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On Sacks Forcing and the Sacks Property

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Abstract. In this survey we explain the general idea of forcing, present Sacks forcing with some of its properties, give an overview of closely related forcing notions, and investigate the influence of some combinatorial principles to the question whether a c.c.c. forcing notion can have the Sacks property. Moreover, we discuss the iterated Sacks model, which is the standard model of set theory in which the Continuum Hypothesis fails, but where all other reasonably defined cardinal characteristics of the continuum have the minimal possible value. We also discuss the countable support side-by-side product of Sacks forcing.

0 Introduction and outline of the paper

The technique of forcing was invented by Paul Cohen [Coh63] in order to produce a model of the commonly accepted system of set-theoretic

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axioms, ZFC, the Zermelo-Fraenkel axioms ZF together with the axiom of choice, in which the Continuum Hypothesis (CH) fails. The existence of such a model shows that CH does not follow from ZFC. Earlier it was shown by Gödel (see [Kun80, Chapter VI]) that CH cannot be refuted from ZFC, assuming of course that ZFC is consistent.

Forcing turned out to be a very powerful tool for proving independence results in mathematics. Starting from a model V, the *ground model*, which satisfies ZFC, a partial order $P \in V$ is fixed which codes the desired properties of the model one wishes to construct. Then forcing with P over V yields a generic extension $V[G]$ where all this information is decoded and ZFC holds as well. $V[G]$ is obtained by adding a certain generic object G (a subset of P) to V. $V[G]$ is the smallest model of ZFC which extends V and contains G.

Sacks forcing is one such partial order and was invented by Gerald Sacks to produce a minimal forcing extension $V[G]$ in the following sense: If W is a model of ZFC such that $V \subseteq W \subseteq V[G]$, then either $W = V$ or $W = V[G]$. In this article we give an overview of the properties and applications of Sacks forcing.

In Section 1 we illustrate the idea of forcing by an example. The reader who is interested in learning the details of forcing is referred to [Kun80], [Jec78], or [Jec02]. Our notation follows [Kun80] and whenever we state a fact about forcing without mentioning a source, this fact can be found in [Kun80] or, if Borel sets and descriptive set theory are involved, in [Jec78]. This section is ment as a sketchy introduction to forcing, and it certainly does not replace some detailed treatment as in the books mentioned above.

After that, in Section 2, we introduce Sacks forcing and study some of its properties. We isolate the so-called Sacks property and mention some of its consequences in Section 3. Section 4 is devoted to relatives of Sacks forcing which share some of its important features. Sacks forcing does not satisfy the countable chain condition (c.c.c.). So in Section 5 we wonder whether there is any c.c.c. forcing that has the Sacks property.

We then turn to iterations of Sacks forcing. Section 6 deals with countable support side-by-side products of Sacks forcing and in Section 7 we discuss the countable support iteration of Sacks forcing of length ω_2.

1 Collapsing the continuum to $\aleph_1$

When Gödel proved that CH is consistent with ZFC, he started from a model V of ZFC and constructed a definable class L, the class of constructible sets, inside V such that L satisfies ZFC+CH. (Actually, Gödel started from a model of ZF and produced a model of ZFC+CH.) Forcing works the opposite way. We start from a model V of ZFC and extend it. As usual, we will frequently talk about *models of set theory*, which just means models of ZFC. Whenever we consider two models V_0, V_1 of set theory such that $V_0 \subseteq V_1$, we will tacitly assume that V_0 is transitive in V_1, i.e., for all $x \in V_0$ and $y \in V_1$, if $y \in x$ in V_1, then $y \in V_0$. This guarantees that V_0 and V_1 agree on elementary properties of sets, for instance whether a set $x \in V_0$ is an ordinal or not. (See [Kun80, IV.3] for more information on this subject.)

Suppose we wish to construct a model of ZFC+CH by forcing. Let V be a model of ZFC. If V satisfies CH, we are done. Otherwise, in V the cardinal $2^{\aleph_0}$, the size of $\mathbb{R}$ and of $\mathcal{P}(\omega)$, is bigger than $\aleph_1$. In this case we try to add a function $f : \aleph_1 \to \mathcal{P}(\omega)$ which is onto. Since V does not satisfy CH, no such function exists in V. However, for every countable ordinal α, $f \restriction \alpha$ could be an element of V. We define a *forcing notion* P, i.e., a partial order, which consists of possible initial segments of a surjective map $f : \aleph_1 \to \mathcal{P}(\omega)$.

Let $P := \{p : \exists \alpha < \omega_1 (p : \alpha \to \mathcal{P}(\omega))\}$. The order on P is reverse inclusion. More precisely, for $p, q \in P$ let $p \leq q$ iff $q \subseteq p$, i.e., if p extends q. If $p \leq q$, we say that p is *stronger* than q. The elements of P are the *forcing conditions* or just *conditions*. Two conditions $p_0, p_1 \in P$ are *compatible* iff there is $q \in P$ such that $q \leq p_0, p_1$. Otherwise p_0 and p_1 are *incompatible*. In this case we write $p_0 \perp p_1$. Our particular forcing notion P is a tree of height $\aleph_1$ (the root is the empty function, and stronger (i.e., smaller with respect to $\leq$) conditions are higher up the tree). In general, a forcing notion can be any partial order. For technical reasons, we assume that every partial order Q has a largest element 1_Q.

We now describe how we get a surjection $f : \aleph_1 \to \mathcal{P}(\omega)$ from P. $G \subseteq P$ is a *filter* iff

(i) for all $p, q \in G$ there is $r \in G$ such that $r \leq p, q$ and
(ii) for all $p \in G$ and all $q \in P$ with $p \leq q$, $q \in G$.

In our case, a filter in P is the same as a branch in the tree P. If $G \subseteq P$ is a filter such that $\{\alpha < \omega_1 : \exists p \in G(p : \alpha \to \mathcal{P}(\omega))\} = \omega_1$, then $f_G := \bigcup G$ is a function from $\aleph_1$ to $\mathcal{P}(\omega)$. However, if G is an element of V, then f_G is not onto $\mathcal{P}(\omega)$ since $2^{\aleph_0} > \aleph_1$ in V. This is where *genericity* comes into play. We have to make sure that G is sufficiently complicated. $D \subseteq P$ is *dense* in P if for all $p \in P$ there is q in D such that $q \leq p$. A filter $G \subseteq P$ is P*-generic* over V if it has nonempty intersection with every $D \in V$ which is dense in P. A set $D \subseteq P$ is *dense below* a condition $p \in P$ if for all $q \in P$ with $q \leq p$ there is $q' \in D$ with $q' \leq q$. It is not difficult to check that if D is dense below p and G is P-generic over V with $p \in G$, then $D \cap G \neq \emptyset$.

Except for trivial cases, filters that are generic over V cannot be elements of V. Thus, we need someone outside V who chooses a P-generic filter over V. It can be shown that for each partial order Q in V it is safe (i.e., it does not lead to a contradiction unless ZFC itself is contradictory) to assume that a Q-generic filter over V exists. More precisely, whenever we consider a specific forcing notion Q it will have a reasonable description. If the existence of a forcing notion with this description is consistent with ZFC, then there is a model V of set theory such that in V we have a partial order Q' satisfying the description of Q and there is a model W of set theory such that $V \subseteq W$ and in W there is a Q'-generic filter G over V.

So, let G be P-generic over V. For every $x \in \mathcal{P}(\omega)$ the set $D_x := \{p \in P : x \in \operatorname{ran}(p)\}$ is dense in P. Since $D_x \in V$ and G is generic, there is $p \in G \cap D_x$, i.e., there is $p \in G$ such that $x \in \operatorname{ran}(p)$. This implies that $f := f_G$ is a function onto $\mathcal{P}(\omega)$. Similarly, for every $\alpha < \omega_1$, the set $D^\alpha := \{p \in P : \alpha \in \operatorname{dom}(p)\}$ is dense in P. It follows that for every $\alpha < \omega_1$ there is $p \in G$ such that $\alpha \in \operatorname{dom}(p)$. This implies

$\mathrm{dom}(f) = \aleph_1$. It follows that $f : \aleph_1 \to \mathcal{P}(\omega)$ is onto.

After constructing the desired map f from G, we have to explain how to get a model $V[G]$ of ZFC from V and G. Roughly speaking, $V[G]$ consists of all sets that are definable from G together with parameters from V. More precisely, a class of so-called *P-names* is defined in V. Then, by means of some simple algorithm, from every P-name $\dot{x} \in V$ a set $\dot{x}_G$ is computed using G. The set $\dot{x}_G$ is the *evaluation of $\dot{x}$ with respect to G*. $V[G]$ consists of all $\dot{x}_G$. Every element y of V has a *canonical P-name* $\check{y}$ with the property that for every P-generic G, $\check{y}_G = y$. In other words, V is included in $V[G]$. Moreover, there is a name $\dot{G}$ such that for every P-generic filter G, $\dot{G}_G = G$. $V[G]$ turns out to be a model of ZFC, and it has the same ordinals as V.

In $V[G]$ we have the function $f = f_G$, as defined above. The function f has at least one name in V, so we can pick one and call it $\dot{f}$. Thus, f is the interpretation of the name $\dot{f}$ under G, and we denote this fact by $f = \dot{f}_G$. We argued that $f : \aleph_1 \to \mathcal{P}(\omega)$ is onto. However, we were talking about the $\aleph_1$ and $\mathcal{P}(\omega)$ of V, not of $V[G]$. Formally, the symbols $\aleph_1$ and $\mathcal{P}(\omega)$ stand for definitions of certain sets. $\mathcal{P}(\omega)$ is the power set of the first limit ordinal and $\aleph_1$ is the first ordinal α such that there is no map from ω onto α. As it turns out, ω is the same in V as in $V[G]$, and this holds for every forcing extension. Let $\aleph_1^V$ and $\aleph_1^{V[G]}$ denote the first uncountable ordinal in V, respectively in $V[G]$. $\mathcal{P}(\omega)^V$ and $\mathcal{P}(\omega)^{V[G]}$ are defined similarly. In order to prove that $V[G]$ is a model of CH it is enough to show $\aleph_1^V = \aleph_1^{V[G]}$ and $\mathcal{P}(\omega)^V = \mathcal{P}(\omega)^{V[G]}$. Both of these statements follow from the fact that $V[G]$ does not contain any countable sequences of ordinals that are not already elements of V (i.e., there are no *new* countable sequences of ordinals), which we prove in a moment.

We need some information on how properties of $V[G]$ are connected to the properties of P. Let $\varphi(x_1, \dots, x_n)$ be a formula in the language of set theory (first order logic with the binary relation symbol $\in$) with all free variables among $x_1, \dots, x_n$. Then for every condition $p \in P$ and P-names $\dot{x}_1, \dots, \dot{x}_n$ we say that *p forces* $\varphi(\dot{x}_1, \dots, \dot{x}_n)$ (where $\varphi(\dot{x}_1, \dots, \dot{x}_n)$ denotes φ with every free x_i replaced by the name $\dot{x}_i$) iff for every P-generic filter G with $p \in G$, $V[G]$ satisfies $\varphi((\dot{x}_1)_G, \dots, (\dot{x}_n)_G)$. In this case we write $p \Vdash_P \varphi(\dot{x}_1, \ldots, \dot{x}_n)$. It is crucial for the theory of forcing that the relation $\Vdash_P$ is actually definable in V. Moreover, we have the following important fact:

Theorem 11 (The Forcing Lemma) *Let Q be a forcing notion and suppose that G is Q-generic over V. Then*

$$V[G] \models \varphi((\dot{x}_1)_G, \ldots, (\dot{x}_n)_G)$$

(i.e., $\varphi((\dot{x}_1)_G, \ldots, (\dot{x}_n)_G)$ is true in $V[G]$) iff there is $q \in G$ such that

$$q \Vdash_P \varphi(\dot{x}_1, \ldots, \dot{x}_n).$$

Thus, nothing in $V[G]$ depends on chance, everything is forced at some point. We say that a condition p *decides* a statement φ if $p \Vdash \varphi$ or $p \Vdash \neg\varphi$. Similarly, if $\dot{x}$ is a name, we say that p *decides* $\dot{x}$ if there is some ground model set x such that $p \Vdash \dot{x} = \check{x}$. It is worth noting that for conditions p and q with $q \leq p$, q forces everything that p forces. Moreover, for every condition p and every statement φ there is $q \leq p$ such that q decides φ.

