THE COINITIALITY OF A COMPACT SPACE

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ABSTRACT. This article deals with the coinitiality of topological spaces, a concept that generalizes the cofinality of a Boolean algebra as introduced by Koppelberg [7]. The compact spaces of countable coinitiality are characterized. This gives a new characterization of Boolean algebras of countable cofinality. We also discuss cofinalities of C^*-algebras and of Banach spaces.

0. INTRODUCTION

In [7] Koppelberg defined the *cofinality* $\mathrm{cof}(B)$ of an infinite Boolean algebra B as the least limit ordinal δ such that there is a strictly increasing sequence $(B_\alpha)_{\alpha<\delta}$ of subalgebras of B such that $B = \bigcup_{\alpha<\delta} B_\alpha$. (See also [2].)

Koppelberg showed that for every Boolean algebra B the cofinality of B is at most $2^{\aleph_0}$. Moreover, every infinite Boolean algebra which is almost σ-complete is of cofinality $\aleph_1$. Almost σ-completeness is a weakening of σ-completeness and is called *countable separation property* nowadays.

The main open question concerning cofinalities of Boolean algebras is

Question 0.1. Is it consistent that there is a Boolean algebra whose cofinality is bigger than $\aleph_1$?

The present article was motivated by this question. We generalize cofinalities of Boolean algebras to coinitialities of compact spaces and characterize compact spaces of countable coinitiality by topological means. This is a modest attempt to add a new perspective to the problem of cofinalities of Boolean algebras.

It should be mentioned that cofinalities of groups and other algebraic structures have been studied extensively in the literature (see [4, 15] and the references therein).

1. BASIC DEFINITIONS AND ELEMENTARY FACTS

Our notation for inverse systems follows [3]. Also, all the topological facts that we use without reference can be found in that book.

Definition 1.1. Let X be a topological space. The *coinitiality* $\mathrm{ci}(X)$ of X is the least limit ordinal δ such X is the limit of an inverse system $\{X_\alpha, \pi_\alpha^\beta, \delta\}$ whose bonding maps π_α^β are onto and not 1-1, provided such an inverse system exists.

Note that the coinitiality of a topological space is defined except for trivial cases. It is defined for infinite compact spaces.

In Definition 1.1, the requirement about the inverse system is rather weak. If X is the limit of an inverse system $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \delta\}$, δ a limit ordinal, such that the canonical maps $\pi_\alpha : X \to X_\alpha$ are not 1-1, i.e., if X is not already determined by a single X_α, then $\mathcal{S}$ can be replaced by a well-ordered inverse system of limit length $\leq \delta$ whose bonding maps are onto and not 1-1.

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Note that for every X, $\mathrm{ci}(X)$ is a regular cardinal. If X is zero-dimensional, let $\mathrm{ci}_0(X)$ be the cardinal invariant obtained in Definition 1.1 when requiring the X_α to be zero-dimensional. By Stone-duality, for every Boolean algebra B, $\mathrm{ci}_0(\mathrm{Ult}(B)) = \mathrm{cof}(B)$ where $\mathrm{Ult}(B)$ denotes the Stone-space of B. Clearly, for every infinite compact zero-dimensional space X, $\mathrm{ci}(X) \leq \mathrm{ci}_0(X)$. We will show that in fact $\mathrm{ci}(X) = \mathrm{ci}_0(X)$, but the argument for this is probably most conveniently stated in terms of C^*-algebras. Following [14], we require every C^*-algebra to have a unit.

Definition 1.2. For an infinite dimensional C^*-algebra A let $\mathrm{cof}_{C^*}(A)$ denote the least limit ordinal δ such that there is a strictly increasing sequence $(A_\alpha)_{\alpha<\delta}$ of (closed) C^*-subalgebras of A such that $\bigcup_{\alpha<\delta} A_\alpha$ is dense in A.

For every compact space X let $C(X)$ denote the C^*-algebra of complex continuous functions on X. Then, essentially by the Stone-Weierstraß theorem, for every infinite compact space X, $\mathrm{cof}_{C^*}(C(X)) = \mathrm{ci}(X)$.

