \omega$ pick an open set $p_n \subseteq {}^{\omega}2$ which is disjoint from d_n such that $\mu(p_n) < \frac{\varepsilon}{2^{n+17}}$. Now $q := p \cup \bigcup_{n \in \omega} p_n$ is a condition below p such that for no $n \in \omega$, $d_n \leq^* q$. Hence D is not *dense below p.

Now consider $\mathbb{LOC}$. Since $\mathbb{A}$ completely embeds into $\mathbb{LOC}$, a generic extension obtained by adding an $\mathbb{LOC}$-generic filter cannot be obtained by adding a Cohen real. The set $C_{\mathbb{LOC}}$ of sequences in $\mathbb{LOC}$ that are eventually constant with value $\emptyset$ is countable and $S(C_{\mathbb{LOC}})$ is easily seen to be dense in $\mathbb{LOC}$. □

Next I consider Hechler forcing and eventually different forcing.

3.1.15. Definition. *Hechler forcing* is the partial order $\mathbb{D}$ consisting of all conditions $p = (f_p, F_p)$ where f_p is a finite sequence of natural numbers and F_p is a finite set of (total) functions from ω to ω. The order is defined as follows: For all $p, q \in P$, $p \leq q$ iff $f_q \subseteq f_p$, $F_q \subseteq F_p$, and for all $n \in \mathrm{dom}(f_p \setminus f_q)$ and all $f \in F_q$, $f_p(n) \geq f(n)$.

Eventually different forcing is the the partial order $\mathbb{E}$ having the same conditions as $\mathbb{D}$ and the following order: For all $p, q \in \mathbb{E}$, $p \leq q$ iff $f_q \subseteq f_p$, $F_q \subseteq F_p$, and for all $n \in \mathrm{dom}(f_p \setminus f_q)$ and all $f \in F_q$, $f_p(n) \neq f(n)$. □

$\mathbb{D}$ is frequently called dominating forcing since it adds a function from ω to ω which dominates all the functions from the ground model. In order to avoid confusion, in the following by a dominating real I mean an element of ${}^{\omega}\omega$ that eventually dominates all functions from ω to ω in the ground model. The dominating real added by Hechler forcing is a Hechler real.

$\mathbb{E}$ adds a real which is eventually different from all functions from ω to ω in the ground model. The generic filter is coded by such a real. $\mathbb{E}$ behaves similarly to $\mathbb{D}$. Like $\mathbb{D}$, it is σ-centered and adds Cohen reals. But it does not add a dominating real. An elegant proof of the latter fact can be found in [1]. An element of ${}^\omega\omega$ that is eventually different from all functions from ω to ω in the ground model is an eventually different real. I will not use a special name for the eventually different real added by $\mathbb{E}$.

3.1.16. Lemma. *$\mathbb{D}$ and $\mathbb{E}$ both meet the conditions in Lemma 3.1.6.*

Proof. I have to struggle with the fact that neither $\mathbb{D}$ nor $\mathbb{E}$ is separative. This means that certain conditions will be identified when passing to the completion of the respective partial order. Call two conditions p and q equivalent if $p \leq^* q$ and $q \leq^* p$ and write $p \sim q$ in this case. With this definition, two conditions are equivalent iff they will be identified in the completion of the respective partial order. For $p \in \mathbb{D}$ or $p \in \mathbb{E}$ (well, actually the underlying sets of both partial orders are the same) such that $p = (f, F)$ write f_p for f and F_p for F.

Claim 1. Two conditions p and q are equivalent in $\mathbb{D}$ iff $f_p = f_q$ and

$$\forall n \in \omega \setminus \operatorname{dom} f_p(\max\{g(n) : g \in F_p\} = \max\{g(n) : g \in F_q\}).$$

Proof of Claim 1: Note that p and q are equivalent iff $\{r \in \mathbb{D} : r \perp p\} = \{r \in \mathbb{D} : r \perp q\}$. Now the claim follows from the fact that $r, s \in \mathbb{D}$ are compatible iff $f_r \subseteq f_s$ or $f_s \subseteq f_r$ and the condition with the larger first coordinate, say r, satisfies

$$\forall n \in \operatorname{dom}(f_r \setminus f_s) \forall g \in F_s(f_r(n) \geq g(n)).$$

Almost the same argument works for $\mathbb{E}$, only the $\geq$ in the last line has to be replaced by $\neq$. Thus the followings holds:

Claim 2. Two conditions p and q are equivalent in $\mathbb{E}$ if $f_p = f_q$ and

$$\forall n \in \omega \setminus \operatorname{dom} f_p(\{g(n) : g \in F_p\} = \{g(n) : g \in F_q\}).$$

For $P = \mathbb{D}$ or $P = \mathbb{E}$ consider the countable set C_P consisting of those conditions $p \in P$ for which $F_p = \{g\}$ for some g that is eventually constant with value 0.

Claim 3. For $P = \mathbb{D}, \mathbb{E}$ the set $S(e[C_P])$ is dense in $\mathrm{ro}(P)$, where $e : P \to \mathrm{ro}(P)$ is the canonical mapping.

Proof of Claim 3: First let $p \in \mathbb{D}$. For $m, n \in \omega$ define

$$g_p^n(m) := \begin{cases} \max\{g(m) : g \in F_p\} & \text{for } m < n, \\ 0 & \text{otherwise.} \end{cases}$$

Having the proof of Claim 1 in mind, it is not difficult to see that p is the greatest lower bound of the set $\{(f_p, \{g_p^n\}) : n \in \omega\}$ in $\mathrm{ro}(\mathbb{D})$.

Now let $p \in \mathbb{E}$. For $n \in \omega$ let

$$F_p^n := \{g \in {}^\omega\omega : \exists h \in F_p(h \restriction n = g \restriction n) \wedge \forall m \geq n(g(m) = 0)\}.$$

Let $z : \omega \to \omega$ be the function which is constant with value 0. Using the arguments from the proof of Claim 2, it is not difficult to see that $(f_p, F_p \cup \{z\}) \leq p$ is the greatest lower bound of the set $\{(f_p, F_p^n) : n \in \omega\}$ in $\mathrm{ro}(\mathbb{E})$, which proves Claim 3.

It remains to show that forcing extensions obtained using $\mathbb{D}$ or $\mathbb{E}$ cannot be obtained by adding one Cohen real. This is immediate for $\mathbb{D}$ since it is well known that Cohen forcing does not add a dominating real. For $\mathbb{E}$ I only have to show that adding a Cohen real does not add an eventually different real. Note that the fact that Cohen forcing does not add a dominating real follows from this since a dominating real is eventually different.

Let $(p_n)_{n \in \omega}$ be an enumeration of a countable dense subset of the Cohen algebra such that every condition is listed infinitely often. Let σ be a name for a function from ω to ω. For each $n \in \omega$ let $U_n := \{m \in \omega : p \Vdash \sigma(n) \neq m\}$. Pick a function $g : \omega \to \omega$ such that $g(n) \notin U_n$ for all n. Let G be a Cohen-generic filter. I show that σ_G is not eventually different from g. Suppose it is. Then there are $n \in \omega$ and $m \geq n$ such that $p_m \Vdash \forall k \geq n(\sigma(k) \neq g(k))$. But now $g(m) \in U_m$, a contradiction. □

I will quickly sketch how to prove similar results for some other notions of forcing.

3.1.17. Definition. The underlying set of *Miller forcing* $\mathbb{M}$ is the set of superperfect trees, i.e. subtrees of ${}^{<\omega}\omega$ in which beyond every node there is one with infinitely many immediate successors. The order is inclusion. $\square$

Miller forcing is proper. A Miller-generic filter can be coded by a single real. Note that Sacks forcing can also be considered as a partial order made up from trees, namely the set of perfect trees. That is, subtrees of ${}^{<\omega}\omega$ in which beyond every node there is one with at least two immediate successors.

3.1.18. Lemma. *Miller forcing meets the conditions in Lemma 3.1.6.*

Proof. For a tree $T \subseteq {}^{<\omega}\omega$ and $s \in T$ let $\mathrm{succ}_T(s) := \{n \in \omega : s^\frown(n) \in T\}$. Consider the set C of trees T in which up to some finite level only finite or cofinite sets occur as $\mathrm{succ}_T(s)$ and beyond that level only ω occurs as $\mathrm{succ}_T(s)$. Clearly, C is countable. It is easy to see that every superperfect tree is the greatest lower bound of some subset of C. For a given superperfect tree T and a countable set D of superperfect trees below T one can inductively thin out T in order to obtain a superperfect tree $T' \subseteq T$ such that no tree from D lies below T'. Thus Lemma 3.1.6 applies to Miller forcing as well. $\square$

3.1.19. Definition. *Grigorieff reals.* A filter $\mathcal{F}$ on ω is called a p-filter if for every countable set $\mathcal{G} \subseteq \mathcal{F}$ there is a set $a \in \mathcal{F}$ which is almost included in every element of $\mathcal{G}$. A filter $\mathcal{F}$ on ω is called unbounded if the set of monotone enumerations of elements of $\mathcal{F}$ is unbounded in ${}^{\omega}\omega$. Note that every ultrafilter is unbounded and that CH (as well as MA) implies that there are p-ultrafilters, i.e. p-points. For an unbounded p-filter $\mathcal{F}$ containing all cofinite sets let Grigorieff forcing $\mathbb{G}_{\mathcal{F}}$ be the set of partial functions f from ω to 2 such that $\omega \setminus \mathrm{dom}\, f \in \mathcal{F}$. The order is reverse inclusion. $\mathbb{G}_{\mathcal{F}}$ is proper and ${}^{\omega}\omega$-bounding.

Prikry-Silver reals. Priky-Silver forcing is the set of partial functions from ω to 2 with co-infinite domains ordered by reverse inclusion. Prikry-Silver forcing is proper.

Infinitely equal forcing. The conditions of infinitely equal forcing $\mathbb{EE}$ are partial functions p from ω to ${}^{<\omega}2$ such that for all $n \in \omega$ the sequence $p(n)$ is an element of ${}^{n}2$ and $\omega \setminus \mathrm{dom}(p)$ is infinite. The order is reverse inclusion. $\mathbb{EE}$ is proper and ${}^{\omega}\omega$-bounding. $\square$

3.1.20. Lemma. *Let P be either Grigorieff forcing, Prikry-Silver forcing, or $\mathbb{EE}$. Then P meets the conditions in Lemma 3.1.6.*

Proof. P is proper. For each condition $p \in P$ there is a condition $q \in P$ such that $p \subseteq q$ and $\mathrm{dom}(q \setminus p)$ is infinite. In fact, there is an uncountable antichain below p. In particular, there is no countable set dense below p. Thus it only remains to construct a suitable subset C_P of P in order to show that P meets the conditions in Lemma 3.1.6. Let C_P consist of all conditions in P that have finite domain. Clearly, C_P is countable. Clearly, every $p \in P$ is the greatest lower bound of some subset of C_P. □

Finally, Lemma 3.1.6 also applies to the countable support iteration of Cohen forcing of length ω.