Returning to our example, how can we see that there are no new countable sequences of ordinals? The forcing notion P is *σ-closed*, that is, if $(p_i)_{i<\omega}$ is a sequence of conditions in P such that $p_{i+1} \leq_P p_i$ for each i, then there is some $p \in P$ such that $p \leq p_i$ for every $i \in \omega$. Just let $p := \bigcup_{i\in\omega} p_i$.

Assume there is a P-name $\dot{h}$ and a condition p which forces that $\dot{h}$ is a function from ω to the ordinals. Let $p' \leq p$ be arbitrary. Then, using Theorem 11 and the remarks following it, we find a decreasing sequence $(q_i)_{i<\omega}$ of conditions such that $q_0 \leq_P p'$ and for each i there is some ordinal α_i such that $q_i \Vdash_P \dot{h}(i) = \alpha_i$. (Formally, the last part should have been stated as $q_i \Vdash_P \dot{h}(\check{i}) = \check{\alpha}_i$ since we have to use names to the right of the forcing relation. However, we identify the elements of V with their canonical names as long as it is clear what we mean.)

Let $q \in P$ be such that $q \leq q_i$ for all $i \in \omega$. Now for all $i \in \omega$, $q \Vdash \dot{h}(i) = \alpha_i$. In other words, q forces that $\dot{h}$ is the sequence $(\alpha_i)_{i\in\omega}$, which is an element of the ground model. Since $q \leq p'$ and p' was arbitrary below p, the set of conditions that force $\dot{h}$ to be a sequence in the ground model is dense below p. It follows that every generic filter G that contains p intersects this set. Thus, whenever G is P-generic over V and $p \in G$, then $\dot{h}_G \in V$. It follows that for every P-generic filter G over V, in $V[G]$ there are no new countable sequences of ordinals and thus, $\aleph_1$ and $\mathcal{P}(\omega)$ are the same in $V[G]$ as in V. But this implies that $V[G]$ is a model of ZFC+CH since in $V[G]$ there is a surjection from $\aleph_1$ onto $\mathcal{P}(\omega)$.

It is worth mentioning that if p forces $\dot{h}$ to be a function from ω to the ordinals, then we may already assume that every condition q forces $\dot{h}$ to be a function from ω to the ordinals. This is because we can replace $\dot{h}$ by a name $\dot{g}$ such that every condition q forces "$\dot{g}$ is a function from ω to the ordinals and if $\check{p} \in \dot{G}$, then $\dot{h} = \dot{g}$". The existence of $\dot{g}$ follows from the next theorem, the so-called *Maximality Principle*.

Theorem 12 *Let $\varphi(x, y_1, \ldots, y_n)$ be a formula in the language of set theory. Let Q be a forcing notion, $q \in Q$, and suppose that $\dot{y}_1, \ldots, \dot{y}_n$ are Q-names such that*

$$q \Vdash \exists x \varphi(x, \dot{y}_1, \ldots, \dot{y}_n).$$

Then there is a Q-name $\dot{x}$ such that

$$q \Vdash \varphi(\dot{x}, \dot{y}_1, \ldots, \dot{y}_n).$$

In particular, if every $q \in Q$ forces $\exists x \varphi(x, \dot{y}_1, \ldots, \dot{y}_n)$, then 1_Q forces this and thus, by Theorem 12, there is a Q-name $\dot{x}$ such that every $q \in Q$ forces $\varphi(\dot{x}, \dot{y}_1, \ldots, \dot{y}_n)$.

2 Sacks forcing and its properties

The forcing notion which we introduced in Section 1 for collapsing $2^{\aleph_0}$ to $\aleph_1$ was a rather special case. As mentioned before, forcing was invented to enlarge $2^{\aleph_0}$, not to make it smaller. Enlarging $2^{\aleph_0}$ can be done by adding new reals, where a "real" is an element of one of the sets $\mathcal{P}(\omega)$, 2^ω, ω^ω, $\mathbb{R}$, and $[0, 1]$. In our context, usually it does not matter whether we look at $\mathcal{P}(\omega)$, 2^ω, ω^ω, $\mathbb{R}$, or $[0, 1]$, and here is why:

The first two spaces can be identified in a natural way. The space ω^ω is homeomorphic to the set of irrational numbers, a subset of $\mathbb{R}$ whose complement is countable and therefore neglegible for our purposes. The compact unit interval is a continuous image of 2^ω via a mapping that fails to be injective only on countably many places, i.e., again on a neglegible set. Finally, $[0,1]$ is homeomorphic to the two-point compactification of $\mathbb{R}$. These mappings, which are almost bijections between the different spaces, are all nicely definable and show that once we know all the elements of one of those spaces, we know all the elements of the other spaces too. Moreover, if we investigate some frequently studied ideals such as the ideal of measure zero sets, of meager sets, or of countable sets, it turns out that, as far as the basic properties of these ideals concerned, it does not matter on which incarnation of "the reals" the respective ideals are considered.

The forcing notion used by Cohen to add new reals is now called *Cohen forcing* and is defined as follows. Let C be the set $2^{<\omega}$ of all finite sequences of 0's and 1's. C is ordered by reverse inclusion. If G is C-generic over the ground model, then the *Cohen real* added by C is just $\bigcup G$, a function from ω to 2. Thus, Cohen forcing uses finite approximations to create a new real. To enlarge the continuum using C one has to use iterate Cohen forcing in order to add many new reals. Iterated forcing will be discussed later.

Let us consider another description of Cohen forcing. Whenever we consider a Boolean algebra B as a forcing notion, we mean the partial order $(B \setminus \{0_B\}, \leq)$ where $\leq$ is the natural order on B. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partial orders. A mapping $e : P \to Q$ is a *dense embedding* iff the following conditions hold:

(i) $\forall p, p' \in P(p \leq_P p' \Rightarrow e(p) \leq_Q e(p'))$
(ii) $\forall p, p' \in P(p \perp_P p' \Rightarrow e(p) \perp_Q e(p'))$
(iii) $\forall q \in Q \exists p \in P(e(p) \leq_Q q)$

Note that we do not demand e to be 1-1. But condition (ii) implies that for all $p, p' \in P$ with $e(p) = e(p')$ and every P-generic filter G we have $p \in G$ iff $p' \in G$. Every partial order P has a so-called *completion*, a complete Boolean algebra B such that P densely embeds into $(B \setminus \{0_B\}, \leq)$. B is unique up to isomorphism.

We call two partial orders P and Q *forcing equivalent* (or just *equivalent*) iff they have isomorphic completions. If P and Q are equivalent, then they produce the same generic extensions. Therefore, equivalent partial orders can be considered the same as far as forcing is concerned.

Many forcing notions adding generic reals can be described in the following way: Let $\mathrm{Bor}(\mathbb{R})$ denote the Boolean algebra of Borel subsets of the reals. If $\mathcal{I} \subseteq \mathrm{Bor}(\mathbb{R})$ is an ideal, then $\mathrm{Bor}(\mathbb{R})/\mathcal{I}$ is a Boolean algebra. As a forcing notion, this Boolean algebra is equivalent to $(\mathrm{Bor}(\mathbb{R})\setminus\mathcal{I}, \subseteq)$. Let $\mathcal{M}$ denote the ideal of meager subsets of the real line. Then Cohen forcing is equivalent to $\mathrm{Bor}(\mathbb{R})/\mathcal{M}$ since $\mathrm{Bor}(\mathbb{R})/\mathcal{M}$ has a dense subset which is isomorphic to C.

Another important ideal in $\mathrm{Bor}(\mathbb{R})$ is the ideal $\mathcal{N}$ of Borel sets of measure zero. The forcing notion $\mathrm{Bor}(\mathbb{R})/\mathcal{N}$ is called *random real forcing* and adds a *random real*. Both forcing notions, Cohen and random real forcing satisfy the *countable chain condition* (c.c.c.), that is, they only have countable *antichains*. Here an antichain is a set of pairwise incompatible elements. Forcing notions with the c.c.c. do not collapse cardinals. More exactly, if P is c.c.c. and G is P-generic over the ground model V, then an ordinal α is a cardinal in $V[G]$ iff it is a cardinal in V.

If $\mathcal{I} \subseteq \mathrm{Bor}(\mathbb{R})$ is a σ-ideal, then the generic real added by $\mathrm{Bor}(\mathbb{R})/\mathcal{I}$ is determined as follows: We work with $\mathrm{Bor}(\mathbb{R})\setminus\mathcal{I}$ instead of $\mathrm{Bor}(\mathbb{R})/\mathcal{I}$. Let G be $\mathrm{Bor}(\mathbb{R}) \setminus \mathcal{I}$-generic over the ground model V. Then $\bigcap G$ is nonempty and contains a unique real x_G, and this real is the *generic real* added by $\mathrm{Bor}(\mathbb{R})/\mathcal{I}$. While x_G is not a generic filter for some partial order, G can be recovered from x_G (i.e., G and x_G are interdefinable), and thus it makes sense to talk about x_G as the generic object added by $\mathrm{Bor}(\mathbb{R})/\mathcal{I}$. In a moment we will see how G can be recovered from x_G.

To describe the properties of x_G, we have to mention *Borel codes* (see [Jec78, page 537]). A Borel set $X \subseteq \mathbb{R}$ is not merely a set of real numbers, but it has a description of how it can be obtained from open sets by taking complements and unions over countable families. This description can be coded as an element of ω^ω, and this code is a Borel code for X. (Note that a Borel code for a given set X is not unique.)

In fact, usually mathematicians use Borel codes when they are talking about Borel sets. E.g., they are talking about the open set $(0, 1)$ instead of thinking about every single real inside this interval. The notation $(0,1)$ can be considered as a Borel code for a certain open set. This Borel code has different interpretations in different models of set theory. However, usually the structure of a set matters, and the individual elements which are inside are unimportant. When we talk about a Borel set X in two different models of set theory, we really choose a Borel code for X and then identify the two Borel sets that are obtained in the respective models by applying the definition of a Borel set coded by the Borel code. This does not depend on the particular Borel code we chose for X.

The generic real x_G has the property that it is not an element of any element of $\mathcal{I}$. More precisely, let $X \in \mathcal{I}$. Then X is Borel and thus has a Borel code. (We work in V, so far.) Now the Borel set in $V[G]$ which has this Borel code does not contain x_G. Roughly speaking, x_G avoids all sets from $\mathcal{I}$.