Lemma 1.3. *Let X be compact and zero-dimensional. Then*

$$\mathrm{ci}(X) = \mathrm{cof}_{C^*}(C(X)) = \mathrm{cof}(\mathrm{Clop}(X)) = \mathrm{ci}_0(X).$$

Proof. We only have to show that $\mathrm{cof}(\mathrm{Clop}(X)) \leq \mathrm{cof}_{C^*}(C(X))$. Let $A = C(X)$ and let $(A_\alpha)_{\alpha<\delta}$ be a strictly increasing sequence of C^*-subalgebras of A such that $\bigcup_{\alpha<\delta} A_\alpha$ is dense in A. Let $B = \mathrm{Clop}(X)$. For every $\alpha < \delta$ let $\sim_\alpha$ denote the equivalence relation on X which identifies two points if they are not separated by a function in A_α and let $f_\alpha : X \to X/\sim_\alpha$ denote the quotient map. Moreover, let B_α be the image of the natural embedding from $\mathrm{Clop}(X/\sim_\alpha)$ into B.

Let $\alpha < \delta$. Since A_α is a proper C^*-subsalgebra of A, $\sim_\alpha$ is not just equality. It follows that B_α is a proper subalgebra of B. Since the sequence $(A_\alpha)_{\alpha<\delta}$ is increasing, also the sequence $(B_\alpha)_{\alpha<\delta}$ is increasing. This implies $\mathrm{cof}(\mathrm{Clop}(X)) \leq \delta$ provided we can show that $B = \bigcup_{\alpha<\delta} B_\alpha$.

Let $b \in B$. Consider the characteristic function $\chi_b : X \to \{0,1\}$ of b. Since b is clopen, χ_b is continuous and thus $\chi_b \in A$. Since $\bigcup_{\alpha<\delta} A_\alpha$ is dense in A, there are $\alpha < \delta$ and $f \in A_\alpha$ such that

$$\sup_{x\in X} |f(x) - \chi_b(x)| < \frac{1}{2}.$$

Now $b = f^{-1}\left[\left(\frac{1}{2}, \frac{3}{2}\right)\right]$ and $X \setminus b = f^{-1}\left[\left(-\frac{1}{2}, \frac{1}{2}\right)\right]$. Therefore $b \in B_\alpha$. It follows that $B = \bigcup_{\alpha<\delta} B_\alpha$. □

We conclude this section with a few observations concerning upper bounds for the coinitiality of compact spaces. For an infinite compact space X let $\mathrm{a}(X)$ denote the *altitude* of X, the least limit ordinal δ such that there is a strictly decreasing sequence $(F_\alpha)_{\alpha<\delta}$ of closed subsets of X with $|\bigcap_{\alpha<\delta} F_\alpha| = 1$. It is clear that the altitude of an infinite space is an infinite regular cardinal. Observe that a compact space has altitude $\aleph_0$ if and only if it contains a non-trivial convergent sequence.

Lemma 1.4. *(a) If X is an infinite closed subspace of a compact space Y, then* $\mathrm{ci}(Y) \leq \mathrm{ci}(X)$.

(b) If X is an infinite compact space, then $\mathrm{ci}(X) \leq \mathrm{a}(X)$.

(c) For every infinite compact space X, $\mathrm{ci}(X) \leq \mathrm{cf}(\mathrm{w}(X))$ *where* $\mathrm{w}(X)$ *denotes the weight of X.*

(d) For every infinite compact space X, $\mathrm{ci}(X) \leq 2^{\aleph_0}$.

For compact zero-dimensional spaces and ci_0, (a), (c) and (d) are due to Koppelberg [7] and (b) is due to van Douwen [2].

Proof. The proofs of (a) and (b) are easy. The proof of (c) is also easy if one considers X as a subspace of $[0,1]^{w(X)}$. (d) Follows from (a), (c), and the fact that every infinite compact space has an infinite closed subspace that is separable and hence has weight $\leq 2^{\aleph_0}$. □

Note that it is consistent that $\mathrm{ci}(X) < 2^{\aleph_0}$ holds for every compact space X. Just take a model of set theory where $2^{\aleph_0}$ is singular. In fact, in such a model we have $\mathrm{a}(X) < 2^{\aleph_0}$ for every compact space X.