3.1.21. Lemma. *Let $P := (\mathrm{Fn}(\omega, 2))^\omega$ be ordered componentwise. Then P meets the conditions in Lemma 3.1.6.*

Proof. Since Cohen forcing is absolute, P is equivalent to the countable support iteration of Cohen forcing of length ω. Thus P is proper. Let C_P be the set of all conditions with finite support. Clearly, C_P is countable. It is easily seen that $S(C_P)$ is dense in P. However, below every element of P there is an uncountable antichain. Thus a generic extension obtained by adding a P-generic filter cannot be obtained by adding a Cohen real. This proves the lemma. □

It follows that Lemma 3.1.6 applies to all notions of forcing mentioned so far, except for Cohen forcing, of course.

3.1.22. Corollary. *Random forcing, Sacks forcing, amoeba forcing, localization forcing, Hechler forcing, eventually different forcing, Miller forcing, Grigorieff forcing, Prikry-Silver forcing, infinitely equal forcing and the countable support iteration of Cohen forcing of length ω yield non-σ-extensions of $\mathfrak{P}(\omega)$.* □

It follows from this corollary together with Lemma 3.1.2 that $\mathfrak{P}(\omega)$ does not have the WFN in many popular models of set theory. I only mention one example.

3.1.23. Corollary. *Let M be a model of* ZFC* +CH. *In M let P be the measure algebra on ${}^{\omega_2}2$. Suppose G is P-generic over M. Then*

$$M[G] \models \neg\,\mathrm{WFN}(\mathfrak{P}(\omega)).$$

Proof. For $\alpha < \omega_2$ let P_α be the measure algebra on ${}^{\omega\cdot\alpha}2$. P_α can be considered as a complete subalgebra of P in a natural way. The sequence $(P_\alpha)_{\alpha<\omega_2}$ is continuous at limit ordinal of uncountable cofinality. For $\alpha < \omega_2$ and a $P_{\alpha+1}$-generic filter G, $M[G] = M[G \cap P_\alpha][r]$ where r is a random real over $M[G \cap P_\alpha]$. Thus it follows from the last corollary that

$$M[G] \models \mathfrak{P}(\omega) \cap M[G \cap P_\alpha] \not\leq_\sigma \mathfrak{P}(\omega).$$

Therefore Lemma 3.1.2 applies. Thus $\Vdash_P \neg\,\mathrm{WFN}(\mathfrak{P}(\omega))$. □

Note that if N is a proper forcing extension of the ground model M, then in N, $\mathbb{R}(\omega)^M$ still meets the conditions in Lemma 3.1.6. It follows that even forcing with a side-by-side product of $\aleph_2$ copies of random forcing over a model of CH gives a model of $\neg\,\mathrm{WFN}(\mathfrak{P}(\omega))$.

The latter model is especially interesting since Fuchino has recently observed that for every regular $\kappa > \aleph_1$ a combinatorial principle called $C^S(\kappa)$ holds in this model ([8]). This principle was introduced in [25] and implies among other things that there is no increasing chain with respect to $\subseteq^*$ in $\mathfrak{P}(\omega)$ of ordertype ω_2 and that there is no socalled $\aleph_2$-Luzin gap. It was shown in [16] respectively in [19] that under WFN($\mathfrak{P}(\omega)$), there is no increasing chain with respect to $\subseteq^*$ in $\mathfrak{P}(\omega)$ of ordertype ω_2 and there is no $\aleph_2$-Luzin gap. Fuchino and Soukup asked whether the latter two statements imply WFN($\mathfrak{P}(\omega)$). They do not.

It should also be noted that Corollary 3.1.23 can be obtained in a different way. After forcing with P over a model of CH, the covering number of the ideal of measure zero subsets of the Cantor space, $\mathrm{cov}(\mathcal{N})$, is $\aleph_2$. But WFN($\mathfrak{P}(\omega)$) implies that $\mathrm{cov}(\mathcal{N})$ is $\aleph_1$. This can be seen as follows. Assume WFN($\mathfrak{P}(\omega)$). Let M be some $V_{\aleph_1}$-like elementary submodel of H_χ for sufficiently large χ. M has size $\aleph_1$. If $\mathrm{cov}(\mathcal{N})$ is larger than $\aleph_1$, then the measure zero subsets of the Cantor space which are in M do not cover the whole space. By Solovay's characterization of random reals, there is a random real x over M. Now $M[x] \models \mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$. By $V_{\aleph_1}$-likeness of M, every countable subset of M is covered by a countable element of M. It follows that in the real world, $\mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$. But this contradicts WFN($\mathfrak{P}(\omega)$). However, Section 3.2 on cardinal invariants of the continuum contains a stronger result, due to Soukup. He showed that even non($\mathcal{M}$), the smallest cardinality of a non-meager subset of the Cantor space, is $\aleph_1$ under WFN($\mathfrak{P}(\omega)$). It is well-known that non($\mathcal{M}$) is larger or equal to cov($\mathcal{N}$).

Adding a Hechler real over ω_2 Cohen reals to a model of CH gives a model of $\neg$ WFN($\mathfrak{P}(\omega)$)

Soukup pointed out to me that forcing with Fn$(\omega, 2)$ preserves WFN($\mathfrak{P}(\omega)$), but cannot introduce it. Koppelberg and Shelah ([31]) constructed a complete subalgebra A of $\mathbb{C}(\aleph_2)$ which is not a Cohen algebra. Soukup also pointed out to me that forcing with A introduces an $\aleph_2$-Luzin gap. As I have mentioned earlier, he and Fuchino have shown in [19] that WFN($\mathfrak{P}(\omega)$) fails if there is an $\aleph_2$-Luzin gap. It follows that forcing with A over a model of CH gives a model of $\neg$ WFN($\mathfrak{P}(\omega)$), but WFN($\mathfrak{P}(\omega)$) can be introduced by some cardinal preserving notion of forcing, namely the quotient forcing $\mathbb{C}(\aleph_2) : A$. Of course, collapsing the continuum to $\aleph_1$ always introduces WFN($\mathfrak{P}(\omega)$). This is the reason why only cardinal preserving notions of forcing are interesting here.

It is clear from the results in the previous section that adding many reals can destroy WFN($\mathfrak{P}(\omega)$). In [37] Shelah proved that $0^\sharp$ is needed in order to destroy CH by adding a single real without collapsing $\aleph_1$. This means that typically, adding only one real to a model of CH preserves WFN($\mathfrak{P}(\omega)$), simply because it preserves CH. Soukup asked me whether adding only one real by some proper, c.c.c., or even σ-centered forcing can destroy WFN($\mathfrak{P}(\omega)$). It can. A Hechler real is sufficient. It would be nice to know some cardinal preserving generic reals, apart from Cohen reals, which preserve WFN($\mathfrak{P}(\omega)$). But I guess these are hard to find. (Provided they exist at all.) Note that adding a Hechler real gives a model where the popular cardinal invariants of the continuum have the same values as in a model with the same size of the continuum that is obtained by adding Cohen reals to a model of CH. This was shown by Brendle, Judah, and Shelah ([7]).

The strategy to show that adding a Hechler real over ω_2 Cohen reals gives a model in which $\mathrm{WFN}(\mathfrak{P}(\omega))$ fails is the following:

First decompose the forcing for first adding ω_2 Cohen reals and then adding a Hechler real into a chain of small forcings indexed by ω_2 instead of $\omega_2 + 1$. Now this iteration can be handled using the techniques developed in the last sections.

I will use an alternative definition of Hechler forcing now, which yields a partial order that is forcing equivalent to the Hechler forcing $\mathbb{D}$ defined before. The definition above was chosen in order to make the similarity between Hechler forcing and eventually different forcing apparent.

3.1.24. Definition. For two partial functions $f, g \subseteq \omega \times \omega$ let $f \leq g$ iff for every $n \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, $f(n) \leq g(n)$.

Let $\mathbb{D}' := \{(\sigma, f) : \sigma \in {}^{<\omega}\omega \wedge f \in {}^{\omega}\omega\}$. For $(\sigma, f), (\tau, g) \in \mathbb{D}'$ let $(\sigma, f) \leq (\tau, g)$ iff $\sigma \supseteq \tau$, $f \geq g$, and $\sigma \setminus \tau \geq g$. □

For a set $F \in [{}^{\omega}\omega]^{<\aleph_0}$ let $\max(F) : \omega \to \omega; n \mapsto \max\{f(n) : f \in F\}$. The mapping $\varphi : \mathbb{D} \to \mathbb{D}'; (\sigma, F) \mapsto (\sigma, \max(F))$ is easily seen to induce an isomorphism between $\mathrm{ro}(\mathbb{D})$ and $\mathrm{ro}(\mathbb{D}')$. This justifies calling $\mathbb{D}'$ Hechler forcing as well.

3.1.25. Definition. Let

$$\dot{\mathbb{D}} := \{(\sigma, \dot{f}) : \sigma \in {}^{<\omega}\omega \text{ and } \dot{f} \text{ is an } \mathrm{Fn}(\omega, 2)\text{-name for a function from } \omega \text{ to } \omega\}. \quad \square$$

$\dot{\mathbb{D}}$ can be regarded as an $\mathrm{Fn}(\omega, 2)$-name for Hechler forcing in a straightforward way.

3.1.26. Definition. Let $\mathbb{P} := \mathrm{Fn}(\omega, 2) * \dot{\mathbb{D}}$ and let

$$\mathbb{Q} := \{(1_{\mathrm{Fn}(\omega,2)}, (\sigma, \check{f})) : \sigma \in {}^{<\omega}\omega \wedge f \in {}^{\omega}\omega\} \subseteq \mathbb{P}. \quad \square$$

$\mathbb{Q}$ is equivalent to ordinary Hechler forcing. When analyzing the relation between $\mathbb{P}$ and $\mathbb{Q}$, it will be necessary to approximate functions in ${}^\omega\omega$ in a generic extension by ground model functions.

3.1.27. Definition. Let P be any notion of forcing. Suppose $\dot{f}$ is a P-name for an element of ${}^\omega\omega$ and $p \in P$. Then a function $g \in {}^\omega\omega$ is *possible* for $\dot{f}$ and p iff for all $n \in \omega$, $p \not\Vdash \dot{f} \restriction n \neq \check{g} \restriction n$. □

Note that for any name $\dot{f}$ for a function from ω to ω and any condition $q \in Q$ there is a possible function $g \in {}^\omega\omega$ for $\dot{f}$. Using this notion, one can show that $\mathbb{Q}$ behaves reasonably well with respect to $\mathbb{P}$.