Now G can be recovered from x_G as follows: Let $(\mathrm{Bor}(\mathbb{R}))^V$ denote the set of ground model Borel sets. Then G is the set of all $X \in (\mathrm{Bor}(\mathbb{R}))^V$ which in $V[G]$ contain x_G. Since x_G avoids all elements of the ideal $\mathcal{I}$, the ground model Borel sets which in $V[G]$ contain x_G are indeed elements of $(\mathrm{Bor}(\mathbb{R}))^V \setminus \mathcal{I}$. By the definition of x_G, every element of G contains x_G (in $V[G]$). On the other hand, if X is a ground model Borel set which in $V[G]$ contains x_G, then X is an element of $(\mathrm{Bor}(\mathbb{R}))^V \setminus \mathcal{I}$ which is compatible with all elements of G since the intersection of X with an arbitrary element of G is again a ground model Borel set that in $V[G]$ contains x_G. It is a general fact about generic filters that they contain a condition iff the condition is compatible with all the elements of the filter. This implies $X \in G$. (Note that all the time we considered $\mathcal{I}$ as the actual set of ground model Borel sets, i.e., as an ideal in $(\mathrm{Bor}(\mathbb{R}))^V$. Usually $\mathcal{I}$ fails to be an ideal in $\mathrm{Bor}(\mathbb{R})$ in $V(G)$. If $\mathcal{I}$ is nicely definable like $\mathcal{M}$ and $\mathcal{N}$, then typically the ground model ideal $\mathcal{I}$ is different from the ideal in $V[G]$ which is obtained by applying the definition of $\mathcal{I}$ in $V[G]$.)

One σ-ideal of $\mathrm{Bor}(\mathbb{R})$ stands out as the smallest σ-ideal containing all the singletons, namely the ideal $\mathtt{countable}$ of countable sets. *Sacks forcing* is $\mathrm{Bor}(\mathbb{R})/\mathtt{countable}$. A *Sacks real* is a generic real for Sacks forcing.

There is another ideal related to Sacks forcing, *Marczewski's ideal* s^0 introduced in [Mar35]. (See [Bre95] for some information on Marczewski's ideal and other similarly defined ideals.) A set $X \subseteq \mathbb{R}$ is *perfect* iff it is nonempty, closed, and has no isolated points. A set $X \subseteq \mathbb{R}$ is in the ideal s^0 iff every perfect set $P \subseteq \mathbb{R}$ has a perfect subset $Q \subseteq P$ which is disjoint from X. Note that no perfect set is in s^0. A set $X \subseteq \mathbb{R}$ is *s-measurable* iff for every perfect set $P \subseteq \mathbb{R}$ there is a perfect set $Q \subseteq P$ such that either $Q \cap X = \emptyset$ or $Q \subseteq X$. The set of s-measurable sets is called the *Marczewski field*.

A well known theorem due to Alexandroff and Hausdorff says that every uncountable Borel set includes a perfect set (see [Jec78, Theorem 94]). It follows that every Borel set is s-measurable and that the Borel sets in s^0 are precisely the countable sets. Moreover, every s-measurable set $X \subseteq \mathbb{R}$ which is not in the ideal s^0 has a perfect subset. And perfect sets are clearly Borel. It follows that there is a dense embedding from the partial order $\mathrm{Bor}(\mathbb{R}) \setminus \texttt{countable}$ into the partial order of s-measurable sets that are not in s^0. This implies that the Boolean algebra $\mathrm{Bor}(\mathbb{R})/\texttt{countable}$ and the quotient of the Boolean algebra of s-measurable sets modulo the ideal s^0 have isomorphic completions and are thus forcing equivalent.

We give a combinatorial description of Sacks forcing. First note that every perfect set $X \subseteq \mathbb{R}$ has a perfect subset which is homeomorphic to 2^ω. (See [Jec78, Lemma 4.2] for this. The statement follows from the usual proof of the fact that every perfect set is of size $2^{\aleph_0}$.) It follows that the copies of 2^ω are dense in $\mathrm{Bor}(\mathbb{R}) \setminus \texttt{countable}$. Therefore forcing with the partial order of perfect subsets of 2^ω gives the same generic extensions as forcing with $\mathrm{Bor}(\mathbb{R})/\texttt{countable}$.

Let X be a subset of 2^ω. Then the set $T(X) := \{x \upharpoonright n : n \in \omega \wedge x \in X\}$ of initial segments of elements of X is a subtree of the tree $(2^{<\omega}, \subseteq)$. For a subtree T of $2^{<\omega}$ let $[T] := \{x \in 2^\omega : \forall n \in \omega(x \upharpoonright n \in T)\}$ be the set of branches through T. For every subtree T of $2^{<\omega}$ the set $[T]$ is closed. For every closed set $X \subseteq 2^\omega$, $[T(X)] = X$. In this translation between closed subsets of 2^ω and subtrees of $2^{<\omega}$ the perfect sets correspond to *perfect trees*. A nonempty subtree T of $2^{<\omega}$ is perfect iff for every $s \in T$ there are $t_0, t_1 \in T$ such that $s \subseteq t_0, t_1$ and t_0 and t_1 are incomparable with respect to $\subseteq$.

This gives rise to our official definition of Sacks forcing S.

Definition 21 *Sacks forcing is the set S of all perfect subtrees of $2^{<\omega}$ ordered by inclusion.*

With this definition of Sacks forcing, whenever we refer to a *Sacks real* we mean the unique element x of $\bigcap_{p\in G}[p]$ where G is an S-generic filter. If x is a Sacks real and this is witnessed by G, then G is the set of all perfect trees p in the ground model such that $x \in [p]$.

It is quite obvious that Sacks forcing does not have the c.c.c. In fact, there are antichains of size $2^{\aleph_0}$: Fix an almost disjoint family $\{A_\alpha \mid \alpha < 2^{\aleph_0}\}$ of subsets of ω, and for each $\alpha < 2^{\aleph_0}$ choose a perfect tree T_α whose splitting levels are exactly the elements of A_α, for example, $T_\alpha := \{s \in 2^{<\omega} : \forall n < |s|(n \notin A_\alpha \to s(n) = 0)\}$. If $\alpha \neq \beta$, then $T_\alpha \cap T_\beta$ includes no perfect tree, so they are incompatible. Nevertheless, S does not collapse $\aleph_1$ either, since it satisfies Baumgartner's Axiom A [Bau83, Section 7]:

Definition 22 *A forcing notion $(P, \leq)$ satisfies Axiom A iff there is a decreasing chain of partial orders $\leq = \leq_0 \supseteq \leq_1 \supseteq \leq_2 \dots$ on P such that*

(i) for every sequence $(p_n)_{n<\omega}$ in P such that for all $n \in \omega$, $p_{n+1} \leq_n p_n$ there is $p \in P$ such that for all $n \in \omega$, $p \leq_n p_n$ (such a sequence $(p_n)_{n\in\omega}$ is called a fusion sequence *and p is a* fusion *of $(p_n)_{n\in\omega}$) and*

(ii) if $A \subseteq P$ is an antichain and $p \in P$, then for every $n \in \omega$ there is $q \leq_n p$ such that q is compatible with at most countably many members of A.

Axiom A can be seen as a generalisation of the c.c.c. since each c.c.c. forcing satisfies Axiom A: Just let $\leq_n$ be equality for all $n > 0$. Axiom A forcings belong to a wider class of forcing notions preserving $\aleph_1$, namely the class of proper forcings [She98, III.1]. Properness was introduced by Shelah as a strengthening of "not collapsing $\aleph_1$" which behaves nicely with respect to iterations. Countable support iterations of proper forcings are again proper while countable support iterations of

$\aleph_1$-preserving forcing notions may collapse $\aleph_1$ [She98, III.3].

Instead of invoking properness it can be seen very quickly that Axiom A suffices to guarantee that $\aleph_1$ is preserved.

Lemma 23 *Let P be a forcing notion satisfying Axiom A. Then for every P-generic filter G over the ground model V the following holds: If x is a countable set of ordinals in $V[G]$, then there is a set $y \in V$ such that $x \subseteq y$ and y is countable in V. In particular, forcing with P does not collapse $\aleph_1$.*

PROOF: Let $\dot{x}$ be a P-name and suppose that $p \in P$ forces that $\dot{x}$ is a countable set of ordinals. We will construct a fusion sequence $(q_n)_{n\in\omega}$ starting with $q_0 = p$ and a countable set y such that for every fusion q of this sequence, $q \Vdash \dot{x} \subseteq \check{y}$. We start by choosing a name $\dot{h}$ for a function from ω into the ordinals such that p forces $\dot{h}$ to be a function from ω onto $\dot{x}$. The existence of $\dot{h}$ is guaranteed by the Maximality Principle. Let $n \in \omega$ and suppose we have already defined q_n. The set of conditions deciding $\dot{h}(n)$ is dense in P. It follows that there is a maximal antichain $A_n \subseteq P$ consisting of conditions that decide $\dot{h}(n)$. By condition (ii) of Axiom A there is $q_{n+1} \leq_n q_n$ such that q_{n+1} is compatible with at most countably many elements of A_n.

Let q be a fusion of $(q_n)_{n\in\omega}$, and put

$$y := \{\alpha : \exists q' \leq q \exists n \in \omega (q' \Vdash \dot{h}(n) = \alpha)\}.$$

Then, whenever G is P-generic over V and $q \in G$, for every $n \in \omega$ there is $q' \in G$ such that $q' \leq q$ and $q' \Vdash \dot{h}(n) = \check{\alpha}$ where $\alpha = \dot{h}_G(n)$. (Recall that $V[G]$ has the same ordinals as V.) It follows that $\dot{x}_G = \dot{h}_G[\omega] \subseteq y$.

It remains to show that y is countable. Let $n \in \omega$ and $q' \leq q$. Since A_n is a maximal antichain, there is $r \in A_n$ such that q' and r are compatible. By the choice of A_n, there is α_r such that $r \Vdash \dot{h}(n) = \check{\alpha}_r$. It follows that if q' decides $\dot{h}(n)$, then q' decides $\dot{h}(n)$ to be α_r. Since $q' \leq q$, r and q are compatible. By the choice of q_{n+1} and since $q \leq q_{n+1}$, q is compatible with at most countably many elements of A_n. It follows that $\{\alpha : \exists q' \leq q(q' \Vdash \dot{h}(n) = \check{\alpha})\}$ is countable. But this implies the countability of y. □

The conclusion of Lemma 23 is useful enough to get a name.

Definition 24 *Let V_0 and V_1 be models of set theory such that $V_0 \subseteq V_1$ and V_0 and V_1 have the same ordinals. We say that V_1 has the* $\aleph_0$-covering property *over V_0 iff for every set $A \in V_1$ of ordinals which is countable in V_1 there is a set $B \in V_0$ of ordinals which is countable in V_0 such that $A \subseteq B$.*

Lemma 23 says that if P is a forcing notion satisfying Axiom A, then for every P-generic filter G over the ground model V, $V[G]$ has the $\aleph_0$-covering property over V. Properness instead of Axiom A is actually sufficient for this (see [She98, III.1.16]).

The definition of Axiom A seems rather technical. However, it is a quite natural property which is typically satisfied by forcing notions where the conditions are trees which are required to split often. Let us show

Lemma 25 *Sacks forcing S satisfies Axiom A.*

PROOF: For $n \in \omega$ and $p \in S$ let p^n consist of those $t \in p$ that are minimal in p (with respect to $\subseteq$) such that t has exactly n proper initial segments that have two immediate successors in p. For $p, q \in S$ let $p \leq_n q$ iff $p \leq q$ and $p^n = q^n$. Suppose $(p_n)_{n\in\omega}$ is a sequence in S such that $p_{n+1} \leq_n p_n$ for all $n \in \omega$. Then $q := \bigcap p_n$ is easily seen to be a perfect tree, i.e., an element of S. Clearly, $q \leq_n p_n$ for every $n \in \omega$. In the context of Sacks forcing we will refer to the intersection of a fusion sequence as *the fusion* of the sequence.