In the zero-dimensional case, much more information has been provided by Koszmider [8], who showed that it is consistent to have $\mathrm{pa}(X) \leq \aleph_1$ for every Boolean space X while $2^{\aleph_0} > \aleph_1$. Here $\mathrm{pa}(X)$ is the *pseudo-altitude* of X, the least character of a non-isolated point in a closed subspace of X. It is easily checked that $\mathrm{pa}(X)$ is an upper bound for $\mathrm{a}(X)$ and hence for $\mathrm{ci}(X)$.

We will go back to this type of results at the end of the next section.

2. Martin's Axiom and convergent sequences

Koppelberg [7] showed that under Martin's Axiom every Boolean algebra of size $< 2^{\aleph_0}$ is of countable cofinality. The following stronger statement (modulo Stone duality) seems to be set-theoretic folklore (see [11]):

Under MA every infinite compact space of weight $< 2^{\aleph_0}$ has a non-trivial convergent sequence.

However, Koppelberg's proof of her result is still useful since turned into topology it actually gives

Theorem 2.1. *Assume Martin's Axiom for countable partial orders. Then every infinite compact space of weight* $< 2^{\aleph_0}$ *has a non-trivial convergent sequence.*

Since it is not completely straight forward to turn Koppelberg's algebraic argument into topology, we give a proof of this theorem.

Proof. First assume that X is scattered. After removing the isolated points from X, we are left with a closed subspace X' of X. Since X is scattered, X' has an isolated point p. Being isolated X', p has an open neighborhood $O \subseteq X$ such that $X' \cap O = \{p\}$. Since X is scattered, X is zero-dimensional and we can choose O to be clopen. The compact set O is a one-point compactification of a discrete space and therefore contains a convergent sequence.

Now assume that X is not scattered. Then by passing to a closed subspace of X if neccessary, we may assume that X has no isolated points.

We construct a closed subspace of X that maps (via a continuous mapping) onto 2^ω. Let $(O_s)_{s\in 2^{<\omega}}$ be a family of nonempty open subsets of X such that for all $s, t \in 2^{<\omega}$,

(1) if s and t are incomparable (with respect to $\subseteq$), then $\mathrm{cl}(O_s)$ and $\mathrm{cl}(O_t)$ are disjoint, and
(2) if $s \subseteq t$, then $O_t \subseteq O_s$.

Since X has no isolated points, one can easily choose the family $(O_s)_{s\in 2^{<\omega}}$ by induction on the length of s.

Let $Y = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} \mathrm{cl}(O_s)$. For every $x \in 2^\omega$ let $f^{-1}(x) = \bigcap_{n\in\omega} \mathrm{cl}(O_{x\restriction n})$. This defines a continuous surjection $f : Y \to 2^\omega$.

By passing to a closed subspace of Y if neccessary, we may assume that f is irreducible, that is, no proper closed subspace of Y is mapped onto 2^ω by f.

We will apply Martin's Axiom to the countable Boolean algebra $\mathrm{Clop}(2^\omega)$ of clopen subsets of 2^ω. Since $\mathrm{w}(X) < 2^{\aleph_0}$, $\mathrm{w}(Y) < 2^{\aleph_0}$. Let $\mathcal{B}$ be a base for the topology on Y of size $< 2^{\aleph_0}$. For every open set $O \subseteq Y$ let

$$D_O = \{A \in \mathrm{Clop}(2^\omega) : f^{-1}[A] \subseteq O \vee f^{-1}[A] \cap O = \emptyset\}.$$

We show that every D_O is dense in $\mathrm{Clop}(2^\omega)$.