3.1.28. Lemma. *$\mathbb{Q}$ is completely embedded into $\mathbb{P}$.*

Proof. According to Kunen's book ([32]), the following points have to be checked:

(i) $\forall p, q \in \mathbb{Q}(p \perp_{\mathbb{Q}} q \Leftrightarrow p \perp_{\mathbb{P}} q)$

(ii) $\forall p \in \mathbb{P} \exists q \in \mathbb{Q} \forall r \in \mathbb{Q}(r \leq q \Rightarrow r \not\perp_{\mathbb{P}} p)$

(i) is easily seen. Therefore in the following I will omit the subscripts on $\perp$. For (ii) let $p \in \mathbb{P}$, say $p = (s, (\sigma, \dot{f}))$ for $s \in \mathrm{Fn}(\omega, 2)$, $\sigma \in {}^{<\omega}\omega$, and an $\mathrm{Fn}(\omega, 2)$-name $\dot{f}$ for a function from ω to ω. Let g be a possible function for $\dot{f}$ and s. $q := (\emptyset, (\sigma, \check{g}))$ works for (ii):

Let $r \in \mathbb{Q}$ be such that $r \leq q$, say $r = (\emptyset, (\tau, \dot{h}))$. Let $v \leq s$ be a condition in $\mathrm{Fn}(\omega, 2)$ which forces that $\dot{f}$ and g are equal on $\mathrm{dom}(\tau)$. This is possible since g is possible for $\dot{f}$ and s. Let $\max(\dot{f}, \dot{h})$ be an $\mathrm{Fn}(\omega, 2)$-name for a function such that for all $n \in \omega$, $\Vdash \max(\dot{f}, \dot{h})(n) = \max(\dot{f}(n), \dot{h}(n))$. Since r extends q, for all $n \in \mathrm{dom}(\tau \setminus \sigma)$, $\tau(n) \geq g(n)$. Thus $(v, (\tau, \max(\dot{f}, \dot{h})))$ is a common extension of r and p. □

Since $\mathbb{Q}$ is completely embedded into $\mathbb{P}$, it makes sense to consider the quotient $\mathbb{P} : \mathbb{Q}$.

3.1.29. Definition. Let $\dot{H}$ be the canonical $\mathbb{Q}$-name for the $\mathbb{Q}$-generic filter and let $\mathbb{P} : \mathbb{Q}$ be a $\mathbb{Q}$-name for a subset of $\mathbb{P}$ s.t.

$$\Vdash_{\mathbb{Q}} \mathbb{P} : \mathbb{Q} = \{p \in \check{\mathbb{P}} : \forall q \in \dot{H}(p \text{ and } q \text{ are compatible})\}.$$

If H is a $\mathbb{Q}$-generic filter, let $\mathbb{P} : H := (\mathbb{P} : \mathbb{Q})_H$. □

It is well-known that forcing with $\mathbb{Q} * (\mathbb{P} : \mathbb{Q})$ is equivalent to forcing with $\mathbb{P}$. The proof of this fact really gives the following:

3.1.30. Lemma. *Let H be $\mathbb{Q}$-generic over the ground model M. If G is $\mathbb{P} : H$-generic over $M[H]$, then G as a subset of $\mathbb{P}$ is $\mathbb{P}$-generic over M and contains H.*

Proof. Let H be $\mathbb{Q}$-generic over M and let G be $\mathbb{P} : H$-generic over $M[H]$. For every $q \in H$, $q \in \mathbb{P} : H$ and q is compatible with every element of G. The $\mathbb{P} : H$-genericity of G implies $q \in G$. Thus $H \subseteq G$. It is clear that G is a filter.

Consider $A := \mathrm{ro}(\mathbb{P})$ and $B := \mathrm{ro}(\mathbb{Q})$. The complete embedding from $\mathbb{Q}$ into $\mathbb{P}$ induces a complete embedding from B into A. Thus I may think of B as a complete subalgebra of A. Clearly, forcing with B is equivalent to forcing with $\mathbb{Q}$. Let $H' \subseteq B$ be the B-generic filter induced by H and let $A : H := \{a \in A : \forall b \in H'(a \cdot b \neq 0)\}$. For each $a \in A$ let $\pi(a) := \prod(B \uparrow a)$. Note that for every $a \in A$, $a \in A : H$ iff $\pi(a) \in H'$. Let $f : \mathbb{P} \to A$ be the canonical mapping. Now $f[\mathbb{P} : H] \subseteq A : H$ and $f^{-1}[A : H] = \mathbb{P} : H$. Suppose $D \in M$ is a dense subset of $\mathbb{P}$.

Claim. $D \cap (\mathbb{P} : H)$ is dense in $\mathbb{P} : H$.

Let $p \in \mathbb{P} : H$. Then $\pi(f(p)) \in H'$. Let $b \in B$ be such that $b \leq \pi(f(p))$. By the definition of π, $b \cdot f(p) \neq 0$. Since $\mathbb{D}$ is dense in $\mathbb{P}$ and by the properties of f, $\{f(p') : p' \in D \wedge p' \leq p)\}$ is dense below $f(p)$ in A. Therefore there is $p' \leq p$ such that $f(p') \leq b$ and $p' \in D$. Clearly, $\pi(f(p')) \leq b$. It follows that the set $\{\pi(f(p')) : p' \in D \wedge p' \leq p\}$ is dense below $\pi(f(p))$ in B. Therefore there is $p' \in D$ such that $p' \leq p$ and $\pi(f(p')) \in H'$. Now $f(p') \in A : H$ and thus $p' \in \mathbb{P} : H$. This proves the claim.

Since G is $\mathbb{P} : H$ generic, the claim implies that G intersects D. Since D was arbitrary, it follows that G is $\mathbb{P}$-generic over M. □

$\mathrm{ro}(\mathbb{P} : H)$ is generated by a name for a real in a nice way.

3.1.31. Lemma. *If H is $\mathbb{Q}$-generic, then in $M[H]$, $\mathrm{ro}(\mathbb{P} : H)$ has a countable subset C such that $S(C)$ is dense in $\mathrm{ro}(\mathbb{P} : H)$.*

Proof. For a function $f : \omega \to \omega$ and a condition $p \in \mathrm{Fn}(\omega, 2)$ let $\dot{f}_p$ be an $\mathrm{Fn}(\omega, 2)$-name for a function such that $p \Vdash (\dot{f}_p = \check{f})$ and for each $q \in \mathrm{Fn}(\omega, 2)$ such that $q \perp p$, $q \Vdash_{\mathrm{Fn}(\omega,2)} (\dot{f}_p$ is constant with value 0). Let

$$C' := \{(p, (\sigma, \dot{f}_q)) \in \mathbb{P} : H : p, q \in \mathrm{Fn}(\omega, 2),\ \sigma \in {}^{<\omega}\omega \text{ and } f \in {}^{\omega}\omega \text{ is eventually constant}\}.$$

Clearly, C' is countable. The image C of C' under the embedding of $\mathbb{P} : H$ into $\mathrm{ro}(\mathbb{P} : H)$ works for the lemma:

Let $(q, (\tau, \dot{g})) \in \mathbb{P} : H$. Let $(r, (\rho, \dot{h})) \in \mathbb{P} : H$ be such that for all $c \in C'$ with $(q, (\tau, \dot{g})) \leq c$, $(r, (\rho, \dot{h})) \leq c$ holds. Now $r \leq q$ and $\rho \supseteq \tau$. I will be done if I can show

$$r \Vdash_{\mathrm{Fn}(\omega,2)} \forall n \in \omega(\dot{h}(n) \geq \dot{g}(n)) \wedge \forall n \in \mathrm{dom}(\rho \setminus \tau)(\rho(n) \geq \dot{g}(n)).$$

But here it is sufficient to prove

$$(*) \quad \forall n \in \omega(r \Vdash_{\mathrm{Fn}(\omega,2)} \dot{h}(n) \geq \dot{g}(n))$$

and

$$(**) \quad \forall n \in \mathrm{dom}(\rho \setminus \tau)(r \Vdash_{\mathrm{Fn}(\omega,2)} \rho(n) \geq \dot{g}(n)).$$

Let $n \in \omega$ and let $s \in \mathrm{Fn}(\omega, 2)$ be such that $s \leq r$ and $s \Vdash (\dot{g}(n) = m)$ for some $m \in \omega$. Let $f : \omega \to \omega$ be the function that has the value m at the place n and is 0 everywhere else. Then $(q, (\tau, \dot{g})) \leq (q, (\tau, \dot{f}_s))$ and $(q, (\tau, \dot{f}_s)) \in C$. Thus $r \Vdash_{\mathrm{Fn}(\omega,2)} (\dot{h}(n) \geq \dot{g}(n))$ since $(r, (\rho, \dot{h})) \leq (q, (\tau, \dot{f}_s))$ for a set of s's dense below r. This shows $(*)$. The proof of $(**)$ is practically the same. □

Next I am going to show that for no $q \in \mathrm{ro}(\mathbb{P} : H)$ there is a countable subset of $\mathrm{ro}(\mathbb{P} : H)$ that is dense below q. This needs some combinatorial preparation.

3.1.32. Lemma. *Let $n \in \omega$. Then ω^n ordered componentwise is wellfounded and every set $A \subseteq \omega^n$ consisting of pairwise incomparable elements is finite. In particular, every subset of ω^n has only finitely many minimal elements.*

Proof. For wellfoundedness let $S \subseteq \omega^n$. For each $b \in \omega^n$ and every $i < n$ let b_i be the i-th coordinate of b. Inductively for $i < n$ pick $a_i \in \omega$ minimal with the property $\exists b \in S \forall j \leq i(b_j = a_j)$. $(a_i)_{i<n}$ is minimal in S.

Now let $A \subseteq \omega^n$ consist of pairwise incomparable elements and assume for contradiction that A is infinite. Let $(a^k)_{k\in\omega}$ be an one-one-enumeration of A. Thinning out this sequence in n steps using the wellfoundedness of ω, one can find an infinite subset S of ω such that for each $i \in n$ the sequence $(a_i^k)_{k\in S}$ is strictly increasing or constant. Since the enumeration of A was chosen to be one-one, $\{a^k : k \in S\}$ is an infinite linearly ordered subset of A. A contradiction. □

Note that this proof works for any other wellordered set instead of ω as well. However, I am not going to use this.

Let me collect some additional facts on $\mathbb{P} : H$.

3.1.33. Lemma. *a) Let H be $\mathbb{Q}$-generic. The* dominating real added by H *is $d := \bigcup\{\tau : \exists g((\emptyset, (\tau, \check{g})) \in H)\}$. For all $(p, (\sigma, \dot{f})) \in \mathbb{P}$, $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$ iff $\sigma \subseteq d$ and for no $n \in \omega$, $p \Vdash \dot{f} \not\leq d \restriction n \setminus \mathrm{dom}(\sigma)$.*

b) Let $(\emptyset, (\tau, \check{g})) \in \mathbb{Q}$ and $(p, (\sigma, \dot{f})) \in \mathbb{P}$. If $\sigma \subseteq \tau$ and

$$p \not\Vdash \dot{f} \restriction \omega \setminus \mathrm{dom}(\sigma) \not\leq \tau \vee \dot{f} \restriction \omega \setminus \mathrm{dom}(\tau) \not\leq \check{g},$$

then $(\emptyset, (\tau, \check{g})) \Vdash_{\mathbb{Q}} (p, (\sigma, \dot{f})) \in \mathbb{P} : \mathbb{Q}$.

If $(\emptyset, (\tau, \check{g})) \Vdash_{\mathbb{Q}} (p, (\sigma, \dot{f})) \in \mathbb{P} : \mathbb{Q}$, then $\sigma \subseteq \tau$.

c) Let H be $\mathbb{Q}$-generic and $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$. Let $\dot{g}$ be an $\mathrm{Fn}(\omega, 2)$-name in the ground model for an element of ${}^\omega\omega$. Then there is $(q, (\tau, \dot{h})) \in \mathbb{P} : H$ such that $(q, (\tau, \dot{h})) \leq (p, (\sigma, \dot{f}))$ and $\Vdash \dot{g} \leq \dot{h}$.