Now let $n \in \omega$, $p \in S$, and suppose that A is an antichain in S. Every $\sigma \in 2^n$ uniquely determines an element t_σ of p^n (using the natural bijection between 2^n and p^n). Let $p * \sigma := \{s \in p : s \subseteq t_\sigma \vee t_\sigma \subseteq s\}$. Clearly, $p * \sigma \in S$ and $p * \sigma \leq p$.

For every $\sigma \in 2^n$ let $q_\sigma \leq p * \sigma$ be such that q_σ is compatible with at most one element of A. Now $q := \bigcup_{\sigma\in 2^n} q_\sigma \in S$ and $q \leq_n p$. Note that with this definition of q, for every $\sigma \in 2^n$ we have $q_\sigma = q * \sigma$. If q is compatible with some $r \in A$, then there is $\sigma \in 2^n$ such that $q * \sigma$ is compatible with r. But every $q*\sigma$ is compatible with at most one element of A. It follows that q is compatible with at most 2^n elements of A. □

Since Sacks forcing satisfies Axiom A, it does not collapse $\aleph_1$. Under CH, S is of size $\aleph_1$ and therefore does not collapse any cardinal above $\aleph_1$, either. However, if CH fails in the ground model, forcing with S could easily collapse $2^{\aleph_0}$. In fact, it is possible to have a ground model V such that CH fails in V and whenever G is S-generic over V, $2^{\aleph_0} = \aleph_1$ in $V[G]$. We will give the argument for this in Section 7. Petr Simon [Sim93] showed that adding a Sacks real collapses $2^{\aleph_0}$ to the *unboundedness number* $\mathfrak{b}$, the least size of a set $U \subseteq \omega^\omega$ such that for all $f : \omega \to \omega$ there is $g \in U$ such that for infinitely many $n \in \omega$, $g(n) > f(n)$. But $\mathfrak{b} < 2^{\aleph_0}$ is not the only reason for Sacks forcing to collapse cardinals.

Judah, Miller, and Shelah [JudMilShe92] showed that it is consistent with Martin's Axiom (which implies $\mathfrak{b} = 2^{\aleph_0}$) that Sacks forcing collapses cardinals. On the other hand, Shelah showed that it is consistent with ZFC+¬CH that Sacks forcing does not collapse cardinals (see [CarLav89]). Moreover, Carlson and Laver [CarLav89] showed that some strengthened form of Martin's Axiom implies that Sacks forcing does not collapse $2^{\aleph_0}$ to $\aleph_1$. This stronger version of Martin's axiom is known to be consistent with ZFC+$2^{\aleph_0} = \aleph_2$. This also implies that it is consistent with ZFC+¬CH that Sacks forcing does not collapse any cardinal.

Let us go back to the proof of Lemma 25. This proof shows that Sacks forcing S actually satisfies a stronger form of Axiom A: Given a condition p, an integer n, and an antichain A, we can find $q \leq_n p$ which is compatible with only finitely many members of A. This already implies that Sacks forcing has the so-called Sacks property. However, we can do even better.

Definition 26 *Let $f : \omega \to \omega \setminus \{0\}$. $C : \omega \to [\omega]^{<\aleph_0}$ is an f*-cone *iff for all $n \in \omega$, $|C(n)| \leq f(n)$. A real $r \in \omega^\omega$ is* covered *by C iff for all $n \in \omega$, $r(n) \in C(n)$. Let V_0 and V_1 be models of set theory such that $V_0 \subseteq V_1$. We say that V_1 has the* Sacks property *over V_0 iff every real $r : \omega \to \omega$ in V_1 is covered by a 2^n-cone $C : \omega \to [\omega]^{<\omega}$ in V_0. (Here 2^n*

is meant as a shortcut for the function $n \mapsto 2^n$.) A forcing notion P has the Sacks property iff for every P-generic filter G over the ground model V, $V[G]$ has the Sacks property over V.

A subtree T of $\omega^{<\omega}$ is binary *iff every $t \in T$ has at most 2 immediate successors in T. A real $r : \omega \to \omega$ is* covered *by T iff $r \in [T]$. Let V_0 and V_1 be as before. Then V_1 has the* 2-localization property *over V_0 iff every real $r \in \omega^\omega$ in V_1 is covered by a binary tree in V_0. A forcing notion P has the* 2-localization property *iff for every P-generic filter G over the ground model V, $V[G]$ has the 2-localization property over V.*

The Sacks property clearly follows from the 2-localization property.

Lemma 27 *Sacks forcing has the 2-localization property. In particular, it has the Sacks property.*

PROOF: Let $p \in S$, and suppose that $\dot{z}$ is an S-name such that $p \Vdash \dot{z} \in \check{\omega}^{\check{\omega}}$. It suffices to find a binary tree T and $q \leq p$ such that q forces $\dot{z}$ to be a branch through T. The condition q will be the fusion of a fusion sequence $(p_n)_{n\in\omega}$. We then define T to be

$$T_q(\dot{z}) := \{s \in \omega^{<\omega} : \exists q' \leq q(q' \Vdash \check{s} \subseteq \dot{z})\},$$

the *tree of q-possibilities for $\dot{z}$*. Note that q forces $\dot{z}$ to be a branch of $T_q(\dot{z})$. While choosing the sequence $(p_n)_{n\in\omega}$ all we have to make sure is that $T_q(\dot{z})$ becomes binary.

The set

$$\{q \in S : q \text{ decides all of } \dot{z}\} \cup \{q \in S : \forall r \leq q(r \text{ does not decide all of } \dot{z})\}$$

is dense in S. If there is some $q \leq p$ which decides all of $\dot{z}$ we are done since in this case $T_q(\dot{z})$ does not have incomparable elements. Thus, we may assume that no condition below p decides all of $\dot{z}$.

For every condition $q \leq p$ let $\dot{z}_q$ be the longest initial segment of $\dot{z}$ that is decided by q. In particular, $q \Vdash (\dot{z}_q)\check{} \subseteq \dot{z}$. Since no condition below p decides all of $\dot{z}$, $\dot{z}_q \in \omega^{<\omega}$. Let $p_0 := p$. Let $n \in \omega$ and suppose that we have already chosen p_n. For $\sigma \in 2^n$ consider the conditions

$p_n * (\sigma ^\frown 0)$ and $p_n * (\sigma ^\frown 1)$ where $s ^\frown i$ denotes the concatenation of s with the sequence of length 1 that has the value i. Since $p_n * (\sigma ^\frown 0)$ and $p_n * (\sigma ^\frown 1)$ do not decide all of $\dot{z}$, there are $q_{\sigma ^\frown 0} \leq p_n * (\sigma ^\frown 0)$ and $q_{\sigma ^\frown 1} \leq p_n * (\sigma ^\frown 1)$ such that $\dot{z}_{q_{\sigma ^\frown 0}}$ and $\dot{z}_{q_{\sigma ^\frown 1}}$ are incomparable with respect to $\subseteq$. Now $p_{n+1} := \bigcup_{i \in 2, \sigma \in 2^n} q_{\sigma ^\frown i} \leq_n p_n$.

Let $q := \bigcap_{n \in \omega} p_n$. It is easily checked by induction on n that for all $n \in \omega$ the finite tree T_n which is generated by (i.e., consists of all initial segments of elements of) $\{\dot{z}_{q*\sigma} : \sigma \in 2^n\}$ has the following properties:

1. T_n is a finite binary tree of height at least n,
2. $T_n \subseteq T = T_q(\dot{z})$, and
3. if $t \in T$ is of length $\leq n$, then $t \in T_n$.

It follows that $T = \bigcup_{n \in \omega} T_n$ and that T is binary. □

Let us analyze this proof. For every $p \in S$ and every name $\dot{z}$ for an element of ω^ω such that no condition below p decides all of $\dot{z}$ we found a condition $q \leq p$ such that $T_q(\dot{z})$ is binary. It follows from the construction of q that

$$h : [q] \to [T_q(\dot{z})]; x \mapsto \bigcup\{\dot{z}_{q*\sigma} : \sigma \in 2^{<\omega} \wedge x \in [q * \sigma]\}$$

is well-defined and in fact a homeomorphism. Obviously, q forces the Sacks real to be a branch of q (every condition in S does this). Moreover, q forces that h maps the Sacks real to $\dot{z}$.

This shows the following: Let G be S-generic over the ground model V. Then for every element z of ω^ω in $V[G]$ with $z \notin V$ there is a homeomorphism $h \in V$ between a perfect subset of 2^ω (the set $[q]$ for some suitable $q \in G$) and a perfect subset of ω^ω (the set $[T_q(\dot{z})]$ for some name $\dot{z}$ for z) such that (in $V[G]$) h maps the Sacks real x added by G to z. The homeomorphism h induces an order isomorphism between the set of perfect subtrees of q and the set of perfect subtrees of $T_q(\dot{z})$. The partial orders of perfect subtrees of q and of $T_q(\dot{r})$ are clearly isomorphic to S. It is easily checked that $\{p \leq q : z \in [p]\}$ is a generic filter for the partial order of perfect subtrees of q. But this implies that $\{p \in S : p \subseteq T_q(\dot{z}) \wedge z \in [p]\}$ is a generic filter for the partial order of perfect subtrees of $T_q(\dot{z})$. From this fact it easily follows that if z happens to be an element of 2^ω, then z is a Sacks real belonging to some

S-generic filter which is usually different from, but interdefinable with G using ground model parameters. It follows that every new element of 2^ω is again a Sacks real. (This was proved by Sacks in [Sac71].)

The existence of a ground model homeomorphism h mapping the original Sacks real x to z implies that x can be reconstructed from z using a ground model parameter, namely h (or rather h^{-1}). This observation already indicates the minimality of the Sacks extension mentioned in the introduction: For every S-generic filter G over the ground model V and every model W of set theory such that $V \subseteq W \subseteq V[G]$, either $W = V$ or $V[G] = W$. A real r which codes (i.e., is interdefinable with) some P-generic filter G over V for some partial order P is called a *minimal real* iff $V[G]$ is a minimal extension of V.

From [Jec78, Lemma 25.2 and Lemma 25.3] we quote the following: If G is a P-generic filter over the ground model V for some forcing notion P and W is a model of ZFC such that $V \subseteq W \subseteq V[G]$, then there is a set A of ordinals such that $W = V[A]$ where $V[A]$ is the smallest model of ZFC which extends V and contains A. Note that it is not obvious that $V[A]$ exists at all for a given set $A \in V[G]$ of ordinals. However, it can be shown that for every set $A \in V[G]$ of ordinals there is a forcing notion $Q \in V$ and a *complete embedding* $e : Q \to P$ such that $V[e^{-1}[G]]$ is the smallest model of ZFC which extends V and contains A.

A mapping $e : Q \to P$ is a *complete embedding* iff for all $q, q' \in Q$,

(i) $q \le q'$ implies $e(q) \le e(q')$ and
(ii) $e(q) \perp e(q')$ iff $q \perp q'$

and

(iii) for all $p \in P$ there is $q \in Q$ such that for all $q' \in Q$ with $q' \le q$, $e(q')$ is compatible with p.

If $e : Q \to P$ is a complete embedding (in the ground model V), then for every P-generic filter G over V, $e^{-1}[G]$ is Q-generic over V.