Let A be a nonempty clopen subset of 2^ω. If $f^{-1}[A]$ is disjoint from O, then $A \in D_O$. If $f^{-1}[A]$ intersects O, then $f^{-1}[A] \cap O$ is a nonempty open subset of Y and thus, by the irreducibility of f, $f \restriction Y \setminus (f^{-1}[A] \cap O)$ is not onto. In particular, there is a nonempty clopen set $A' \subseteq 2^\omega$ such that $A' \subseteq 2^\omega \setminus f[Y \setminus (f^{-1}[A] \cap O)]$. Now it is easily checked that $A' \subseteq A$ and $f^{-1}[A'] \subseteq O$. This shows the density of D_O.

By Martin's Axiom, there is an ultrafilter $H \subseteq \mathrm{Clop}(2^\omega)$ that intersects D_O for every $O \in \mathcal{B}$. Let p be the element of 2^ω corresponding to H, i.e., let p be the unique element of $\bigcap H$. Let $y \in f^{-1}(p)$. We will show that y is of countable character in Y. This clearly implies that there is a sequence in Y, and therefore in X, that converges to y.

We show that $\{f^{-1}[A] : A \in H\}$ is a local base at y. Let $U \subseteq Y$ be a neighborhood of y. Then there is $O \in \mathcal{B}$ such that $y \in O \subseteq U$. By the choice of H, there is $A \in H \cap D_O$. By the choice of y, $y \in f^{-1}[A]$. Since $A \in D_O$ and $y \in O$, $f^{-1}[A] \subseteq O$. □

Note that this proof actually shows that the weight of a compact space without a non-trivial convergent sequence is at least $\mathrm{cov}(\mathcal{M})$, the smallest size of family of nowhere dense sets that covers 2^ω.

Another formulation of Theorem 2.1 is the following dichotomy:

Corollary 2.2. *Assume Martin's Axiom for countable partial orders. Then every infinite compact space X either has an infinite closed subspace of countable weight or all of its infinite closed subspaces are of weight at least $2^{\aleph_0}$*

Without some instance of Martin's Axiom, this dichotomy can fail even in the Boolean case. Just and Koszmider [6] showed that it is consistent that there is a Boolean algebra of size $< 2^{\aleph_0}$ without a countably infinite homomorphic image. Dualizing this we obtain a compact zero-dimensional space of weight $< 2^{\aleph_0}$ without an infinite closed subspace of countable weight.

By Lemma 1.4, Theorem 2.1 implies

Corollary 2.3. *Assume Martin's Axiom for countable partial orders. Then every infinite compact space of weight $< 2^{\aleph_0}$ is of coinitiality $\aleph_0$.*

Corollary 2.4. *The following statement is consistent with ZFC:*

$$\textit{For every compact space } X,\ \mathrm{ci}(X) \leq \aleph_1 < 2^{\aleph_0}.$$

Proof. Just add $\aleph_{\omega_1}$ Cohen reals to a model of CH. In the resulting model we have $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$ and Martin's Axiom holds for countable partial orders. If X is a compact space of weight $2^{\aleph_0}$, then $\mathrm{ci}(X) \leq \aleph_1$ by Lemma 1.4. If X is a compact space of weight $< 2^{\aleph_0}$, then $\mathrm{ci}(X) \leq \aleph_0$ by Corollary 2.3. □

Note that this result is close to, but incomparable (at least without further work) with Koszmider's result about pseudo-altitudes of Boolean spaces mentioned at the end of Section 2. Also note that in the proof of Corollary 2.4 we really need that in Corollary 2.3 we do not assume all of MA since MA implies that $2^{\aleph_0}$ is regular.

3. Compact spaces of countable coinitiality

Our next goal is to characterize the compact spaces of countable coinitiality.

A sequence $(x_n)_{n\in\omega}$ in a topological space X is *discrete* if the sequence is 1-1 and $\{x_n : n \in \omega\}$ is discrete with respect to the subspace topology inherited from X. If X is regular Hausdorff, a sequence $(x_n)_{n\in\omega} \in X^\omega$ is discrete iff there is a family $(U_n)_{n\in\omega}$ of pairwise disjoint open sets such that for all $n \in \omega$, $x_n \in U_n$.

