SEPARABLE LINEAR ORDERS AND UNIVERSALITY

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Abstract. This is mostly a note to myself to clear up some confusion.

In various places in the literature, including [2], it is stated that every separable linear order embeds into the real line. This is, however, not the case, at least not with respect to the usual definition of separability.

Definition 1. Let $(L, \leq)$ be a linear order. $D \subseteq L$ is *dense* in L if for all $a, b \in L$ with $a < b$ there is $d \in D$ with $a < d < b$. L is *separable* if it has a countable dense subset.

Two points $x, y \in L$ form a *jump* if $x < y$ and the open interval (x, y) is empty.

Lemma 2. *Let L be a suborder of $\mathbb{R}$. Then L is separable and has only countably many jumps.*

Proof. Let $x_0 < y_0$ and $x_1 < y_1$ be two different jumps in L. Then there are $q_0, q_1 \in \mathbb{Q}$ such that for all $i \in 2$, $x_i < q_i < y_i$. Since the two jumps are different, $y_0 \leq x_1$ or $y_1 \leq x_0$. In either case, $q_0 \neq q_1$. It follows that there are not more jumps than rationals, i.e., there are only countably many jumps.

To see that L is separable, choose a countable set $D \subseteq L$ such that for all $q \in \mathbb{Q}$ and all $n > 0$ the following holds: if $L \cap (q - 1/n, q + 1/n) \neq \emptyset$, then $D \cap (q - 1/n, q + 1/n) \neq \emptyset$. It is easily checked that D is dense in L. $\square$

Example 3. Consider the set $\mathbb{R} \times 2$ ordered lexicographically. Then $\mathbb{Q} \times 2$ is dense in $\mathbb{R} \times 2$ and hence $\mathbb{R} \times 2$ is separable. But $\mathbb{R} \times 2$ has uncountably many jumps and therefore does not embed into $\mathbb{R}$.

Theorem 4. *If L is any separable linear order, then there is an order embedding $e : L \to \mathbb{R} \times 2$.*

Proof. Let D be a countable dense subset of L. We may assume that D contains the first and last element of L provided they exist. By the saturation of $\mathbb{Q}$, there is an order embedding $i : D \to \mathbb{Q}$. For each $x \in L$ let

$$e_1(x) = \sup\{i(d) : d \in D \wedge d \leq x\}.$$

Date: July 20, 2012.

Now $e_1 : L \to \mathbb{R}$ preserves $\leq$, but we have $e_1(x) = e_1(y)$ if

$$x > y = \sup\{d \in D : d \leq x\}.$$

Note that this can only happen if x is the successor of y in L and $x \notin D$.

To correct this failure of injectivity, we embed into $\mathbb{R} \times 2$ rather than $\mathbb{R}$. For $x \in L$ let

$$e(x) = \begin{cases} (e_1(x), 0), \text{ if } x = \sup\{d \in D : d \leq x\} \text{ and} \\ (e_1(x), 1), \text{ if } x > \sup\{d \in D : d \leq x\}. \end{cases}$$

□

The proof of Theorem 4 suggests the following notion:

Definition 5. Let L be a linear order. A set $D \subseteq L$ is *left dense* if for all $x \in L$, $x = \sup\{d \in D : d \leq x\}$. We define *right dense* analogously, using inf instead of sup.

L is *left* (*right*) *separable* if L has a countable left (right) dense subset.

Theorem 6. *For every linear order L the following are equivalent:*

(1) *L is left separable.*
(2) *L is right separable.*
(3) *L is separable and has only countably many jumps.*
(4) *L order embeds into $\mathbb{R}$.*

Proof. If L is left separable, then the map e_1 in the proof of Theorem 4 embeds L into $\mathbb{R}$. This shows the implication from (1) to (4). The implication from (2) to (4) now follows symmetrically. If L embeds into $\mathbb{R}$, then L is separable and has only countably many jumps by Lemma 2. Hence (4) implies (3)

If L is separable and has only countably many jumps, let $D \subseteq L$ be countable, dense, and such that for each jump $x < y$, $x, y \in D$. It is easily checked that D is both left and right dense. Hence (3) implies both (1) and (2) □

Note that if $x < y$ is a jump of a linear order L and there is an automorphism φ of L that maps x to y, then y and hence x is isolated. It follows that a separable homogeneous linear order either has no jumps and therefore embeds into $\mathbb{R}$ or all its elements are isolated and hence the linear order is isomorphic to the integers $\mathbb{Z}$. It follows that $\mathbb{R}$ is universal for homogeneous separable linear orders, but no universal separable linear order is homogeneous.