Lemma 28 *Sacks reals are minimal.*

PROOF: Let $\dot{A}$ be a name for a set of ordinals. Then there is an ordinal α such that 1_S forces $\dot{A}$ to be a subset of α. Let $\dot{z}$ be a name for the characteristic function of $\dot{A}$ from α to 2. Let $p \in S$ and let $\dot{r}$ be a name for the Sacks real added by the S-generic filter. We have to find $q \leq p$ such that q decides all of $\dot{z}$ or such that for every S-generic filter G over the ground model V with $q \in G$, $\dot{r}_G$ can be reconstructed from $\dot{z}_G$ using parameters in V.

We proceed essentially as in the proof of Lemma 27. Again we may assume that no condition below p decides all of $\dot{z}$ since otherwise we are done. For every condition $q \leq p$ let $\dot{z}_q$ be the longest initial segment of $\dot{z}$ that is decided by q, as before. Since no condition below p decides all of $\dot{z}$, the domain of $\dot{z}_q$ is an ordinal $< \alpha$. Let $p_0 := p$. Let $n \in \omega$ and suppose that we have already chosen p_n. For $\sigma \in 2^n$ consider the conditions $p_n * (\sigma^\frown 0)$ and $p_n * (\sigma^\frown 1)$. Since $p_n * (\sigma^\frown 0)$ and $p_n * (\sigma^\frown 1)$ do not decide all of $\dot{z}$, there are $q_{\sigma^\frown 0} \leq p_n * (\sigma^\frown 0)$ and $q_{\sigma^\frown 1} \leq p_n * (\sigma^\frown 1)$ such that $\dot{z}_{q_{\sigma^\frown 0}}$ and $\dot{z}_{q_{\sigma^\frown 1}}$ are incomparable with respect to $\subseteq$. Let $p_{n+1} := \bigcup_{i \in 2, \sigma \in 2^n} q_{\sigma^\frown i} \leq_n p_n$.

Let $q := \bigcap_{n \in \omega} p_n$. Now if G is S-generic over V and $q \in G$, then $\dot{r}_G$ is the unique element of $[q]$ such that for all $\sigma \in 2^{<\omega}$ with $\dot{r}_G \in [q * \sigma]$, $\dot{z}_{q*\sigma} \subseteq \dot{z}_G$. This shows that $q \in G$ implies $\dot{r}_G \in V[\dot{A}_G]$ and we are done. □

Brendle [Bre00] proved some interesting theorems about the set of all Sacks reals over the ground model V in a generic extension $V[G]$. He showed that after adding a single Sacks real over V, the set of ground model reals is in the ideal s^0. Since every new real is a Sacks real, this means that the set of Sacks reals is big with respect to s^0, i.e., its complement is in s^0. It follows that after adding a Sacks real x_0 over V and then adding a Sacks real x_1 over $V[x_0]$, in $V[x_0][x_1]$ the set of Sacks reals over V is in s^0.

The results about Sacks forcing mentioned so far show that when adding a single Sacks real, we have a very good control about all the new reals. Let us mention another result along these lines. Let $\mathcal{U}$ be an ultrafilter on ω in the ground model V. Suppose G is P-generic over V for some forcing notion P. Then $\mathcal{U}$ is *destroyed* by adding G if the filter generated by $\mathcal{U}$ in $V[G]$ is not an ultrafilter anymore. Otherwise the ultrafilter is *preserved*. It can be shown that adding any new real destroys some ultrafilter from the ground model [BarJud95, Theorem 6.2.2]. However, some ground model ultrafilters are preserved when adding a Sacks real.

An ultrafilter $\mathcal{U}$ on ω is a *p-point* if for every countable family $\mathcal{F} \subseteq \mathcal{U}$ there is a set $U \in \mathcal{U}$ which is almost included in all elements of $\mathcal{F}$. U is called a *pseudo-intersection of* $\mathcal{F}$. In [BarJud95, Theorem 7.3.48] it was shown that Miller forcing (see Section 4 for the definition of Miller forcing) preserves p-points. The same proof works for Sacks forcing.

Lemma 29 *If $\mathcal{U}$ is a p-point in the ground model V and G is S-generic over V, then $\mathcal{U}$ generates a p-point in $V[G]$. In other words, Sacks forcing preserves p-points.*

It should be pointed out that it is very easy to destroy every ground model ultrafilter by adding a new real. A real $r \in 2^\omega$ is called a *splitting real* over the ground model V if for $A := r^{-1}(1)$ and all $B \in \mathcal{P}(\omega) \cap V$ that are neither finite nor cofinite, $A \cap B$ and $B \setminus A$ are both infinite. Both random reals and Cohen reals are splitting. It is easily checked that adding a splitting real destroys all ultrafilters from the ground model.

To conclude this section, we mention a result about the Π^1_3-theory of $L[x]$ where x is a Sacks real over Gödel's universe L of constructible sets. Note that the minimality of the Sacks extension implies that by adding a Sacks real x to L, one obtains a minimal model of $V \neq L$, i.e., for every class $C \subseteq L[x]$ that is a model of ZFC, either $C = L[x]$ or $C = L$, and in the latter case $C \models V = L$.

Recall that $\varphi(x)$ is a Σ^1_2-formula if $\varphi(x)$ is of the form

$$\exists r_1 \in \omega^\omega \ \forall r_2 \in \omega^\omega \ \psi(x, r_1, r_2)$$

where ψ is arithmetical, i.e., only has quantifiers ranging over natural numbers (see, e.g., [Jec78, 7.40]). Shoenfield's Absoluteness Theorem (see [Jec78, Theorem 98]) implies that Σ^1_2-formulas are absolute for forcing extensions. That is, if $V[G]$ is a forcing extension of the ground model V, $\varphi(x)$ is a Σ^1_2-formula, and r is an element of ω^ω in V, then $\varphi(r)$ holds in V iff it holds in $V[G]$.

A Π^1_3-statement is a statement of the form

$$\forall r_0 \in \omega^\omega\ \exists r_1 \in \omega^\omega\ \forall r_2 \in \omega^\omega\ \psi(r_0, r_1, r_2)$$

where ψ is arithmetical.

Lemma 210 *If φ is a Π^1_3-statement which holds in a universe containing a non-constructible real, then it holds in $L[G]$ where G is S-generic over L.*

It follows that if there is any model V which is different from L such that in V every real r satisfies the Σ^1_2-formula

$$\exists r_1 \in \omega^\omega\ \forall r_2 \in \omega^\omega\ \psi(r, r_1, r_2),$$

then Sacks forcing over L adds no real for which this statement fails.

As Woodin remarked in [Woo99, Theorem 1.5], Lemma 210 is the consequence of the following result of Mansfield (see [Jec78, Theorem 99]):

Theorem 211 *Let A be a set of reals defined by a Σ^1_2-formula with constructible parameters. If A contains a non-constructible element, then it includes a constructibly coded perfect subset.*

To see why this implies Lemma 210, let V be a model of ZFC containing a non-constructible real, let $\varphi(x)$ be a Σ^1_2-formula with only constructible parameters, and suppose that $V \models \forall r \varphi(r)$. Fix a generic extension $V[G]$ in which $\aleph_1^L$ is countable. Then in particular, all constructible dense sets of S are countable in $V[G]$. We have already seen that in the Sacks extension every new real is itself a Sacks real. Therefore it suffices to show that the Sacks generic real satisfies φ in $L[G]$ where G is S-generic over L.

In order to show this, let $T \in S \cap L$. In $V[G]$, $A = \{r \in [T] \mid \varphi(r)\}$ is a Σ^1_2-set containing a non-constructible element (from V), and thus Theorem 211 yields a perfect tree $T' \in L$ such that $[T'] \subseteq A$. By the choice of $V[G]$, $[T']$ also contains a Sacks real over L, and since Σ^1_2-statements are absolute, neither T' nor any perfect subtree of it can force $\neg\varphi(\dot r)$.

As Brendle and Löwe observed in [BreLöw99], a slight generalization of Theorem 211 can be used to show the equivalence of the statements "every $\mathbf{\Sigma}^1_2$-set of reals is s-measurable" and "every $\mathbf{\Delta}^1_2$-set of reals is s-measurable". Here a set of reals is $\mathbf{\Sigma}^1_2$ iff it is the set of reals satisfying a fixed Σ^1_2-formula with a fixed additional real parameter. A set of reals is $\mathbf{\Delta}^1_2$ iff the set and its complement are $\mathbf{\Sigma}^1_2$. The class of $\mathbf{\Sigma}^1_2$-sets can also be defined topologically (see [Jec78, page 500]). The $\mathbf{\Sigma}^1_1$-sets are the *analytic* sets, i.e., the projections of Borel sets, the $\mathbf{\Pi}^1_1$-sets are the complements of $\mathbf{\Sigma}^1_1$-sets, and the $\mathbf{\Sigma}^1_2$-sets are the projections of $\mathbf{\Pi}^1_1$-sets.

3 Consequences of the Sacks property

Before discussing consequences of the Sacks property, let us observe that in the definition of the Sacks property it is not essential that the size of $C(n)$ is bounded by 2^n. 2^n could be replaced by any other non-decreasing unbounded function from ω to ω. This can be seen as follows:

Let V_0 and V_1 be models of set theory with $V_0 \subseteq V_1$. Let $f, g : \omega \to \omega \setminus \{0\}$ be non-decreasing and unbounded functions in V_0. Suppose that every real in V_1 is covered by an f-cone in V_0. In other words, suppose that V_1 has the Sacks property over V_0 with 2^n replaced by f. We show that every new real in V_1 is covered by a g-cone from V_0. Note that we may assume $f(0) \leq g(0)$ since every f-cone is the coordinatewise union of finitely many f-cones C with $|C(0)| = 1$.

Let $r : \omega \to \omega$ be a real in V_1. In V_0 choose a strictly increasing sequence $(m_n)_{n\in\omega}$ of natural numbers such that for every $n \in \omega$, $g(m_n) \geq f(n)$. By our assumption $f(0) \leq g(0)$, we may assume $m_0 = 0$. Identifying ω with $\omega^{<\omega}$ for a moment, we find an f-cone $C : \omega \to [\omega^{<\omega}]^{<\aleph_0}$ in V_0 which covers $(r \restriction m_n)_{n\in\omega}$. Let $D : \omega \to [\omega]^{<\aleph_0}$ be defined as follows: For every $n \in \omega$ let $D(n) := \{s(n) : s \in C(n) \wedge n \in \mathrm{dom}(s)\}$. Now for every $n \in \omega$, if m is in the interval $[m_n, m_{n+1})$, then $|D(m)| \leq f(n) \leq g(m_n) \leq g(m)$. It follows that D is a g-cone in V_0 covering r.

The Sacks property implies that the new reals in the generic extension are very well controlled by ground model reals. For example, if V_1 has the Sacks property over V_0, then the $2^{\aleph_0}$ of V_0 is still uncountable in V_1. This is because in V_1, countably many 2^n-cones do not suffice to cover all of ω^ω, but the 2^n-cones from V_0 do cover all of ω^ω.

We mention two more examples of this: If a forcing notion P has the Sacks property and G is P-generic over the ground model V, then the ideal $\mathcal{N}^{V[G]}$ of measure zero sets in the generic extension $V[G]$ is generated by the measure zero Borel sets from the ground model. (Whether a Borel code codes a measure zero set is absolute, that is, it does not depend on the model of set theory in which we evaluate the Borel code. The analogue is true for Borel codes of nowhere dense sets and of meager sets. See [Jec78, Lemma 42.4].) Similarly, the ideal of nowhere dense subsets of 2^ω in $V[G]$ is generated by the nowhere dense Borel sets in V. Both of these consequences of the Sacks property follow from the following lemma.