For an ultrafilter p over ω and a sequence $(x_n)_{n\in\omega}$ in a compact space X let $\lim_p(x_n)_{n\in\omega}$ denote the *p-limit* of $(x_n)_{n\in\omega}$, i.e., the unique element of $\bigcap_{A\in p}\mathrm{cl}(\{x_n : n \in A\})$.

Lemma 3.1. *Let X be an infinite compact space such that for every discrete sequence $(x_0, y_0, x_1, y_1, \dots) \in X^\omega$ there is a free ultrafilter p over ω such that*

$$\lim_p(x_n)_{n\in\omega} \neq \lim_p(y_n)_{n\in\omega}.$$

Then $\mathrm{ci}(X) > \aleph_0$.

Proof. We first observe that X does not contain a non-trivial convergent sequence.

Suppose that $(a_n)_{n\in\omega}$ is sequence in X that converges to some point $a \in X$ and is not eventually constant. After thinning out this sequence, we may assume that it is discrete. Putting $x_n = a_{2n}$ and $y_n = a_{2n+1}$ we obtain two sequences such that for every free ultrafilter p over ω,

$$\lim_p(x_n)_{n\in\omega} = a = \lim_p(y_n)_{n\in\omega},$$

contradicting our assumption on X. This shows that X has no non-trivial converging sequences.

Now suppose that X is the limit of an inverse system $\mathcal{S} = \{X_n, \pi^m_n, \omega\}$ whose bonding maps are onto and not 1-1. For each $n \in \omega$ let π_n denote the canonical map from X to X_n.

In X we choose sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ such that, for some strictly increasing sequence $(k_n)_{n\in\omega}$ of natural numbers, for every $n \in \omega$, $\pi_{k_n}(x_n) = \pi_{k_n}(y_n)$ but $\pi_{k_{n+1}}(x_n) \neq \pi_{k_{n+1}}(y_n)$.

Claim 3.2. There are open sets $U_0, V_0 \subseteq X$ such that $x_0 \in U_0$, $y_0 \in V_0$, and for infinitely many $n \in \omega$, $\{x_n, y_n\}$ is disjoint from the closure of $U_0 \cup V_0$.

We pick U_0 first. Suppose there is no open neighborhood U of x_0 such that for infinitely many $n \in \omega$, x_n and y_n are outside U. Then for every $m \in \omega$ the sequences $(\pi_m(x_n))_{n\in\omega}$ and $(\pi_m(y_n))_{n\in\omega}$, which eventually agree, converge to $\pi_m(x_0)$. It follows that the sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ converge to x_0. At least one of the two sequences is not eventually constant. But this contradicts the fact that in X there are no non-trivial converging sequences.

It follows that for some open neighborhood U of x_0 there are infinitely many $n \in \omega$ such that x_n and y_n are outside U. Let U_0 be an open neighborhood of x_0 such that $\mathrm{cl}(U_0) \subseteq U$. After thinning out the sequence $((x_n, y_n))_{n\in\omega}$, we may assume that all x_n and all y_n, $n > 0$, are outside $\mathrm{cl}(U_0)$. By the same argument as for U_0, we can pick an open neighborhood V_0 of y_0 as required. This finishes the proof of the claim.

By iterated application of the claim, we can thin out the sequence $((x_n, y_n))_{n\in\omega}$ such that, after renumbering, the sequence $(x_0, y_0, x_1, y_1, \dots)$ is discrete. Now let p be any free ultrafilter over ω.

For every $m \in \omega$ we have

$$\lim_p(\pi_m(x_n))_{n\in\omega} = \lim_p(\pi_m(y_n))_{n\in\omega}$$

since the sequences $(\pi_m(x_n))_{n\in\omega}$ and $(\pi_m(y_n))_{n\in\omega}$ eventually agree. It is easily checked that this implies

$$\lim_p(x_n)_{n\in\omega} = \lim_p(y_n)_{n\in\omega},$$

contradicting our assumption on X. □

Let us call a sequence $(x_n)_{n\in\omega}$ a *double sequence* if it is discrete and for every free ultrafilter p over ω,

$$\lim_p(x_{2n})_{n\in\omega} = \lim_p(x_{2n+1})_{n\in\omega}.$$

Theorem 3.3. *Let X be a compact space. Then* $\mathrm{ci}(X) = \aleph_0$ *iff X contains a double sequence.*

Proof. If X is a compact space of countable coinitiality, then X contains a double sequence by Lemma 3.1.