Proof. For a) let $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$. Suppose $\sigma \not\subseteq d$. Then there is $(\emptyset, (\tau, \check{g})) \in H$ such that $\tau \cup \sigma$ is not a function. Clearly, $(\emptyset, (\tau, \check{g}))$ and $(p, (\sigma, \dot{f}))$ are incompatible in $\mathbb{P}$. A contradiction. Thus $\sigma \subseteq d$. Now suppose for some $n \in \omega \setminus \mathrm{dom}(\sigma)$, $p \Vdash \dot{f} \not\leq d \restriction n \setminus \mathrm{dom}(\sigma)$. Let G be $\mathbb{P}$ generic over the ground model such that $H \subseteq G$ and $(p, (\sigma, \dot{f})) \in G$. G exists by Lemma 3.1.30. By genericity of G, there is $q \in \mathrm{Fn}(\omega, 2)$ such that $q \leq p$, $(q, (\sigma, \dot{f})) \in G$, and for some $m \in n \setminus \mathrm{dom}(\sigma)$, $q \Vdash \dot{f}(m) > d(m)$. There is $g : \omega \to \omega$ in the ground model such that $(\emptyset, (d \restriction n, \check{g})) \in H$. Now $q \Vdash (\sigma, \dot{f}) \perp_{\dot{\mathbb{D}}} (d \restriction n, \check{g})$ and thus $(q, (\sigma, \dot{f})) \perp_{\mathbb{P}} (\emptyset, (d \restriction n, \check{g}))$. But this is impossible since $(\emptyset, (d \restriction n, \check{g})), (q, (\sigma, \dot{f})) \in G$.

For the other direction of a) let $(p, (\sigma, \dot{f})) \in \mathbb{P}$ be as in the right-hand-side of the statement. Let $(\emptyset, (\tau, \check{g})) \in H$. There is $q \in \mathrm{Fn}(\omega, 2)$ such that $q \leq p$ and $q \Vdash \dot{f} \restriction \mathrm{dom}(\tau \setminus \sigma) \leq \tau$. Let $\max(\dot{f}, \check{g})$ be an $\mathrm{Fn}(\omega, 2)$-name for an element of ${}^{\omega}\omega$ such that for all $n \in \omega$, $\Vdash \max(\dot{f}, \check{g})(n) = \max(\dot{f}(n), \check{g}(n))$. Since $\tau \subseteq d$, $\sigma \cup \tau$ is a function. Now $(q, (\tau \cup \sigma, \max(\dot{f}, \check{g})))$ is a common extension of $(p, (\sigma, \dot{f}))$ and $(\emptyset, (\tau, \check{g}))$. It follows that $(p, (\sigma, \dot{f}))$ is compatible with all elements of H and therefore $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$.

For b) note that $(\emptyset, (\tau, \check{g}))$ forces that the dominating real added by the $\mathbb{Q}$-generic filter starts with τ and is larger or equal to g on $\omega \setminus \mathrm{dom}(\tau)$. Together with a) this implies the first part of b). The second part is straightforward and uses arguments already given above.

Finally let H, $(p, (\sigma, \dot{f}))$, and $\dot{g}$ be as in c). Let G be a $\mathbb{P}$-generic filter extending H that contains $(p, (\sigma, \dot{f}))$. The set of conditions $(q, (\tau, \dot{h})) \leq (p, (\sigma, \dot{f}))$ such that $\Vdash \dot{g} \leq \dot{h}$ is dense below $(p, (\sigma, \dot{f}))$. Thus G contains such a condition. This condition is compatible with all elements of H and therefore lies in $\mathbb{P} : H$. □

By Lemma 3.1.31, a $\mathbb{P} : H$-generic filter can be coded by a single real. But it cannot be coded by a Cohen real.

3.1.34. Lemma. *Let H be $\mathbb{Q}$-generic. For no $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$ there is a countable subset of $\mathbb{P} : H$ which is *dense below it. In particular, for no $a \in \mathrm{ro}(\mathbb{P} : H)^+$, $\mathrm{ro}(\mathbb{P} : H) \restriction a$ has a countable dense subset.*

Proof. Assume on the contrary that $(p_n, (\sigma_n, \dot{f}_n))_{n \in \omega}$ enumerates a subset of $\mathbb{P}$ which contains a subset of $\mathbb{P} : H$ that is dense below $(p, (\sigma, \dot{f}))$. Since the formulation here is carefully chosen and $\mathbb{P}$ has c.c.c., I may assume that $(p_n, (\sigma_n, \dot{f}_n))_{n \in \omega}$ is an element of the ground model. By part c) of Lemma 3.1.33, I may also assume that each $\dot{f}_n$ is a name for a new function. Let

$(\emptyset, (\tau, \check{g}))$ be a condition in H such that

$$(\emptyset, (\tau, \check{g})) \Vdash_{\mathbb{Q}} (p, (\sigma, \dot{f})) \in \mathbb{P} : \mathbb{Q} \text{ and } \{(p_n, (\sigma_n, \dot{f}_n)) : n \in \omega\}$$
$$\text{contains a set *dense below } (p, (\sigma, \dot{f})) \text{ in } \mathbb{P} : \mathbb{Q}.$$

By part b) of Lemma 3.1.33, $\tau \supseteq \sigma$. By Lemma 3.1.32, for each $n \in \omega$ there are only finitely many minimal restrictions of possible functions for $\dot{f}_n$ and p_n to $X := (n \cup \operatorname{dom}(\tau)) \setminus \operatorname{dom}(\sigma_n)$. For each such restriction fix an element of $\operatorname{Fn}(\omega, 2)$ below p_n deciding $\dot{f}_n$ on X accordingly and let A_n be the set of the chosen conditions. A_n is a finite antichain in $\operatorname{Fn}(\omega, 2)$. For each $a \in A_n$ let $p^a_{n,0} := a$ and let f^a_n be a possible function for $\dot{f}_n$ and a. Suppose $p^a_{n,m}$ has been constructed for some $m \in \omega$. Let $q^a_{n,m}, p^a_{n,m+1} \leq p^a_{n,m}$ be such that f^a_n is possible for $\dot{f}_n$ and $p^a_{n,m+1}$ and such that $q^a_{n,m}$ and $p^a_{n,m+1}$ decide a larger initial segment of $\dot{f}_n$ than $p^a_{n,m}$ does, but the way $p^a_{n,m+1}$ decides an initial segment of $\dot{f}_n$ is inconsistent with the way in which $q^a_{n,m}$ decides an initial segment of $\dot{f}_n$. This can be done since $\dot{f}_n$ is a name for a new function. Let $f^a_{n,m}$ be possible for $\dot{f}_n$ and $q^a_{n,m}$. Now $(q^a_{n,m})_{m\in\omega}$ is an antichain below a for each $a \in A_n$.

Note that for all $m \in \omega$ and all $k \geq m$, $f^a_{n,k} \restriction m + 1 = f^a_n \restriction m + 1$. Let $h \in {}^{\omega}\omega$ be defined as follows:

$$\forall k \in \omega (h(k) := \max(\{f^a_{n,m}(k) : n, m \leq k \wedge a \in A_n\} \cup \{g(k)\}))$$

Clearly, $(\emptyset, (\tau, \check{h})) \leq (\emptyset, (\tau, \check{g}))$.

Claim. $(\emptyset, (\tau, \check{h})) \nVdash_{\mathbb{Q}} \{(p_n, (\sigma, \dot{f}_n)) : n \in \omega\}$ contains a subset of $\mathbb{P} : \mathbb{Q}$ which is *dense below $(p, (\sigma, \dot{f}))$.

Proof of the claim: Pick $q \leq p$ such that $q \Vdash \dot{f} \restriction \operatorname{dom}(\tau \setminus \sigma) \leq \tau$. This is possible since $(\emptyset, (\tau, \check{g}))$ and $(p, (\sigma, \dot{f}))$ are compatible. Fix a partition $(X^a_n)_{n\in\omega}$ of ω into infinite pieces. Define a $\operatorname{Fn}(\omega, 2)$-name $\dot{e}$ for a function from ω to ω in such a way that

$$\forall n, m \in \omega \forall a \in A_n \forall k \in X^a_n (q^a_{n,m} \Vdash_{\operatorname{Fn}(\omega,2)} \dot{e}(k) \geq m).$$

By the choice of q, $(q,(\tau,\dot{e})) \leq (p,(\sigma,\dot{f}))$. $(q,(\tau,\dot{e}))$ and $(\emptyset,(\tau,\dot{h}))$ are compatible, so let G be a $\mathbb{P}$-generic generic filter containing both conditions. Let $H' := G \cap \mathbb{Q}$. Let $n \in \omega$ be such that the dominating real d added by H' extends σ_n and $p_n \leq q$. For all n which do not satisfy these conditions, $(p_n,(\sigma_n,\dot{f}_n)) \notin \mathbb{P} : H'$ or $(p_n,(\sigma_n,\dot{f}_n)) \not\leq^* (q,(\tau,\dot{e}))$. Let $a \in A_n$ be such that a forces $\dot{f}_n$ to be below τ on $n \setminus \operatorname{dom}(\sigma_n)$. This is possible if $(p_n,(\sigma_n,\dot{f}_n)) \in \mathbb{P} : H'$. Again, if the latter does not hold, this n is not interesting.

Subclaim. For all $m \in \omega$, $(q^a_{n,m},(\sigma_n,\dot{f}_n)) \in \mathbb{P} : H'$.

Proof of the subclaim: Let $m \in \omega$. For $k \geq n$, $h(k) \geq f^a_{n,m}(k)$ since $f^a_{n,m}(k) = f^a_n(k)$ for $m \geq k$. By construction, $f^a_{n,m}$ is possible for $\dot{f}_n$ and $q^a_{n,m}$. By choice of a, $f^a_{n,m} \restriction (\operatorname{dom}(\tau) \cup n) \setminus \operatorname{dom}(\sigma_n) \leq d$. By part b) of Lemma 3.1.33,

$$(\emptyset,(\tau,\check{h})) \Vdash_{\mathbb{Q}} (q^a_{n,m},(\sigma_n,\dot{f}_n)) \in \mathbb{P} : \mathbb{Q}.$$

Now the subclaim follows from $(\emptyset,(\tau,\check{h})) \in \mathbb{P} : H'$.

Pick $k \in X^a_n \setminus \operatorname{dom}(\tau)$. Then $(q^a_{n,d(k)+1},(\sigma_n,\dot{f}_n)) \leq (p_n,(\sigma_n,\dot{f}_n))$, but $(q^a_{n,d(k)+1},(\tau,\dot{e})) \notin \mathbb{P} : H'$ since $q^a_{n,d(k)+1} \Vdash \dot{e} \restriction \omega \setminus \operatorname{dom}(\tau) \not\leq d$. Thus $(q^a_{n,d(k)+1},(\sigma_n,\dot{f}_n)) \perp_{\mathbb{P}:H'} (q,(\tau,\dot{e}))$ and therefore $(p_n,(\sigma_n,\dot{f}_n)) \not\leq^* (q,(\tau,\dot{e}))$. This proves the claim and the claim contradicts the choice of $(\emptyset,(\tau,\check{g}))$. □

Using Lemma 3.1.6 and Lemma 3.1.2, it is now easy to prove

3.1.35. Theorem. *Adding a Hechler real over ω_2 Cohen reals to a model of CH gives a model of* $\neg\,$WFN($\mathfrak{P}(\omega)$).