Lemma 31 *Let V_0 and V_1 be models of set theory such that $V_0 \subseteq V_1$ and V_0 and V_1 have the same ordinals. Suppose that V_1 has the Sacks property over V_0. Then*

a) for every measure zero set $A \subseteq \mathbb{R}$ in V_1 there is a Borel set $B \subseteq \mathbb{R}$ in V_0 of measure zero such that $A \subseteq B$ holds in V_1 and

b) for every nowhere dense set $A \subseteq 2^\omega$ in V_1 there is a nowhere dense Borel set $B \subseteq 2^\omega$ in V_0 such that $A \subseteq B$ holds in V_1.

PROOF: a) Let λ denote the Lebesgue measure on $\mathbb{R}$. Recall that if $A \subseteq \mathbb{R}$ is of measure zero, then for every sequence $(\varepsilon_n)_{n\in\omega}$ of positive real numbers there is a sequence $(U_n)_{n\in\omega}$ of finite unions of open intervals with rational endpoints such that $A \subseteq \bigcup_{n\in\omega} U_n$ and for all $n \in \omega$, $\lambda(U_n) < \varepsilon_n$.

We do some preparations in V_0. Let $(O_n)_{n\in\omega}$ be an enumeration of all finite unions of open intervals with rational endpoints, let $e : \omega \times \omega \to \omega$ be a bijection, and fix a matrix $(\varepsilon_{ij})_{i,j\in\omega}$ of positive real numbers such that for all $i \in \omega$,

$$\sum_{j\in\omega} \varepsilon_{ij} < \frac{1}{2^i}.$$

Now for every $i \in \omega$ there is $f_i \in \omega^\omega$ in V_1 such that $A \subseteq \bigcup_{j\in\omega} O_{f_i(j)}$ and for every $j \in \omega$,

$$2^{e(i,j)} \cdot \lambda(O_{f_i(j)}) < \varepsilon_{ij}.$$

Consider the real $r : \omega \to \omega$ defined by $r(n) := m$ iff $m = f_i(j)$ and $n = e(i, j)$. By the Sacks property of P there is a 2^n-cone $C : \omega \to [\omega]^{<\aleph_0}$ in V_0 that covers r. By the definition of r, A is a subset of

$$B := \bigcap_{i\in\omega} \bigcup_{j\in\omega} \bigcup \{O_m : m \in C(e(i, j)) \wedge 2^{e(i,j)} \cdot \lambda(O_m) < \varepsilon_{ij}\}.$$

Since only ground model parameters are used in the definition of B, B is a Borel set coded in the ground model. For all $i, j \in \omega$, the set $C(e(i, j))$ has at most $2^{e(i,j)}$ elements. It follows that the measure of $\bigcup\{O_m : m \in C(e(i, j)) \wedge 2^{e(i,j)} \cdot \lambda(O_m) < \varepsilon_{ij}\}$ is not greater than ε_{ij}. Therefore, and by the choice of $(\varepsilon_{ij})_{i,j\in\omega}$, for every $i \in \omega$,

$$\lambda\Big(\bigcup_{j\in\omega} \bigcup \{O_m : m \in C(e(i, j)) \wedge 2^{e(i,j)} \cdot \lambda(O_m) < \varepsilon_{ij}\}\Big) < \frac{1}{2^i}.$$

It follows that B is of measure zero.

b) Note that if $A \subseteq 2^\omega$ is nowhere dense, then for all $n \in \omega$ there is $m > n$ and $t : [n, m) \to 2$ such that $\{x \in 2^\omega : t \subseteq x\}$ is disjoint from A. (Here $[n, m)$ denotes the interval $\{n, \ldots, m - 1\}$ in ω.) This can be seen as follows: Let $n \in \omega$ and let $(s_i)_{i<2^n}$ be an enumeration of 2^n. Since A is nowhere dense, there is a sequence $(t_i)_{i<2^n}$ in $2^{<\omega}$ such that for all $i < 2^\omega$ the set $\{x \in 2^\omega : s_i^\frown t_0^\frown \ldots ^\frown t_i \subseteq x\}$ is disjoint from A (recall that $s^\frown t$ is the concatenation of the two finite sequences s and t). Now it is easy to check that $t := t_0^\frown \ldots ^\frown t_{2^n}$ has the desired property.

Using this observation we can choose a strictly increasing sequence $(n_i)_{i\in\omega}$ of natural numbers and a sequence $(t_i)_{n\in\omega}$ such that for all $i \in \omega$, $t_i \in 2^{[n_i,n_{i+1})}$ and $\{x \in 2^\omega : t_i \subseteq x\}$ is disjoint from A. In other words, letting $z := t_0{}^\frown t_1{}^\frown \ldots$ we have the following: For all $i \in \omega$ the set $\{x \in 2^\omega : x \restriction [n_i, n_{i+1}) = z \restriction [n_i, n_{i+1})\}$ is disjoint from A. This property of z does not change if we replace the sequence $(n_i)_{i\in\omega}$ by another strictly increasing sequence $(m_i)_{i\in\omega}$, provided for every $i \in \omega$

there is $j \in \omega$ such that $[n_j, n_{j+1}) \subseteq [m_i, m_{i+1})$. By the Sacks property, there is a 2^n-cone $C : \omega \to [\omega]^{<\omega}$ in V_0 covering $i \mapsto n_i$. Putting $m_0 := \max(C(0))$ and $m_{i+1} := \max(C(m_i + 1))$ for every $i \in \omega$, we obtain sequence in V_0 such that for every $i \in \omega$ the set $\{x \in 2^\omega : x \restriction [m_i, m_{i+1}) = z \restriction [m_i, m_{i+1})\}$ is disjoint from A.

Now let $(j_i)_{i\in\omega}$ be a sequence in V_0 of natural numbers such that for all $i \in \omega$, $j_{i+1} - j_i > 2^i$. By the Sacks property, there is a 2^n-cone $C : \omega \to [2^{<\omega}]^{<\aleph_0}$ in V_0 covering $i \mapsto z \restriction m_{j_{i+1}}$. We may assume that for every $i \in \omega$ every $t \in C(i)$ has $\mathrm{dom}(t) = m_{j_{i+1}}$. Since $j_{i+1} - j_i > 2^i$ and $|C(i)| \leq 2^i$, there is $u_i : [m_{j_i}, m_{j_{i+1}}) \to 2$ such that for every $t \in C(i)$ there is $k \in [j_i, j_{i+1})$ such that t agrees with u_i on $[m_k, m_{k+1})$. Since $z \restriction m_{j_{i+1}} \in C(i)$, no $x \in A$ extends u_i. The sequence $(u_i)_{i\in\omega}$ can be chosen in V_0. Now the set $\{x \in 2^\omega : \exists i \in \omega (u_i \subseteq x)\}$ is coded in V_0 and disjoint from A in V_1. Its complement B is a nowhere dense Borel set in the ground model which covers A in V_1. □

The last lemma implies that forcing notions with the Sacks property neither add Cohen reals nor random reals. Moreover, we can compute the *cofinality* of $\mathcal{N}$, the least size of a family $\mathcal{F} \subseteq \mathcal{N}$ such that for all $A \in \mathcal{N}$ there is $B \in \mathcal{F}$ with $A \subseteq B$, in forcing extensions with the Sacks property over the ground model.

Corollary 32 *Let V_0 and V_1 be models of set theory such that $V_0 \subseteq V_1$, V_0 and V_1 have the same ordinals, and V_1 has the Sacks property over V_0. Then in V_1 the cofinality of $\mathcal{N}$ (ordered by $\subseteq$) is at most $|(2^{\aleph_0})^{V_0}|$ where $(2^{\aleph_0})^{V_0}$ denotes the ordinal which is the size of 2^ω in V_0. In particular, if V_0 satisfies CH, then the cofinality of $\mathcal{N}$ is $\aleph_1$ in V_1.*

PROOF: By Lemma 31, the Borel measure zero sets coded in V_0 are cofinal in $\mathcal{N}^{V_1}$. In V_0 there is a bijection between $2^{\aleph_0}$ and all Borel subsets of 2^ω. Since $(2^{\aleph_0})^{V_0}$ is an ordinal, and since V_0 and V_1 have the same ordinals, in V_1 there is a bijection between $|(2^{\aleph_0})^{V_0}|$ and all Borel sets in 2^ω which are coded in V_0. Therefore, in V_1 the cofinality of $\mathcal{N}$ is at most $|(2^{\aleph_0})^{V_0}|$. Now suppose that V_0 satisfies CH. Then $V_1 \models |(2^{\aleph_0})^{V_0}| \leq \aleph_1$. But the cofinality of $\mathcal{N}$ is uncountable and therefore it must be $\aleph_1$ in V_1. □

The cofinality of $\mathcal{N}$ is the largest cardinal that appears in Cichoń's diagram (see [BarJud95, Chapter 2]). Thus, Corollary 32 implies that if one starts from a model of CH and then adds a P-generic filter over the ground model for some forcing notion P which has the Sacks property, then one ends up with a model where all the cardinals in Cichoń's diagram are $\aleph_1$. Of course this is only interesting if there is a forcing notion with the Sacks property that increases $2^{\aleph_0}$. We will go back to this when we discuss iterated forcing. The cardinals in Cichoń's diagram, as well as other cardinal characteristics of the continuum, are studied in depth in [BarJud95]. In this book the values of many cardinal characteristics of the continuum are computed in various models of set theory. Corollary 32 and further characterizations of the Sacks property for proper forcing notions can be found there. For a nice overview of cardinal characteristics of the continuum see [Bla0*].

4 Variants of Sacks forcing

We give a brief overview of some relatives of Sacks forcing. Except for forcing with finitely branching trees, all the forcing notions we mention in this section are discussed in [BarJud95].

All the variants of Sacks forcing in this section satisfy Axiom A and have the *Laver property*, a weakening of the Sacks property. A forcing notion P has the Laver property iff for every P-generic filter G over the ground model V and every $r : \omega \to \omega$ in $V[G]$ the following holds: If r is *bounded* by a ground model real, i.e., if there is $b : \omega \to \omega$ in V such that for all but finitely many $n \in \omega$ we have $r(n) \leq b(n)$ in $V[G]$, then r is covered by a 2^n-cone from the ground model.

Since the Sacks property implies that every $r : \omega \to \omega$ in the generic extension is bounded by some ground model real, the Sacks property is equivalent to the Laver property together with the property that every element of ω^ω in the extension is bounded by a ground model real. A forcing notion with the latter property is called *ω^ω-bounding*. (For example, random real forcing is ω^ω-bounding, Cohen forcing is not.) Random real forcing and Cohen forcing do not have the Laver property, because the Laver property implies that any new real in the generic extension can be covered by a meager set of measure zero. In particular, forcing notions with the Laver property neither add random nor Cohen reals. Except for

Mathias forcing, all the variants of Sacks forcing in this section give minimal generic extensions like Sacks forcing does.

We first summarize the properties of **Sacks forcing** S. S consists of all perfect subtrees of $2^{<\omega}$, ordered by inclusion. It has the Sacks property (even the 2-localization property) and preserves p-points. Moreover, if φ is a Π^1_3-statement which holds in some model of set theory with a non-constructible real, then it holds in $L[G]$ where G is S-generic over L.

Forcing with finitely branching trees. There are some forcing notions which are very closely related to Sacks forcing in that the conditions are also finitely branching trees with certain perfectness properties. For example, for $n \in \omega$ with $n \geq 2$ a subtree p of $n^{<\omega}$ is *n-perfect* iff every $s \in p$ has an extension which has exactly n immediate successors in p. The forcing notion consisting of n-perfect trees ordered by inclusion has the n-localization property (which is obtained from the 2-localization property by replacing "binary tree" by "n-ary tree" where n-ary trees are those trees in which every node has at most n immediate successors) but not the $(n-1)$-localization property [NewRos93]. Note that the n-localization property implies the Sacks property.