Now suppose that X contains a double sequence $(x_n)_{n\in\omega}$. By Lemma 1.4 it suffices to show that $Y := \mathrm{cl}(\{x_n : n \in \omega\})$ is of countable coinitiality. For every $k \in \omega$ let $\sim_k$ denote the equivalence relation on Y that for every $m \geq k$ identifies x_{2m} and x_{2m+1}. For every $x \in Y$ let $[x]_k$ denote the $\sim_k$-class of x.

We show that $Y/\sim_k$ is Hausdorff for every $k \in \omega$. Let $k \in \omega$ and let $x, y \in Y$ be such that $x \not\sim_k y$. In order to show that $[x]_k$ and $[y]_k$ have disjoint open neighborhoods in $Y/\sim_k$, we have to show that x and y have disjoint open neighborhoods that are unions of $\sim_k$-classes.

If one of the points x and y is a member of the sequence $(x_n)_{n\in\omega}$, then this is easily checked using the fact that the x_n are isolated points of Y since the sequence $(x_n)_{n\in\omega}$ is discrete.

If neither x nor y is an element of $\{x_n : n \in \omega\}$, then we choose disjoint open neighborhoods U and V of x and y, respectively. We call a subset A of Y *symmetric* if for every $m \in \omega$,

$$x_{2m} \in A \quad \Leftrightarrow \quad x_{2m+1} \in A.$$

Note that every symmetric set is the union of $\sim_k$-classes.

Now consider the sets

$$S = \{m \in \omega : x_{2m} \in U \wedge x_{2m+1} \notin U\}$$

and

$$T = \{m \in \omega : x_{2m+1} \in U \wedge x_{2m} \notin U\}$$

Since U is disjoint from $\{x_{2m+1} : m \in S\}$, $x \notin \mathrm{cl}(\{x_{2m+1} : m \in S\})$. Since $(x_n)_{n\in\omega}$ is a double sequence, we also have $x \notin \mathrm{cl}(\{x_{2m} : m \in S\})$. By the same argument, $x \notin \mathrm{cl}(\{x_{2m+1} : m \in T\})$.

Therefore $x \in U_0 = U \setminus \mathrm{cl}(\{x_{2m} : m \in S\} \cup \{x_{2m+1} : m \in T\})$. The set U_0 is symmetric. In the same way we can obtain a symmetric open neighborhood $V_0 \subseteq V$ of y. Now U_0 and V_0 are disjoint open neighborhoods of x and y, respectively, and they are unions of $\sim_k$-classes. This shows that $Y/\sim_k$ is Hausdorff.

Clearly, Y is the inverse limit of the spaces $Y/\sim_k$, $k \in \omega$. It follows that $\mathrm{ci}(Y) = \aleph_0$. □

Corollary 3.4. *For every infinite F-space X,* $\mathrm{ci}(X) = \aleph_1$.

Proof. If X is an infinite compact F-space and $(x_n)_{n\in\omega}$ is a discrete sequence in X, then for every $A \subseteq \mathbb{N}$, $\mathrm{cl}(\{x_n : n \in A\})$ is disjoint from $\mathrm{cl}(\{x_n : n \in \omega \setminus A\})$. In particular, $(x_n)_{n\in\omega}$ is not a double sequence. Now by Theorem 3.3, $\mathrm{ci}(X) \geq \aleph_1$.