Proof. For $X \subseteq \omega_2$ let $\mathbb{P}_X := \operatorname{Fn}(X,2) * \dot{\mathbb{D}}_X$, where

$$\dot{\mathbb{D}}_X := \{(\sigma,\dot{f}) : \sigma \in {}^{\omega}\omega \text{ and } \dot{f} \text{ is an } \operatorname{Fn}(X,2)\text{-name for an element of } {}^{\omega}\omega\}$$

is considered as an $\operatorname{Fn}(X,2)$-name for Hechler forcing. By the same argument as in Lemma 3.1.28, for $X \subseteq Y \subseteq \omega_2$, $\mathbb{P}_X$ is completely embedded into $\mathbb{P}_Y$. The sequence $(\mathbb{P}_{\omega\cdot\alpha})_{\alpha\leq\omega}$ is increasing and continuous at limits of uncountable cofinality. Each $\mathbb{P}_{\omega\cdot\alpha}$ is of size $\leq \aleph_1$ and satisfies c.c.c. Applying Lemma 3.1.6 together with Lemma 3.1.31 and Lemma 3.1.34, it follows that for each

$\alpha < \omega_2$,

$$\Vdash_{P_{\omega\cdot(\alpha+1)}} \mathfrak{P}(\omega) \cap M[\dot{G} \cap \check{P}_\alpha] \not\leq_\sigma \mathfrak{P}(\omega).$$

Now the theorem follows from Lemma 3.1.2. □

A characterization of Cohen forcing

In this section I consider σ-extensions of $\mathfrak{P}(2^{\aleph_0})$ since this will give a characterization of Cohen forcing.

3.1.36. Theorem. *Let M be a transitive model of* ZFC* *and let A be an atomless complete c.c.c. Boolean algebra in M. Then the following are equivalent:*

(i) For any A-generic filter G over M,

$$M[G] \models (\mathfrak{P}(2^{\aleph_0}))^M \leq_\sigma \mathfrak{P}((2^{\aleph_0})^M).$$

(ii) A is isomorphic to $\mathbb{C}(\omega)$. □

For the proof of this theorem it is convenient to introduce the cardinal invariant τ of complete Boolean algebras.

3.1.37. Definition. For a complete Boolean algebra A let $\tau(A)$ be the least cardinal λ such that A is completely generated by a subset of size λ. □

The first approximation of the theorem is

3.1.38. Lemma. *Let M be a transitive model of* ZFC* *and let A be an atomless complete c.c.c. Boolean algebra in M. Then the following statements are equivalent:*

(i) For any A-generic filter G over M,

$$M[G] \models (\mathfrak{P}(2^{\aleph_0}))^M \leq_\sigma \mathfrak{P}((2^{\aleph_0})^M).$$

(ii) For any A-generic filter G over M and any $x \in M[G] \setminus M$ such that $x \subseteq (2^{\aleph_0})^M$ there is a Cohen real r over M such that $x \in M[r]$.

(iii) Any complete subalgebra B of A with $\tau(B) \leq 2^{\aleph_0}$ has a countable dense subset.

Proof. (i) implies (iii): First note that any subalgebra of A which is completely generated by a set X of size at most $2^{\aleph_0}$ has size at most $2^{\aleph_0}$. This is because the closure of X under countable operations has size at most $2^{\aleph_0}$ and is already complete since A satisfies c.c.c.

Claim. Let B be a complete subalgebra of A with $\tau(B) \leq 2^{\aleph_0}$. Then B has a dense subset of elements a such that $B \restriction a$ has a countable dense subset.

First I show how (iii) follows from the claim: Take a maximal antichain K consisting of elements $a \in B$ such that $B \restriction a$ has a countable dense subset. K is countable since B satisfies c.c.c. For each $a \in K$ let D_a be a countable dense subset of $B \restriction a$. Now $\bigcup\{D_a : a \in K\}$ is a countable dense subset of B.

Proof of the claim: I argue like in the proof of Lemma 3.1.6. Let $\sigma : 2^{\aleph_0} \to B$ be onto. Consider σ as a name for a subset of $2^{\aleph_0}$. Let G be B-generic over M. Since $I := \{x \in (\mathfrak{P}(2^{\aleph_0}))^M : x \subseteq \sigma_G\}$ is countably generated and B satisfies c.c.c., there is a countable set $C \in M$ such that $C \cap I$ is cofinal in I. It is forced by some $a \in G$ that C has this property. For each $x \in C \cap I$ there is some $b_x \in B$ such that $b_x \leq a$ and $b_x \Vdash x \subseteq \sigma$. W.l.o.g. I may assume that for each $x \in C$ there exists $b_x \leq a$ such that $b_x \Vdash x \subseteq \sigma$. I may also assume that $b_x = a \cdot \prod\{\sigma(\alpha) : \alpha \in x\}$ and $x = \{\alpha \in 2^{\aleph_0} : \sigma(\alpha) \geq b_x\}$ hold for all $x \in C$. Now suppose that $\{b_x : x \in C\}$ is not dense below a. Then there is $b \leq a$ such that no element of $\{b_x : x \in C\}$ lies below b. Now $b \Vdash \sigma^{-1}(b) \subseteq \sigma$, but no $x \in C$ includes $\sigma^{-1}(b)$. This contradicts the fact that a forces that $C \cap I$ is cofinal in I. Hence $B \restriction a$ has a countable dense subset.

It follows that the set D of $a \in B$ such that $B \restriction a$ has countable dense subset is predense in B. But since every relative algebra of a Boolean algebra with a countable dense subset has a countable dense subset as well, D is even dense in B.

(iii) implies (ii): Let G be A-generic over M an let $x \in M[G] \setminus M$ such that $x \subseteq (2^{\aleph_0})^M$. Let σ be a name for X. By c.c.c., I may assume that σ uses only $2^{\aleph_0}$ conditions. Let B be the complete subalgebra of A that is completely generated by the conditions used by σ. Let $a \in B$ be the complement of the sum of all atoms in B. Since x is a new subset of $(2^{\aleph_0})^M$, G does not contain an atom of B. Thus B is not atomic and therefore $a \neq 0$. By (iii), the algebra B has a countable dense subset. Hence $B \restriction a$ has a countable dense subset. Since $B \restriction a$ is atomless, $G \cap B \restriction a$ is a Cohen-generic filter which can be coded by a Cohen real $r \in M[G]$. Clearly, one can recover $G \cap B$ from $G \cap B \restriction a$. Thus $x \in M[G \cap B] = M[r]$.

(ii) implies (i): Let G be A-generic over M and let $x \in M[G] \setminus M$ such that $x \subseteq (2^{\aleph_0})^M$. Pick a Cohen real $r \in M[G]$ over M such that $x \in M[r]$. By the same argument as for $\mathfrak{P}(\omega)$ in the proof of Lemma 3.1.4 or in [16], one can see that $I := (\mathfrak{P}(2^{\aleph_0}))^M \restriction x$ is countably generated in $M[r]$. Hence I is countably generated in $M[G]$. □

Koppelberg noticed that statement (iii) in Lemma 3.1.38 already characterizes $\mathbb{C}(\aleph_0)$. I give a slight generalization of her argument.

3.1.39. Definition. For a complete Boolean algebra A let

$$\sigma_\tau(A) := \{\tau(B) : B \text{ is a complete subalgebra of } A\}$$

be the *τ-spectrum* of A. □

3.1.40. Lemma. *Let A be a complete Boolean algebra and let κ be an uncountable regular cardinal such that A satisfies the κ-c.c. Suppose there is $\lambda \in \sigma_\tau(A)$ such that $\kappa \leq \lambda$. Then $\kappa \in \sigma_\tau(A)$.* □

The proof of this lemma uses

3.1.41. Lemma. *(Vladimirov, see [30].) Let A be complete and B a complete subalgebra of A. Assume that for no $b \in B^+$, $B \cap A \restriction b$ is dense in $A \restriction b$. Then there is $a \in A$ such that a is independent over B, i.e. for all $b \in B^+$, $a \cdot b \neq 0$ and $b - a \neq 0$.* □

Proof of Lemma 3.1.40. By passing from A to a complete subalgebra of A if necessary, I may assume $\lambda := \tau(A) \geq \kappa$. Note that τ is monotone in the sense that $\tau(A \restriction a) \leq \tau(A)$ for every $a \in A^+$. Call $a \in A^+$ *τ-homogeneous* iff for all $b \in (A \restriction a)^+$, $\tau(A \restriction b) = \tau(A \restriction a)$. Since the cardinals are wellfounded, the set of τ-homogeneous elements of A is dense in A. Let C be a maximal antichain in A consisting of τ-homogeneous elements. By κ-c.c., $|C| < \kappa$. By $\kappa \leq \lambda$ and since κ is regular, there is $a \in C$ such that $\tau(A \restriction a) \geq \kappa$. Define a chain $(B_\alpha)_{\alpha<\kappa}$ of complete subalgebras of $A \restriction a$ as follows:

Let $B_0 := \{0, a\}$. Let $\alpha < \kappa$ and assume for all $\beta < \alpha$, B_β has been defined such that $\tau(B_\beta) < \kappa$. Let B'_α be the complete subalgebra of $A \restriction a$ generated by $\bigcup_{\beta<\alpha} B_\beta$. Since κ is regular, $\tau(B'_\alpha) < \kappa$. Now for all $b \in (B'_\alpha)^+$, $B'_\alpha \restriction b$ is not dense in $A \restriction b$ since $\tau(A \restriction b) \geq \kappa$ while $\tau(B'_\alpha \restriction b) \leq \tau(B'_\alpha) < \kappa$. Therefore Vladimirov's Lemma applies. Let $a_\alpha \in A \restriction a$ be such that a_α is independent over B'_α in $A \restriction a$. Let B_α be the complete subalgebra of $A \restriction a$ generated by B'_α and a_α.

Let $B := \bigcup_{\alpha<\kappa} B_\alpha$. By κ-c.c., B is a complete subalgebra of $A \restriction a$. Since $\tau(B_\alpha) < \kappa$ for every $\alpha < \kappa$, $\tau(B) \leq \kappa$. Since every subset of B of size less than κ is included in some B_α and $B_\alpha \neq B$ for all $\alpha < \kappa$, $\tau(B) = \kappa$. Let B' be the complete subalgebra of A generated by B. Now $\tau(B') = \kappa$ and thus $\kappa \in \sigma_\tau(A)$. □

Using Lemma 3.1.40, it is now easy to finish the

Proof of Theorem 3.1.36. (ii)⇒(i) follows immediately from (iii)⇒(i) in Lemma 3.1.38.