Laver forcing LT [Lav76] consists of all subtrees p of $\omega^{<\omega}$ such that almost all $s \in p$ have infinitely many immediate successors in p. LT is ordered by inclusion. Laver forcing adds a *dominating real*, i.e., a function $d : \omega \to \omega$ which bounds all ground model functions from ω to ω. This implies that Laver forcing destroys every ultrafilter from the ground model. If G is LT-generic over the ground model V, then in $V[G]$ the set $\mathbb{R} \cap V$ has outer measure one.

Modifying Mansfield's proof of Theorem 211, one can show that if φ is a Π^1_3-statement which holds in a universe with a dominating real over L, then φ holds in $L[G]$ where G is LT-generic over L. For this notice that if there is a dominating real over L, then there is a so-called *strongly dominating* real over L (see [GolRepSheSpi95]). Moreover, if A is a Σ^1_2-set in constructible parameters which contains a strongly dominating real over L, then A contains all the branches of a constructibly coded Laver condition. This can be shown using ideas from [GolRepSheSpi95].

Miller forcing or rational perfect set forcing RP [Mil84] consists of all subtrees p of $\omega^{<\omega}$ such that every $s \in p$ has an extension $t \in p$ which has infinitely many immediate successors in p. The elements of RP are sometimes called *superperfect trees*. Miller forcing is ordered by inclusion. It preserves p-points and therefore does not add a dominating real. If G is RP-generic over the ground model V, then $\mathbb{R} \cap V$ is neither meager nor of measure zero in $V[G]$.

As for Laver forcing, a variant of Theorem 211 is true for Miller forcing. A real $r : \omega \to \omega$ is *unbounded* over a class C iff it is not bounded by a function in $\omega^\omega \cap C$. Using ideas from [Kec77] it is possible to show that every constructibly coded Σ^1_2-set which contains an unbounded real over L contains all the branches of a constructibly coded Miller condition. This implies that every Π^1_3-statement which holds in a model of set theory with an unbounded real over L also holds in every model of the form $L[G]$ where G is RP generic over L.

Mathias forcing M [Mat77] consists of all pairs (s, A) such that $s \subseteq \omega$ is finite and $A \subseteq \omega \setminus \bigcup s$ is infinite. M is ordered as follows: $(s, A) \leq (t, B)$ iff $t \subseteq s$, $A \subseteq B$ and $s \setminus t \subseteq B$. Mathias forcing adds a dominating real. If G is M-generic over the ground model V, then $\mathbb{R} \cap V$ is a meager set of measure zero in $V[G]$.

Of course, we can only mention the most popular relatives of Sacks forcing. More forcing notions which use trees as conditions, usually called tree forcings, are studied in [Bre95]. A general framework for constructing tree forcings (creature forcing with tree creatures) was developed by Rosłanowski and Shelah in [RosShe99].

Some forcing notions can be considered as suborders of Sacks forcing, but have not been mentioned in this section since they seem to be based on a different philosophy. Cohen forcing is an example of this: Every Cohen condition p, a finite partial function from ω to 2, determines a perfect tree

$$T_p := \{q \in 2^{<\omega} : p \cup q \text{ is a function}\}.$$

The mapping $p \mapsto T_p$ is an embedding of Cohen forcing into Sacks forcing, but this embedding is obviously not a complete embedding. In the same way, Silver forcing and Gregorief forcing can be considered as suborders of Sacks forcing. (See [She98, VI.4] for the definitions of these two forcing notions. Shelah used Gregorief forcing to construct a model of set theory without p-points.) However, Silver forcing and Gregorief forcing are much closer to Sacks forcing than Cohen forcing is since they use as their conditions partial functions from ω to 2 whose domains may be infinite. They are both proper. Gregorief forcing is ω^ω-bounding, but Silver forcing is not. Note that the Silver forcing in [She98, VI.4] is different from the similar Prikry-Silver forcing as defined in [Bau83, Section 7]. Prikry-Silver forcing consists of all functions from co-infinite subsets of ω to 2 and is ordered by reverse inclusion. Prikry-Silver forcing satisfies Axiom A.

In the next section we mention some results about possible c.c.c. relatives of Sacks forcing.

5 The Sacks property and the c.c.c.

In the past many people investigated the question how close or how far away a c.c.c. forcing notion can be from Cohen forcing or random real forcing. Here "being close" simply means that Cohen forcing, respectively random real forcing can be completely embedded into the given forcing notion. If so, then such a forcing also adds a Cohen, respectively random real. One instance of this question, namely the question whether random real forcing completely embeds into every ω^ω-bounding forcing notion that is c.c.c. goes back to the Scottish Book [Mau81] problem of von Neumann about measure algebras.

Since the Sacks property implies that neither Cohen nor random reals are added, it is a good test question to ask when a c.c.c. forcing can have the Sacks property. Concerning reasonably definable partial orders, Shelah showed in [She94] that no c.c.c. Souslin forcing has the Laver property where a forcing notion is Souslin iff its underlying set, the order, and the incompatibility relation are analytic, i.e., Σ^1_1. So in particular, a c.c.c. Souslin forcing cannot have the Sacks property. Moreover, Shelah showed that if a c.c.c. Souslin forcing adds an unbounded real, then it adds a Cohen real. It is unknown whether there is a model of ZFC where every c.c.c. forcing adding an unbounded real adds a Cohen real.

There are some combinatorical statements which imply or negate the existence of c.c.c. forcings (not necessarily nicely definable) with the Sacks property. We mention the combinatorial statements involved and give a summary of results.

$\diamondsuit$: There is a sequence $(A_\alpha)_{\alpha<\omega_1}$ such that for all $\alpha < \omega_1$, $A_\alpha \subseteq \alpha$ and for every $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is stationary.

In [Jen70], Jensen constructed a c.c.c. forcing with the Sacks property from $\diamondsuit$. The principle $\diamondsuit$, which implies CH, is used in an inductive construction of length ω_1 to predict initial parts of antichains of the final forcing notion already at a countable stage of the construction. During the construction one makes sure that the predicted antichains cannot be extended to uncountable antichains in the final forcing notion, so that the final forcing notion becomes c.c.c. This is similar to the standard construction of a Souslin tree from $\diamondsuit$ as presented in [Kun80, Theorem 7.8].

CCC(S): For every family $\mathcal{D}$ of size at most $2^{\aleph_0}$ of dense subsets of Sacks forcing S there is a c.c.c. suborder P of S such that $D \cap P$ is dense in P for each $D \in \mathcal{D}$.

This principle together with a large size of the continuum was shown to be consistent with ZFC by Veličković [Vel91]. From CCC(S) one can easily extract a c.c.c. suborder of S which has the Sacks property.

OCA (Open Coloring Axiom): Let X be a subset of the reals and suppose that K_0 and K_1 are disjoint subsets of $[X]^2$ with $[X]^2 = K_0 \dot\cup K_1$ such that K_0 is open in the topology that $[X]^2$ inherits from the product topology on X^2. Then either there is an uncountable $Y \subseteq S$ such that $[Y]^2 \subseteq K_0$, or $S = \bigcup_{i\in\omega} Y_i$ where $[Y_i]^2 \subseteq K_1$ for every $i \in \omega$.

From this principle, which implies that the size of the continuum is at least $\aleph_2$, Veličković [Vel0*] deduced that no c.c.c. forcing adding a new real has the Sacks property.

Principle ($\star$) for κ: Let $\mathcal{I}$ be a p-ideal on $[\kappa]^{\leq\aleph_0}$ where $\mathcal{I} \subseteq ([\kappa]^{\leq\aleph_0})$ is a p-ideal if for every countable set $\mathcal{C} \subseteq \mathcal{I}$ there is $X \in \mathcal{I}$ such that every member of $\mathcal{C}$ is almost included in X. Then either there is an uncountable $Y \subseteq \kappa$ such that $[Y]^\omega \subseteq \mathcal{I}$, or $\kappa = \bigcup_{i\in\omega} Y_i$ where $[Y_i]^\omega \cap \mathcal{I} = \emptyset$ for every $i \in \omega$.

The second author showed in [Qui02] that this principle also implies that no c.c.c. forcing has the Sacks property. Since ($\star$) restricted to $\kappa = \aleph_1$ is consistent with CH [AbrTod97], this also shows that CH is not strong enough to decide the existence of c.c.c. forcings with the Sacks property.

The case of the Laver property is different: CH already implies the existence of a c.c.c. forcing with the Laver property. In [She01] it is stated that Mathias forcing defined relatively to a Ramsey ultrafilter is an example (see [Mat77] for the definition of Mathias forcing relative to a Ramsey ultrafilter). A model where such forcings do not exist was found by Shelah [She01]. He showed that the countable support iteration of Mathias forcing of length ω_2 yields such a model.

To conclude this section, we list some questions in this area.

1. Which combinatorical principles imply or deny the existence of a c.c.c. forcing preserving p-points?
2. Are there forcing notions not preserving p-points, but not adding splitting reals?
3. Which combinatorial principles imply or deny the existence of a c.c.c. forcing not adding splitting reals?

The motivation for the second question is to look for less “brutal” methods to destroy ultrafilters than adding a splitting real. The other questions are variants of von Neumann’s problem. As mentioned before, both Cohen and Random real forcing add a splitting real and therefore destroy every ultrafilter in the ground model.

6 Products of Sacks forcing

If one wants to add many Sacks reals to a model of set theory, for example to increase $2^{\aleph_0}$, then there are essentially two possibilities: The first possibility is to use some kind of side-by-side product of many copies of S to add many Sacks reals over ground model at once (in a parallel way). The second possibility is to iterate Sacks forcing, i.e., to add one Sacks real after the other. We will describe the second possibility in the next section. Forcing with side-by-side products has two advantages over the iteration: First, it is technically easier to deal with and second, it is possible to construct models with an arbitrarily large size of the continuum. The iteration, at least the countable support iteration, which is discussed in the next section, only gives models where the size of the continuum is at most $\aleph_2$. However, the countable support iteration of Sacks forcing usually yields stronger consistency results than forcing with side-by-side products.

Definition 61 *The* countable support (side-by-side) product P *of a family* $(Q_i)_{i\in I}$ *of forcing notions is the set of all* $p \in \prod_{i\in I} Q_i$ *with countable support* $\mathrm{supt}(p) := \{i \in I : p(i) \neq 1_{Q_i}\}$. *$P$ is ordered componentwise, i.e., for* $p, q \in P$, $p \leq q$ *iff for all* $i \in I$, $p(i) \leq q(i)$.

Of course one can define products with finite supports or with uncountable supports. However, countable supports are what is appropriate for forcing notions satisfying Axiom A. Something that all reasonable products P of a family $(Q_i)_{i\in I}$ (like the finite support product and the countable support product) have in common, is the following: For every $i \in I$ the embedding $e_i : Q_i \to P$ that maps every $q \in Q_i$ to the sequence which has q as its i-th coordinate and is 1 everywhere else is a complete embedding. In particular, if G is P generic over the ground model V, then for every $i \in I$, $e^{-1}[G]$ is Q_i-generic over V. In other words, if P is the countable support product of κ copies of Sacks forcing, then P adds κ Sacks reals. An easy density argument shows that the Sacks reals added by different factors of P are indeed different.