Moreover, for every discrete sequence $(x_n)_{n\in\omega}$ in X, $\mathrm{cl}(\{x_n : n \in \omega\}) \cong \beta\omega$. By the results of Koppelberg [7], $\mathrm{ci}(\beta\omega) = \mathrm{cof}(\mathcal{P}(\omega)) = \aleph_1$. By Lemma 1.4, $\mathrm{ci}(X) \leq \aleph_1$. □

Note that the compact spaces of the form $Y = \mathrm{cl}(\{x_n : n \in \omega\})$ where $(x_n)_{n\in\omega}$ is a double sequence can be described as follows. Let Z be a compact space with a countably infinite, discrete, dense subset C. Since C is discrete and $Z = \mathrm{cl}(C)$, every $z \in C$ is an isolated point of Z. Let $D(Z, C)$ be the disjoint union of $Z \setminus C$ and $2 \times C$. We describe a topology on $D(Z, C)$ by defining the basic open sets.

For all $z \in C$ and all $i \in 2$ the singleton $\{(i, z)\}$ is open in $D(Z, C)$. For every open set $O \subseteq Z$ the set $(O \setminus C) \cup 2 \times (O \cap C)$ is open. It is easily checked that $D(Z, C)$ is compact. Moreover, if $(z_n)_{n\in\omega}$ is a 1-1 enumeration of C, then the sequence $((0,z_0),(1,z_0),(0,z_1),(1,z_1),\dots)$ is a double sequence. In particular, $\mathrm{ci}(D(Z,C))=\aleph_0$.

Obviously, there is a largest space of the form $D(Z,C)$, namely $D(\beta\omega,\omega)$. Every other space $D(Z,C)$ is a continuous image of $D(\beta\omega,\omega)$ via a continuous map that is induced by a 1-1 map from ω into C.

Corollary 3.5. *For every compact space X, $\mathrm{ci}(X)=\aleph_0$ iff there is an injective map $f:2\times\omega\to X$ that has a continuous extension to $D(\beta\omega,\omega)$.*

Let D be the subalgebra of $\mathcal{P}(\omega)\times\mathcal{P}(\omega)$ consisting of all pairs (a,b) such that the symmetric difference of a and b is finite. Using Stone-duality, Corollary 3.5 implies

Corollary 3.6. *For every Boolean algebra A, $\mathrm{cof}(A)=\aleph_0$ iff there is a homomorphism $h:A\to D$ whose image contains all atoms of D.*

Koppelberg [7] constructed a Boolean algebra of countable cofinality without a countably infinite quotient. A Boolean algebra has a countably infinite quotient iff its Stone space has a non-trivial converging sequence. Since $\beta\omega$ does not have a non-trivial converging sequence, neither does $D(\beta\omega,\omega)$. So by Corollary 3.6, D is the canonical example of a Boolean algebra of countable cofinality but without a countably infinite quotient.

Since a compact space is of countable altitude iff it has a non-trivial converging sequence, both Boolean algebras, Koppelberg's example and D, answer the following question of Monk positively:

Is there a Boolean algebra of countable cofinality, whose Stone space is of uncountable altitude?

We can say a bit more about the altitude of $D(\beta\omega,\omega)$. By a result of Balcar, Simon, and Vojtas [1, Theorem 3.5], $\mathrm{a}(\beta\omega)=\aleph_1$. Since $\beta\omega$ is a closed subspace of $D(\beta\omega,\omega)$, $\mathrm{a}(D(\beta\omega,\omega))=\aleph_1$.

4. Remark on the cofinality of a Banach space

Definition 4.1. For an infinite dimensional Banach space X let $\mathrm{cof}(X)$ denote the least limit ordinal δ such that there is a strictly increasing chain $(X_\alpha)_{\alpha<\delta}$ of closed subspaces of X such that $\bigcup_{\alpha<\delta}X_\alpha$ is dense in X.

In [13] Odell asks whether (in our notation) every infinite dimensional Banach space is of countable cofinality. Johnson and Rosenthal [5] showed that a Banach space is of cofinality $\aleph_0$ iff it has an infinite dimensional separable quotient. Thus, Odell's question is equivalent to the famous *separable quotient problem* for Banach spaces, which asks whether every infinite dimensional Banach space has an infinite dimensional separable quotient.