For (i)⇒(ii) it is sufficient to show that (iii) of Lemma 3.1.38 already characterizes $\mathbb{C}(\omega)$. Let A be a complete c.c.c. Boolean algebra as in (iii) of Lemma 3.1.38. Suppose $\tau(A) > \aleph_0$. Then by c.c.c. and Lemma 3.1.40, A has a complete subalgebra B with $\tau(B) = \aleph_1$. By the properties of A, B has a countable dense subset and therefore $\tau(B) = \aleph_0$. A contradiction. Hence $\tau(A) = \aleph_0$. Again by the properties of A, A itself has a countable dense subset and thus $A \cong \mathbb{C}(\omega)$. □

3.2 WFN($\mathfrak{P}(\omega)$) and cardinal invariants of the continuum

The reason for studying the question whether certain forcing extensions yield σ-extension or not is to provide an easy way to recognize those models of ZFC in which WFN($\mathfrak{P}(\omega)$) fails. But this only works well for models which have been obtained by adding reals to some model and thereby enlarging the continuum. Another approach is to determine the values of cardinal invariants of the continuum under the assumption WFN($\mathfrak{P}(\omega)$). The arguments here often can be phrased in terms of σ-embeddedness of $\mathfrak{P}(\omega) \cap M$ in $\mathfrak{P}(\omega)$ for some model M of ZFC*. Here M will be either an elementary submodel of H_χ for sufficiently large χ or a transitive class. In order to spare notation, I take the following definition:

3.2.1. Definition. A pair (M, N) is *convenient* iff one of the following holds:

(i) N and M are transitive classes satisfying ZFC* such that $M \subseteq N$, M is a definable class in N, M and N have the same ordinals, and every in N countable set of ordinals is covered by a set in M which is countable in M.

(ii) N is a (possibly class-) model of ZFC* and M is an elementary submodel of H_χ^N for some sufficiently large χ such that $M \cap [M]^{\aleph_0}$ is cofinal in $[M]^{\aleph_0}$. □

Cichoń's diagram: The small cardinals

The first explicit result on the effect of WFN($\mathfrak{P}(\omega)$) on cardinal invariants of the continuum was the result of Fuchino, Koppelberg, and Shelah ([16]) that the unboundedness number $\mathfrak{b}$ is $\aleph_1$ under WFN($\mathfrak{P}(\omega)$). This can also be proved in the following way: Using the argument in the proof of Lemma 1.4.11, it is not difficult to see that if (M, N) is a convenient pair and $({}^\omega\omega)^N$ contains a function dominating $({}^\omega\omega)^M$, then $N \models \mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$. Now if M is a $V_{\aleph_1}$-like elementary submodel of H_χ for some sufficiently large χ and $\mathfrak{b} > \aleph_1$, then (M, V) is convenient and there is a function $f : \omega \to \omega$

dominating ${}^\omega\omega \cap M$. Thus $\mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$ and therefore WFN($\mathfrak{P}(\omega)$) fails.

However, one can do better. Soukup proved the following ([14]):

3.2.2. Theorem. *Assume* WFN($\mathfrak{P}(\omega)$). *Let M be a $V_{\aleph_1}$-like elementary submodel of H_χ for some sufficiently large χ. Then ${}^\omega 2 \cap M$ is not meager. In particular, the minimal cardinality of a non-meager subset of ${}^\omega 2$ is $\aleph_1$.*

Proof. I show that for every countable family $\mathcal{I}$ of dense ideals of clop(${}^\omega 2$), $\bigcap\{\bigcup I : I \in \mathcal{I}\}$ intersects M. This implies that $M \cap {}^\omega 2$ is not meager. Let $\mathcal{I}$ be a countable family of dense ideals. By WFN($\mathfrak{P}(\omega)$) and Theorem 1.4.4, $M \cap \mathfrak{P}(\text{clop}({}^\omega 2)) \leq_\sigma \mathfrak{P}(\text{clop}({}^\omega 2))$. Thus for each $I \in \mathcal{I}$, $(\mathfrak{P}(\text{clop}({}^\omega 2)) \cap M) \uparrow I$ has a countable coinitial subset. By $V_{\aleph_1}$-likeness of M, there is a countable family $\mathcal{J} \in M$ of dense open subsets of clop(${}^\omega 2$) such that for each $I \in \mathcal{I}$ and each $I' \in \mathfrak{P}(\text{clop}({}^\omega 2))$ with $I \subseteq I'$, there is $J \in \mathcal{J}$ such that $I \subseteq J \subseteq I'$. Since $\mathcal{J} \in M$ and by Baire's Theorem, $\bigcap\{\bigcup J : J \in \mathcal{J}\} \cap M \neq \emptyset$. Let $x \in M \cap \bigcap\{\bigcup J : J \in \mathcal{J}\}$ and assume $x \notin \bigcap\{\bigcup I : I \in \mathcal{I}\}$. Let $I' := \{a \in \text{clop}({}^\omega 2) : x \notin a\}$. Clearly, $I' \in M$. Since $x \notin \bigcap\{\bigcup I : I \in \mathcal{I}\}$, there is $I \in \mathcal{I}$ such that $I \subseteq I'$. By the choice of $\mathcal{J}$, there is $J \in \mathcal{J}$ such that $I \subseteq J \subseteq I'$. This implies $x \notin \bigcap\{\bigcup J : J \in \mathcal{J}\}$, a contradiction. It follows that $\bigcap\{\bigcup J : J \in \mathcal{J}\} \cap M$ is non-empty. ☐

It follows that WFN($\mathfrak{P}(\omega)$) implies that all cardinal invariants in the left half of Cichoń's diagram are $\aleph_1$. Recall that ${}^\omega\omega$ is homeomorphic to the space of irrational numbers of the unit interval and the unit interval is homeomorphic to ${}^\omega 2/\sim$, where $\sim$ identifies every sequence that is eventually, but not everywhere 1 with its successor with respect to the lexicographical order on ${}^\omega 2$. Looking at these homeomorphisms more closely, it follows that ${}^\omega\omega$ is homeomorphic to ${}^\omega 2 \setminus X$ for a countable set X. Since there is a definable homeomorphism proving this, ${}^\omega 2 \cap M$ is meager iff ${}^\omega\omega \cap M$ is. Observing that for a function $f : \omega \to \omega$ the set of functions in ${}^\omega\omega$ which are eventually different from f is meager in ${}^\omega\omega$ and using Borel codes and the absoluteness of their elementary properties, it turns out that the proof of Theorem 3.2.2 gives the following:

If (M,N) is a convenient pair of models of ZFC* and $({}^{\omega}\omega)^N$ contains a function that is eventually different from every function in $({}^{\omega}\omega)^M$, then $N \models \mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$. This shows that adding an eventually different real yields a non-σ-extension of $\mathfrak{P}(\omega)$.

Cichoń's diagram: The big cardinals

On the other hand, WFN($\mathfrak{P}(\omega)$) implies that all cardinal invariants on the right half of Cichoń's diagram are large. My first argument along this line only showed that WFN($\mathfrak{P}(\omega)$) implies that the dominating number is large and was derived from the proof of Lemma 3.1.4. The argument used some tree of closed subsets of ${}^{\omega}2$. Soukup noticed that this tree could be replaced by a certain family of closed covers of ${}^{\omega}2$, simplifying my original proof, and that his argument even gives that the eventually different number, which is just $\mathrm{cov}(\mathcal{M})$, is large under WFN($\mathfrak{P}(\omega)$). The dual of an eventually different real is an infinitely equal real.

3.2.3. Definition. Let M be a set or a class. $f \in {}^{\omega}\omega$ is an *infinitely equal real* over M iff for all $g \in {}^{\omega}\omega \cap M$, $\{n \in \omega : f(n) = g(n)\}$ is infinite. □

Using this notion, I can state the key lemma for determining $\mathrm{cov}(\mathcal{M})$ under WFN($\mathfrak{P}(\omega)$).

3.2.4. Lemma. *Let (M,N) be convenient. Suppose $({}^{\omega}2)^N \setminus M$ is non-empty and $N \models \mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$. Then for every real $x \in N$ there is an infinitely equal real f over M such that $M[x] = M[f]$.* □

Note that one half of '$M[x] = M[f]$' is cheating since it is well-known that an infinitely equal real can code every other real:

3.2.5. Lemma. *Let M be either an elementary submodel of H_χ for some sufficiently large χ or a transitive model of* ZFC* *. If there is an infinitely equal real over M, then for every $g \in {}^{\omega}\omega$ there is an infinitely equal real h over M such that $g \in M[h]$.*

Proof. Let f be infinitely equal over M and $g \in {}^{\omega}\omega$. For every $n \in \omega$ let $h(2n) := f(n)$ and $h(2n+1) := g(n)$. Clearly, $g \in M[h]$. It remains to show that h is infinitely equal over M. Let $e \in {}^\omega\omega \cap M$. For each $n \in \omega$ let $e'(n) := e(2n)$. By the choice of f, e' and f agree on an infinite subset of ω. By the definition of h, for every $n \in \omega$, $e'(n) = f(n)$ iff $e(2n) = h(2n)$. Therefore e and h agree on an infinite set. It follows that h is infinitely equal over M. $\square$

Proof of Lemma 3.2.4. I argue in N and pretend that M is an elementary submodel of H_χ^N for some sufficiently large χ. But the whole argument can be done using Borel codes instead of subsets of ${}^\omega 2$ as well. Let $x \in {}^\omega 2 \setminus M$. By Lemma 3.2.5, it is sufficient to show that there is an infinitely equal real f over M such that $f \in M[x]$.

Consider $F_x := \{a \in \mathrm{clop}({}^\omega 2) : x \in a\}$. Since $\mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$, also $\mathfrak{P}(\mathrm{clop}({}^\omega 2)) \cap M \leq_\sigma \mathfrak{P}(\mathrm{clop}({}^\omega 2))$. By convenience, there is $A \in M$ such that $A \subseteq \mathfrak{P}(\mathrm{clop}({}^\omega 2))$ and for all $G \in M$ with $G \subseteq F_x$ there is $F \in A$ such that $G \subseteq F \subseteq F_x$. W.l.o.g. I may assume that A consists of filters. Let $C := \{\bigcap F : F \in A\}$. Then $C \in M$ is a set of closed subsets of ${}^\omega 2$ with the following property:

($*$) Whenever $a \in M$ is a closed subset of ${}^\omega 2$ containing x, then there is $c \in C$ such that $x \in b \subseteq c$.

Since $x \notin M$, I may assume that all members of C are infinite. Let $(c_n)_{n\in\omega}$ be an enumeration of C in M. For each $n \in \omega$ pick a family $(U_n^m)_{m\in\omega}$ of pairwise disjoint open sets intersecting c_n and covering ${}^\omega 2$ except for one point $y \in M$. (In fact, since c_n is closed, $y \in c_n \cap M$.) This is possible by infinity of c_n. Now let $f : \omega \to \omega$ be the function such that for each $n \in \omega$ the point x is contained in $U_n^{f(n)}$. This is possible since $x \notin M$. Clearly, $f \in M[x]$.

Suppose f is not infinitely equal over M. Let $g \in {}^\omega\omega \cap M$ be eventually different from f. Since ${}^\omega\omega \cap M$ is closed under finite changes, I may even assume that g is everywhere different from f. Thus $a := \bigcap_{n\in\omega}({}^\omega 2 \setminus U_n^{g(n)})$ is a closed set in M containing x, but not including any $c \in C$. This contradicts ($*$). $\square$

Lemma 3.2.4 easily gives

3.2.6. Theorem. *Assume* $\mathrm{WFN}(\mathfrak{P}(\omega))$.

a) If $0^\sharp$ *does not exist, then* $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$.

b) If $\kappa < 2^{\aleph_0}$ *is such that* $\mathrm{cf}([\kappa]^{\aleph_0}) = \kappa$*, then the* $\mathrm{cov}(\mathcal{M}) > \kappa$. *In particular, if* $n \in \omega$ *is such that* $\aleph_n < 2^{\aleph_0}$*, then* $\mathrm{cov}(\mathcal{M}) > \aleph_n$.