Another way to analyze the situation is this: given a linear order L, we consider two subsets definable without parameters, namely the set $J_\ell(L)$ of left partners of a jump and the set $J_r(L)$ of right partners of a jump. Also, there is a binary relation that can be defined without parameters, namely the relation $J(L)$ where $(x, y) \in J(L)$ if $x < y$ or $y < x$ is a jump.

Every automorphism of L preserves $J_\ell(L)$, $J_r(L)$, and the relation $J(L)$. Therefore it makes sense to define homogeneity as follows:

Definition 7. A linear order L is *jump homogeneous* if for all finite sets $A, B \subseteq L$ every bijection $b : A \to B$ that preserves the relations $<$, $J_\ell(L)$, $J_r(L)$, and $J(L)$ extends to an automorphism of L.

Lemma 8. *The linear order* $\mathbb{R} \times 2$ *is jump homogeneous.*

Proof. Let $A, B \subseteq \mathbb{R}$ be finite sets and let $f : A \to B$ a bijection that preserves $<$, $J_\ell(L)$, $J_r(L)$, and $J(L)$. Note that $J(L)$ is an equivalence relation. Since f preserves $J(L)$, it induces a a bijection $\overline{f} : A/J(L) \to B/J(L)$. Since $\mathbb{R}$ is homogeneous and $(\mathbb{R} \times 2)/J(L) \cong \mathbb{R}$, $\overline{f}$ extends to an automorphism $\overline{g}$ of $(\mathbb{R} \times 2)/J(L)$. Now $\overline{g}$ lifts to an automorphism g of $\mathbb{R} \times 2$. Since f preserves $J_\ell(L)$ and $J_r(L)$, g extends f. $\square$

We now discuss the existence of a universal separable linear order of size $\aleph_1 < 2^{\aleph_0}$.

Definition 9. A set $S \subseteq \mathbb{R}$ is $\aleph_1$-dense if for all $x, y \in \mathbb{R}$ with $x < y$, $S \cap (x, y)$ is of size $\aleph_1$.

Baumgartner showed the following [1]:

Theorem 10. *It is consistent with* $2^{\aleph_0} = \aleph_2$ *that any two* $\aleph_1$*-dense set of reals are order isomorphic.*

Corollary 11. *It is consistent that there is a universal separable linear order of size* $\aleph_1 < 2^{\aleph_0}$.

Proof. By Baumgartner's result, we can assume that any two $\aleph_1$-dense sets of reals are isomorphic and $\aleph_1 < 2^{\aleph_0}$. Let L be a separable linear order of size $\aleph_1$. By Theorem 4, L is isomorphic to a subset of $\mathbb{R} \times 2$. Let $S \subseteq \mathbb{R}$ be of size $\aleph_1$ such that L embeds into $S \times 2$. By enlarging S if necessary, we may assume that S is $\aleph_1$-dense.

It follows that every separable linear order of size $\aleph_1$ embeds into an order of the form $S \times 2$ where S is an $\aleph_1$-dense subset of $\mathbb{R}$. But by our assumption, any two $\aleph_1$-dense subsets of $\mathbb{R}$ and therefore also any two linear orders of the form $S \times 2$ with $S \subseteq \mathbb{R}$ $\aleph_1$-dense are isomorphic. It follows that every linear order of the form $S \times 2$ with S $\aleph_1$-dense is universal for separable linear orders of size $\aleph_1$. $\square$

References

[1] J. Baumgartner, *All* $\aleph_1$*-dense sets of reals can be isomorphic*, Fund. Math. 79 (1973), no. 2, 101–106

[2] S. Geschke, M. Kojman, *Metric Baumgartner theorems and universality*, Math. Res. Lett. 14 (2007), no. 2, 215–226

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