The current state of this problem seems to be as follows: For all standard (infinite dimensional) Banach spaces it is known that they have an infinite dimensional separable quotient and thus, their cofinality is $\aleph_0$ (see [12]). The general case, however, is open. It follows from Theorem 1.1 in [10] that for every infinite compact space X, the Banach space $C(X)$ has an infinite dimensional separable quotient. This was pointed out by Lacey [9].

A direct proof might go like this: By the argument in the proof of Theorem 2.1, an infinite compact space X has a non-trivial convergent sequence or there is a closed subspace of X that maps onto 2^ω. In the first case the space c of convergent sequences of real numbers is a quotient of $C(X)$. In the second case the (Haar-)measure on 2^ω can be pulled back to a measure μ on X. The space $L^1(2^\omega)\cong L^1(X,\mu)$ embeds into the dual of $C(X)$. It is well known that ℓ^2 embeds into $L^1([0,1])\cong L^1(2^\omega)$. Hence ℓ^2 embeds into the dual of $C(X)$. It follows that

ℓ^2 is a quotient of $C(X)$. In both cases we obtain a separable, infinite dimensional quotient of $C(X)$.

By Corollary 3.4, it follows that

$$\aleph_0 = \mathrm{cof}(C(X)) < \mathrm{cof}_{C^*}(C(X)) = \aleph_1$$

for every infinite compact F-space X.

The characterization of Banach spaces of countable cofinality in terms of separable quotients fails for C^*-algebras: Let $X = D(\beta\omega, \omega)$. Then $\mathrm{cof}_{C^*}(C(X)) = \aleph_0$ by Corollary 3.5. As mentioned above, X does not have a non-trivial converging sequence. Dually, $C(X)$ has no infinite dimensional separable quotient (as a C^*-algebra).

Let us conclude by mentioning some simple facts about cofinalities of Banach spaces. For a topological space X let $\mathrm{d}(X)$ denote the *density* of X, the least size of a dense subset of X. The following lemma is the parallel of Lemma 1.4 for Banach spaces.

Lemma 4.2. *Let X be an infinite dimensional Banach space.*

(a) $\mathrm{cof}(X) \leq \mathrm{cf}(\mathrm{d}(X))$

(b) If Y is an infinite dimensional Banach space and $f : X \to Y$ is a continuous epimorphism, then $\mathrm{cof}(X) \leq \mathrm{cof}(Y)$.

(c) $\mathrm{cof}(X) \leq 2^{\aleph_0}$

Proof. (a) is easy. For (b) consider two cases. If $\mathrm{cof}(Y) = \aleph_0$, then Y has an infinite dimensional separable quotient by the result of Johnson and Rosenthal [5] mentioned above. It follows that X has an infinite dimensional separable quotient as well. By the other direction of the Johnson-Rosenthal result, X is of countable cofinality.

If $\mathrm{cof}(Y) > \aleph_0$, then Y has a dense subset of the form $\bigcup_{\alpha<\mathrm{cof}(Y)} Y_\alpha$ where each Y_α is a closed subspace of Y. But since $\mathrm{cof}(Y)$ is a cardinal of uncountable cofinality, $\bigcup_{\alpha<\mathrm{cof}(Y)} Y_\alpha$ is actually closed in Y and hence $Y = \bigcup_{\alpha<\mathrm{cof}(Y)} Y_\alpha$. For each $\alpha < \mathrm{cof}(Y)$ let $X_\alpha = f^{-1}[Y_\alpha]$. Now $(X_\alpha)_{\alpha<\mathrm{cof}(Y)}$ witnesses $\mathrm{cof}(X) \leq \mathrm{cof}(Y)$.

For (c) note that for every infinite dimensional Banach space X there is a continuous homomorphism $f : X \to \ell^\infty$ whose image is infinite dimensional. Just fix a sequence $(f_n)_{n\in\omega}$ of linearly independent functionals of norm 1 on X and put $f(x) := (f_n(x))_{n\in\omega}$ for every $x \in X$.

The image Y of f equipped with the quotient norm is an infinite dimensional Banach space of size $2^{\aleph_0}$. Now $\mathrm{cof}(X) \leq 2^{\aleph_0}$ follows from (a) and (b). □

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