Proof. First note that by a result of Bartoszyński ([1]), $\mathrm{cov}(\mathcal{M})$ is the minimal cardinality of a family $E \subseteq {}^\omega\omega$ such that for every function $f : \omega \to \omega$ there is $g \in E$ such that g is eventually different from f. The latter cardinal invariant is the eventually different number. E is called an eventually different family. a) and b) are handled by the same argument. Let $\kappa < 2^{\aleph_0}$ be such that $\mathrm{cf}([\kappa]^{\aleph_0}) = \kappa$. If $0^\sharp$ does not exist, then by Jensens covering lemma, any κ with $\mathrm{cf}\,\kappa > \omega$ has this property. Let χ be a sufficiently large cardinal. For $\alpha < \omega_1$ let M_α be an elementary submodel of H_χ of size κ including κ such that $M_\alpha \cap [\bigcup_{\beta<\alpha} M_\beta]^{\aleph_0}$ is cofinal in $[\bigcup_{\beta<\alpha} M_\beta]^{\aleph_0}$. Let $M := \bigcup_{\alpha<\omega_1} M_\alpha$. Then (M, V) is convenient. By $\mathrm{WFN}(\mathfrak{P}(\omega))$, $\mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$ and thus, using Lemma 3.2.4, there is an infinitely equal real over M. Thus $M \cap {}^\omega\omega$ is not an eventually different family. Assume $\mathrm{cov}(\mathcal{M}) \leq \kappa$. By elementarity, M contains an enumeration of an eventually different family. But since κ is a subset of M, M includes an eventually different family. A contradiction. $\square$

It should be pointed out that the argument Bartoszyński used for showing that the eventually different number equals $\mathrm{cov}(\mathcal{M})$ does not give a direct correspondence between infinitely equal reals and Cohen reals. From a Cohen real one can easily define an infinitely equal real, but according to Blass ([4]), it is an open problem whether forcing notions adding an infinitely equal real also add a Cohen real. However, it is known that if x is infinitely equal over M and y is infinitely equal over $M[x]$, then $M[x][y]$ contains a real that is Cohen over M. There seems to be no simple way to strengthen Lemma 3.2.4 by replacing the infinitely equal real by a Cohen real. But of course, a large value of $\mathrm{cov}(\mathcal{M})$ implies that there are Cohen reals over small sets.

Modulo the assumption $\neg 0^\sharp$ used in the last theorem, this closes the book on the effect of $\mathrm{WFN}(\mathfrak{P}(\omega))$ on cardinal invariants in Cichoń's diagram. Fuchino proved that the minimal size $\mathfrak{a}$ of a maximal almost disjoint family of subsets of ω is $\aleph_1$ under $\mathrm{WFN}(\mathfrak{P}(\omega))$ ([14]). Investigating the various diagrams in Blass' article ([4]), it turns out that there is one cardinal invariant defined in that paper for which no bounds have been determined here yet, and that is

The groupwise density number $\mathfrak{g}$

3.2.7. Definition. The *standard* topology on $\mathfrak{P}(\omega)$ is the topology $\mathfrak{P}(\omega)$ inherits from ${}^{\omega}2$ when each subset of ω is identified with its characteristic function. A family $\mathcal{G} \subseteq [\omega]^{\aleph_0}$ is called *groupwise dense* if $\mathcal{G}$ is non-meager with respect to the standard topology on $\mathfrak{P}(\omega)$ and closed under taking almost subsets. $\mathfrak{g}$ is the smallest number of groupwise dense families with empty intersection. □

Actually, Blass uses a different definition of groupwise dense families, but he proves that the two definitions are equivalent. He has shown that $\mathfrak{g}$ is $\aleph_1$ in the Cohen model ([5]). Thus it should be $\aleph_1$ under WFN($\mathfrak{P}(\omega)$). And indeed, this is true.

3.2.8. Theorem. WFN($\mathfrak{P}(\omega)$) *implies that the groupwise density number* $\mathfrak{g}$ *is* $\aleph_1$.

Proof. Let M be an $V_{\aleph_1}$-like submodel of H_χ for some sufficiently large χ. Let $x \in [\omega]^{\aleph_0}$. By WFN($\mathfrak{P}(\omega)$), there is a countable set $A \subseteq [\omega]^{\aleph_0}$ in M such that for each $y \in \mathfrak{P}(\omega) \cap M$ with $x \subseteq y$ there is $a \in A$ such that $x \subseteq a \subseteq y$. I may assume that A is closed under finite changes. Let $\mathcal{G}_A := \{z \in [\omega]^{\aleph_0} : \exists c \in \mathfrak{P}(\omega) \cap M(z \subseteq^* c \wedge \forall a \in A(a \not\subseteq^* c))\}$. Obviously, $\mathcal{G}_A$ is closed under taking almost subsets. $\mathcal{G}_A$ does not contain x by the choice of A. From Theorem 3.2.2 it follows that $[\omega]^\omega \cap M$ is non-meager. For each $a \in A$ let $F_a := \{b \subseteq \omega : a \subseteq b\}$. Each F_a is closed and nowhere dense by infinity of a. Thus $C := ([\omega]^{\aleph_0} \cap M) \setminus \bigcup_{a \in A} F_a$ is non-meager. Since A is closed under finite changes, $C \subseteq \mathcal{G}_A$. Hence $\mathcal{G}_A$ is groupwise dense. Now $\bigcap_{A \in [[\omega]^{\aleph_0}]^{\aleph_0} \cap M} \mathcal{G}_A = \emptyset$ and thus $\mathfrak{g} = \aleph_1$. □

Assuming $\neg 0^\sharp$, WFN($\mathfrak{P}(\omega)$) therefore implies that the values of all cardinal invariants of the continuum considered in [4] are the precisely as in the Cohen model, that is, $2^{\aleph_0}$ for all invariants $\geq \operatorname{cov}(\mathcal{M})$ and $\aleph_1$ for all invariants below $\operatorname{non}(\mathcal{M})$, $\mathfrak{a}$, or $\mathfrak{g}$.

3.3 More complete Boolean algebras with the WFN

This section contains some results which show that at least assuming $\neg 0^\sharp$, $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies the WFN of several complete c.c.c. Boolean algebras, among them the measure algebras. However, Soukup ([14]) has shown that if the existence of a supercompact cardinal is consistent with ZFC, then the existence of a complete c.c.c. Boolean algebra without the WFN is consistent with ZFC+GCH. Moreover, he proved that adding $\aleph_2$ Cohen reals to a model of CH gives a model where there is a complete c.c.c. Boolean algebra of size $2^{\aleph_0}$ without the WFN while $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds. Lemma 3.3.2 below gives that in that model there is even a countably generated complete c.c.c. Boolean algebra without the WFN. These examples show that it is not possible to extend the results of this section very far.

The measure algebra of the reals

To commence, I show that $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies $\mathrm{WFN}(\mathbb{R}(\omega))$.

3.3.1. Lemma. *The measure algebra* $\mathbb{R}(\omega)$ *is an order retract of* $\mathfrak{P}(\omega)$*. In particular, if* $\mathrm{WFN}(\mathfrak{P}(\omega))$ *holds, then so does* $\mathrm{WFN}(\mathbb{R}(\omega))$*.*

Proof. By Corollary 1.4.9, it is sufficient to construct an order embedding e from $\mathbb{R}(\omega)$ into $\mathfrak{P}(\omega)$.

In order to construct e it is convenient to replace $\mathfrak{P}(\omega)$ by the isomorphic algebra $\mathfrak{P}(\operatorname{clop}({}^\omega 2) \times \omega)$. As usual, I identify $\operatorname{clop}({}^\omega 2)$ with a subalgebra of $\mathbb{R}(\omega)$ in the obvious way. For $a \in \mathbb{R}(\omega)$ let

$$e(a) := \{(c, n) \in \operatorname{clop}({}^\omega 2) \times \omega : \mu(c - a) < \frac{1}{2^n}\}.$$

It is clear that e is monotone. Let $a, b \in \mathbb{R}(\omega)$ such that $a \not\leq b$. Let $n \in \omega$ be such that $\frac{1}{2^n} < \mu(a - b)$. There is a clopen set $c \subseteq {}^\omega 2$ such that

$\mu((a-c)+(c-a)) < \frac{1}{2^{n+1}}$. In particular $(c, n+1) \in e(a)$. But $\mu(c-b) > \frac{1}{2^{n+1}}$ and thus $(c, n+1) \notin e(b)$. Therefore $e(a) \not\subseteq e(b)$. This shows that e is an order embedding. □

Getting the WFN from the WFN of small complete subalgebras

To extend the last result to larger measure algebras, I need the following theorem which is already interesting at its own. The argument in the proof of the $\neg 0^\sharp$-case is basically the same as an argument used by Fuchino and Soukup in an older, unpublished version of [19] that was kindly explained to me by Soukup. However, the theorem stated here does not seem to follow easily from their results.

3.3.2. Theorem. *Let A be a complete c.c.c. Boolean algebra.*

a) If A is completely generated by a set of less than $\aleph_\omega$ generators, then A has the WFN iff every countably generated complete subalgebra B of A does.

b) Assume $0^\sharp$ does not exist. Then A has the WFN iff every countably generated complete subalgebra B of A does.

The proof of the $\neg 0^\sharp$-part of the theorem uses

3.3.3. Lemma. *Let μ be a singular cardinal of cofinality κ with $\mathrm{cf}([\mu]^\kappa) = \mu^+$. Let X be a set of size μ. Assume $\Box_\mu$ holds. Then there is a matrix $(X_{\alpha,\nu})_{\alpha<\mu^+,\nu<\kappa}$ of subsets of X s.t.*

(i) $(X_{\alpha,\nu})_{\nu<\kappa}$ is increasing for all $\alpha < \mu^+$;

(ii) $|X_{\alpha,\nu}| < \mu$ for all $\alpha < \mu^+$ and all $\nu < \kappa$;

(iii) For $\alpha < \mu^+$ let $\mathcal{X}_\alpha := \bigcup_{\nu<\kappa}[X_{\alpha,\nu}]^{\leq\kappa}$. Then $(\mathcal{X}_\alpha)_{\alpha<\mu^+}$ is increasing and continuous at limit ordinals with cofinality $> \kappa$;

(iv) Every $Y \in [X]^\kappa$ is included in some $X_{\alpha,\nu}$.

Proof. Let $\{Y_\alpha : \alpha < \mu^+\}$ be a cofinal subset of $[X]^\kappa$. Let *lim* be the class of limit ordinals. By $\Box_\mu$, there is a sequence $(\mathcal{C}_\alpha)_{\alpha<\mu^+,\alpha\in lim}$ such that the following hold for all limit ordinals $\alpha < \mu^+$:

(1) $\mathcal{C}_\alpha$ is club in α,

(2) $\operatorname{otp}(\mathcal{C}_\alpha) < \mu$,

(3) If $\beta < \alpha$ is a limit point of $\mathcal{C}_\alpha$, then $\mathcal{C}_\beta = \beta \cap \mathcal{C}_\alpha$.

Note that (2) usually reads '$\operatorname{cf}(\alpha) < \mu \Rightarrow \operatorname{otp}(\mathcal{C}_\alpha) < \mu$', but this is not necessary here, since μ is singular. Fix an increasing cofinal sequence $(\mu_\nu)_{\nu<\kappa}$ of regular cardinals larger than κ in μ. Define $(X_{\alpha,\nu})_{\alpha<\mu^+,\nu<\kappa}$ as follows:

For $\nu < \kappa$ let $X_{0,\nu} := \emptyset$. For $\alpha = \beta + 1 < \mu^+$ and $\nu < \kappa$ let $X_{\alpha,\nu} := X_{\beta,\nu} \cup Y_\beta$. For a limit ordinal $\alpha < \mu^+$, $\nu < \kappa$ let $X_{\alpha,\nu} := \emptyset$ if $\mu_\nu < |\mathcal{C}_\alpha|$ and $X_{\alpha,\nu} := \bigcup_{\beta\in\mathcal{C}_\alpha} X_{\beta,\nu}$ if $\mu_\nu \geq |\mathcal{C}_\alpha|$.

It is clear from the construction that the matrix $(X_{\alpha,\nu})_{\alpha<\mu^+,\nu<\kappa}$ satisfies (iv).

Claim 1. $|X_{\alpha,\nu}| \leq \mu_\nu$ for all $\alpha < \mu^+$ and $\nu < \kappa$.

The proof proceeds by induction on α. For $\alpha = 0$ the statement is true since $X_{0,\nu}$ is empty. Let $\alpha = \beta + 1$. By the inductive hypothesis, $|X_{\beta,\nu}| \leq \mu_\nu$. By construction, $X_{\alpha,\nu} = X_{\beta,\nu} \cup Y_\beta$ and $|Y_\beta| = \kappa$. Since μ_ν was chosen to be larger than κ, it follows that $|X_{\alpha,\nu}| \leq \mu_\nu$. Finally let α be a limit ordinal. If $|\mathcal{C}_\alpha| > \mu_\nu$, then $X_{\alpha,\nu}$ is empty. If $|\mathcal{C}_\alpha| \leq \mu_\nu$, then $X_{\alpha,\nu} = \bigcup_{\beta\in\mathcal{C}_\alpha} X_{\beta,\nu}$ and thus, by the inductive hypothesis, $|X_{\alpha,\nu}| \leq \mu_\nu$.

This claim immediately gives (ii). (i) is easily seen by induction on α. In order to show (iii), I need

Claim 2. For $\alpha \leq \beta < \mu^+$ and $\nu < \kappa$ there is $\rho \in [\nu, \kappa)$ such that $X_{\alpha,\rho} \subseteq X_{\beta,\rho}$.

The proof proceeds by induction on β, parallel for all ν. For $\alpha = \beta$ there is nothing to show. Suppose $\beta > \alpha$ and $\beta = \gamma + 1$. By the inductive hypothesis, there is $\rho \in [\nu, \kappa)$ such that $X_{\alpha,\rho} \subseteq X_{\gamma,\rho}$. By construction, $X_{\gamma,\rho} \subseteq X_{\beta,\rho}$. Now suppose β is a limit ordinal and $\beta > \alpha$. Pick $\gamma \in \mathcal{C}_\beta$ such that $\alpha \leq \gamma$. By the inductive hypothesis, there is $\rho \in [\nu, \kappa)$ such that $X_{\alpha,\rho} \subseteq X_{\gamma,\rho}$ and $|\mathcal{C}_\gamma| \leq \mu_\rho$. By construction, $X_{\beta,\rho} := \bigcup_{\delta\in\mathcal{C}_\beta} X_{\delta,\rho}$. Thus $X_{\gamma,\rho} \subseteq X_{\beta,\rho}$.

Now let $(\mathcal{X}_\alpha)_{\alpha<\mu^+}$ be defined as in (iii). Suppose $\alpha \leq \beta < \mu^+$ and $Y \in \mathcal{X}_\alpha$. Pick $\nu < \kappa$ with $Y \subseteq X_{\alpha,\nu}$. By Claim 2, there is $\rho \in [\nu, \kappa)$ such that $X_{\alpha,\rho} \subseteq X_{\beta,\rho}$. By (i), $Y \subseteq X_{\alpha,\rho} \subseteq X_{\beta,\rho}$ and thus $Y \in \mathcal{X}_\beta$. This shows that $(\mathcal{X}_\alpha)_{\alpha<\mu^+}$ is increasing.

Suppose $\alpha < \mu^+$ is a limit ordinal of cofinality $> \kappa$ and $Y \in \mathcal{X}_\alpha$. Fix $\nu < \kappa$ such that $Y \subseteq X_{\alpha,\nu}$. Since $X_{\alpha,\nu}$ is nonempty, $X_{\alpha,\nu} = \bigcup_{\beta\in\mathcal{C}_\alpha} X_{\beta,\nu}$ and $|\,\mathcal{C}_\alpha\,|\leq \mu_\nu$. Since $\mathrm{cf}(\alpha) > \kappa$, there is a limit $\beta < \alpha$ of $\mathcal{C}_\alpha$ such that $Y \subseteq \bigcup_{\gamma\in\mathcal{C}_\alpha\cap\beta} X_{\gamma,\nu}$. Now $\mathcal{C}_\beta = \mathcal{C}_\alpha \cap \beta$ and $|\,\mathcal{C}_\beta\,|\leq|\,\mathcal{C}_\alpha\,|\leq \mu_\nu$. Therefore $Y \subseteq \bigcup_{\gamma\in\mathcal{C}_\beta} X_{\gamma,\nu} = X_{\beta,\nu}$. Hence $Y \in \mathcal{X}_\beta$. This shows that $(\mathcal{X}_\alpha)_{\alpha<\mu^+}$ is continuous at limit ordinals of cofinality $> \kappa$ and thus establishes (iii). □

Proof of the theorem. The proof of part b) does not use $\neg 0^\sharp$ unless A is not completely generated by a subset of size less than $\aleph_\omega$. Therefore a) will follow from the proof of b). Every complete subalgebra of A is a retract of A and thus has the WFN if A does. This shows the easy direction of a) and b).

The proof of the other direction proceeds by induction on the size of a set completely generating A. If A is countably generated, then there is nothing to prove. Let A be completely generated by a subset $X = \{a_\alpha : \alpha < \mu\}$ for some uncountable cardinal μ and assume that for each subset Y of X of size less than μ the subalgebra A_Y of A completely generated by Y has the WFN. If $\mathrm{cf}\,\mu > \omega$, then by c.c.c., $A = \bigcup_{\alpha<\mu} A_{\{a_\beta:\beta<\alpha\}}$. Every $A_{\{a_\beta:\beta<\alpha\}}$ is a σ-subalgebra of A and $\mathrm{WFN}(A_{\{a_\beta:\beta<\alpha\}})$ holds by the inductive hypothesis. This implies $\mathrm{WFN}(A)$.

Now assume $\mathrm{cf}\,\mu = \aleph_0$. By $\neg 0^\sharp$ and Jensen's Covering Lemma, $\mathrm{cf}([\mu]^{\aleph_0}) = \mu^+$ and $\square_\mu$ holds. (See [10] for these things.) So let $(X_{\alpha,\nu})_{\alpha<\mu^+,\nu<\omega}$ be the matrix of subsets of X guaranteed by Lemma 3.3.3. For all $\alpha < \mu^+$ and $\nu < \omega$ let $A^{\alpha,\nu} := A_{X_{\alpha,\nu}}$. For each $\alpha < \mu^+$ let $A^\alpha := \bigcup_{\nu<\omega} A^{\alpha,\nu}$. By property (i) of the matrix, A^α is a subalgebra of A. Note that A^α is even a σ-subalgebra of A, because it is a countable union of complete subalgebras. By property (ii) of the matrix together with the inductive hypothesis, $\mathrm{WFN}(A^{\alpha,\nu})$ holds for all α and ν. Thus for every α, $\mathrm{WFN}(A^\alpha)$ holds. By c.c.c., there is a function $\mathrm{supp} : A \to [X]^{\aleph_0}$ such that for all $a \in A$, $a \in A_{\mathrm{supp}(a)}$. Since $\mathrm{supp}(a)$ is included in some $X_{\alpha,\nu}$ for each $a \in A$, $A = \bigcup_{\alpha\leq\mu^+} A^\alpha$. By property (iii) of the matrix, $(A^\alpha)_{\alpha<\mu}$ is increasing and continuous at limit ordinals of cofinality $> \aleph_0$. This implies $\mathrm{WFN}(A)$. □

The larger measure algebras

Under $\neg 0^\sharp$ the last theorem together with Lemma 3.3.1 and Maharam's Theorem will give complete information on the WFN of measure algebras. Note that it was already proved in [19] that under $\neg 0^\sharp$ every Cohen algebra has the WFN iff WFN($\mathfrak{P}(\omega)$) holds. This also immediately follows from the last theorem since every countably generated complete subalgebra of a Cohen algebra is a complete subalgebra of a countably generated Cohen algebra.

3.3.4. Definition. A *measure algebra* is a complete Boolean algebra A together with a function $\mu : A \to [0,1]$ such that

(i) forall $A \in A$, $\mu(a) = 0$ iff $a = 0$ and

(ii) for every countable antichain $C \subseteq A$, $\mu(\sum C) = \sum\{\mu(a) : a \in C\}$. □

A Boolean algebra A is called *measurable* iff there is a function $\mu : A \to [0,1]$ such that (A, μ) is a measure algebra. By the usual abuse of notation, I will write only 'measure algebra' when I mean 'measurable algebra'.

Note that the measure algebras in the definition above are frequently called totally finite measure algebras. For measure algebras usually μ is not assumed to be bounded. However, since I will consider only totally finite measure algebras, I call them just measure algebras.

3.3.5. Corollary. *Let A be an infinite measure algebra.*

a) If A is completely generated by strictly less than $\aleph_\omega$ generators, then WFN(A) *holds iff* WFN($\mathfrak{P}(\omega)$) *does.*

b) If $0^\sharp$ does not exist, then WFN(A) *holds iff* WFN($\mathfrak{P}(\omega)$) *does.*

Proof. The 'only if'-part of a) and b) follows from the fact that $\mathfrak{P}(\omega)$ is a retract of every infinity complete Boolean algebra. The proof of the 'if'-part is the almost same for a) and b), too. The only difference is that for a) part a) of Theorem 3.3.2 is used, and for b) part b) of Theorem 3.3.2 is used. So let B be a countably generated complete subalgebra of A. The restriction of the measure on A to B is a measure on B. By Maharam's Theorem, there are $\nu \leq \omega$ and a sequence $(B_n)_{n<\nu}$ of measure algebras such that $B \cong \prod_{n<\nu} B_n$, where B_n is isomorphic to the measure algebra $\mathbb{R}(\omega)$ or trivial, i.e. $= \{0, 1\}$. it follows that B trivial or isomorphic to a product $C \times D$ where C is either trivial or isomorphic to $\mathbb{R}(\omega)$ and D is the powerset of an at most countable set. Assume WFN($\mathfrak{P}(\omega)$). By Lemma 3.3.1, C has the WFN. Obviously, D has the WFN. It follows that B has the WFN. Now WFN(A) follows from Theorem 3.3.2 $\square$

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