Under CH, countable support products of copies of Sacks forcing do not collapse cardinals [Bau85]. The proof of this fact consists of two parts: First one shows that $\aleph_1$ is not collapsed by a countable support product of copies of Sacks forcing. Then one uses CH to show that in countable support products of forcing notions of size at most $\aleph_1$ all antichains are of size at most $\aleph_1$. This can be done using a simple Δ-system argument. It follows that no cardinal above $\aleph_1$ is collapsed (see [Kun80, Lemma 6.9]). We only show that countable support products of copies of Sacks forcing do not collapse $\aleph_1$. In the proof presented here we get the 2-localization property more or less for free. The following lemma is essentially taken from [NewRos93].

Lemma 62 *Countable support products of copies of Sacks forcing have the* 2*-localization property. In particular, they have the Sacks property and do not collapse* $\aleph_1$.

The proof of this lemma follows closely the proof of Lemma 27. But we have to deal with countably supported κ-sequences of conditions in S this time. For this we need a lot of notation. Fix a cardinal κ and let P be the countable support product of κ copies of S. (The elements of P are functions from κ to S or equivalently, κ-sequences of elements of S.) In order to show that P has the 2-localization property, we have to extend the notions "fusion" and "fusion sequence" to P. For this it is convenient to pass from our original definition of fusion sequences in S to a more flexible one.

Now a sequence $(p_n)_{n\in\omega}$ in S is a *fusion sequence* iff there is a non-decreasing unbounded function $f:\omega\to\omega$ such that for all $n\in\omega$, $p_{n+1}\leq_{f(n)} p_n$. It is easily checked that a sequence $(p_n)_{n\in\omega}$ which is a fusion sequence according to the new definition has a subsequence which is a fusion sequence according to the old definition. If $(p_n)_{n\in\omega}$ is a fusion sequence in according to the new definition, its *fusion* $\bigcap_{n\in\omega} p_n$ is again a condition in S, as before.

Definition 63 *For a finite set* $F\subseteq\kappa$ *and* $\eta:F\to\omega$ *let the relation* $\leq_{F,\eta}$ *on* P *be defined as follows: For all* $p,q\in P$ *let* $p\leq_{F,\eta} q$ *if* $p\leq q$ *and for all* $\alpha\in F$, $p(\alpha)\leq_{\eta(\alpha)} q(\alpha)$. *For* p, F, *and* η *as before and* $\sigma\in\prod_{\alpha\in F} 2^{\eta(\alpha)}$ *let* $p*\sigma$ *be such that for all* $\alpha\in F$, $(p*\sigma)(\alpha)=p(\alpha)*\sigma(\alpha)$ *and for all* $\alpha\in\kappa\setminus F$, $(p*\sigma)(\alpha)=p(\alpha)$.

A sequence $(p_n)_{n\in\omega}$ *of conditions in* P *is a* fusion sequence *if there is an increasing sequence* $(F_n)_{n\in\omega}$ *of finite subsets of* κ *and a sequence* $(\eta_n)_{n\in\omega}$ *such that for all* $n\in\omega$, $\eta_n:F_n\to\omega$, *for all* $i\in F_n$ *we have* $\eta_n(i)\leq\eta_{n+1}(i)$, $p_{n+1}\leq_{F_n,\eta_n} p_n$, *and for all* $\alpha\in\mathrm{supt}(p_n)$ *there is*

$m \in \omega$ such that $\alpha \in F_m$ and $\eta_m(\alpha) \geq n$. If $(p_n)_{n\in\omega}$ is a fusion sequence in P, then for each $\alpha \in \kappa$, $(p_n(\alpha))_{n\in\omega}$ is a fusion sequence in S or constant with value 1_S. This shows that

$$p_\omega := \left(\bigcap_{n\in\omega} p_n(\alpha)\right)_{\alpha\in\kappa}$$

is a condition in P, the fusion *of the sequence $(p_n)_{n\in\omega}$.*

For the proof of Lemma 62 it is enough to show

Lemma 64 *For every $p \in P$ and every name $\dot{z}$ for an element of ω^ω there are a condition $q \leq p$ and a binary tree $T \subseteq \omega^{<\omega}$ such that q forces $\dot{z}$ to be a branch of T.*

PROOF: Let p and $\dot{z}$ be as above. Our goal is to construct $q \leq p$ such that

$$T_q(\dot{z}) = \{s \in \omega^{<\omega} : \exists r \leq q(r \Vdash \check{s} \subseteq \dot{z})\}$$

is binary. We may assume that no condition below p decides all of $\dot{z}$.

Let F be a finite subset of κ and let $\eta : F \to \omega$ be a function. A condition $r \in P$ is (F, η)-*faithful* iff for all $\sigma, \tau \in \prod_{\alpha\in F} 2^{\eta(\alpha)}$ and all $r' \leq_{F\eta} r$ the following holds: If $\dot{z}_{r'*\sigma}$ and $\dot{z}_{r'*\tau}$ are incomparable elements of $\omega^{<\omega}$ (with respect to $\subseteq$), then already $\dot{z}_{r*\sigma}$ and $\dot{z}_{r*\tau}$ are incomparable.

Claim 65 *Suppose $r \in P$ is (F, η)-faithful.*

a) Let $j \in \kappa \setminus F$ and define $F' := F \cup \{j\}$ and $\eta' := \eta \cup \{(j, 0)\}$. Then r is (F', η')-faithful.

b) Let $\beta \in F$, and let η' be such that for all $\alpha \in F \setminus \{\beta\}$, $\eta'(\alpha) = \eta(\alpha)$ and $\eta'(\beta) = \eta(\beta) + 1$. Then there is a condition $r' \leq_{F,\eta} r$ such that r' is (F, η')-faithful.

Part a) of this claim follows immediately from the definitions. For the proof of b) fix an enumeration $\{\sigma_1, \ldots, \sigma_m\}$ of $\prod_{i\in F} 2^{\eta(i)}$. We define a $\leq_{F,\eta}$-descending sequence $(r_k)_{k\leq m}$ below r. For $k \in \{1, \ldots, m\}$ and $l \in 2$ let σ_k^l be such that for all $\alpha \in F \setminus \{\beta\}$, $\sigma_k^l(\alpha) = \sigma_k(\alpha)$ and $\sigma_k^l(\beta) = \sigma_k(\beta)^\frown l$. Let $r_0 := r$. Suppose we have constructed r_k for some $k < m$ and we want to build r_{k+1}. Let $r_{k+1} \leq_{F,\eta'} r_k$ be such that $\dot{z}_{r_{k+1}*\sigma^0_{k+1}}$ and $\dot{z}_{r_{k+1}*\sigma^1_{k+1}}$ are incomparable if such a condition exists. Otherwise, let $r_{k+1} := r_k$. Finally, let $r' := r_m$. It follows from the construction that r' is (F, η')-faithful. This shows part b) of the claim.

Using a) and b) of Claim 65 together with some bookkeeping, we construct a sequence $(p_m, F_m, \eta_m, j_m)_{m\in\omega}$ such that

1. $(p_m)_{m\in\omega}$ is a fusion sequence in $\mathbb{P}$ witnessed by $(F_m, \eta_m)_{m\in\omega}$ where $F_0 = \eta_0 = \emptyset$,
2. for all $m \in \omega$, p_m is (F_m, η_m)-faithful and $p_m \leq p$,
3. for all $m \in \omega$ and all $i \in F_m$, $\eta_{m+1}(i) \leq \eta_m(i) + 1$,
4. for all $m \in \omega$, j_m is the unique element of F_{m+1} such that either $j_m \in F_m$ and $\eta_{m+1}(j_m) = \eta_m(j_m)+1$ or $j_m \notin F_m$ and $\eta_{m+1}(j_m) = 1$, and
5. for all $m \in \omega$ and all $\sigma \in \prod_{i\in F_m} 2^{\eta_m(i)}$, $p_m * \sigma$ decides at least $\dot{z} \restriction m$.

Let q be the fusion of the p_m, $m \in \omega$, and let $T := T_q(\dot{z})$. Note that for all $m \in \omega$, $q \leq_{F_m,\eta_m} p_m$. For each $m \in \omega$ let T_{m+1} be the tree generated by

$$\left\{ \dot{z}_{p_m*\sigma} : \sigma \in \prod_{i\in F_m} 2^{\eta_m(i)} \right\},$$

and let T_0 be the "tree" generated by $\dot{z}_{p_0}$.
Now each T_m is a subtree of T. We show that $T = \bigcup_{m\in\omega} T_m$.

Let $t \in T$ and $m := \mathrm{dom}(t)$. Let $r \leq q$ be a condition which forces that t is an initial segment of $\dot{z}$. The set $\{q * \sigma : \sigma \in \prod_{i\in F_m} 2^{\eta_m(i)}\}$ is a maximal antichain below q. It follows that there is $\sigma \in \prod_{i\in F_m} 2^{\eta_m(i)}$ such that r is compatible with $q*\sigma$. Since $q*\sigma \leq p_m*\sigma$, r is compatible with $p_m * \sigma$. This implies $t \subseteq \dot{z}_{p_m*\sigma}$ and thus $t \in T_m$.

It remains to prove that T is binary. We first show that for every $m \in \omega$ and $t \in T_m$, if there is $s \in T$ such that s and t are incomparable, then there is $s' \in T_m$ such that $s' \subseteq s$ and already s' and t are incomparable. Let s, t, and m be as before. Let $r \leq q$ be a condition which forces s to be an initial segment of $\dot{z}$. Then there is $\sigma \in \prod_{i\in F_m} 2^{\eta_m(i)}$ such that r is compatible with $p_m * \sigma$. Let $\tau \in \prod_{i\in F_m} 2^{\eta_m(i)}$ be such that $t \subseteq \dot{z}_{p_m*\tau}$. Clearly, $\sigma \neq \tau$. Let $m' \leq m$ minimal such that

$$\sigma' := (\sigma(\alpha) \restriction \eta_{m'}(\alpha))_{\alpha\in F_{m'}}$$

and

$$\tau' := (\tau(\alpha) \restriction \eta_{m'}(\alpha))_{\alpha \in F_{m'}}$$

are different. Now $s' := \dot{z}_{p_{m'} * \sigma'}$ and $\dot{z}_{p_{m'} * \tau'}$ are incomparable by the $(F_{m'}, \eta_{m'})$-faithfulness of $p_{m'}$. Clearly, $s' \in T_m$, and s' and t are incomparable.

Now suppose that T is not binary. Then there is $t \in T$ such that t has three distinct immediate successors t_0, t_1, and t_2 in T. Let $m \in \omega$ be minimal such that T_m contains at least two of the t_j, $j < 3$. By what we proved just before, T_m in fact contains all the t_j. For every $j < 3$ let $\sigma_j \in \prod_{i \in F_m} 2^{\eta_m(i)}$ be such that $t_j \subseteq \dot{z}_{p_m * \sigma_j}$. For every $n \leq m$ and every $j < 3$ let $\sigma_j^n := (\sigma_j(\alpha) \restriction \eta_n(\alpha))_{\alpha \in F_n}$. Now let $n \leq m$ be minimal such that the set $\{\sigma_j^n : j < 3\}$ has at least two elements. Then $\{\sigma_j^n : j < 3\}$ has exactly 2 elements and $n < m$ by the construction of F_n and η_n. Without loss of generality we may assume that σ_0^n and σ_1^n are different. Since m is minimal such that T_m contains at least two of the t_j and since $n < m$, $\dot{z}_{p_n * \sigma_0^n}$ and $\dot{z}_{p_n * \sigma_1^n}$ are comparable. However, this contradicts the (F_n, η_n)-faithfulness of p_n. □

Corollary 66 *Let V be a model of set theory satisfying CH and suppose that κ is a cardinal V. Then there is a generic extension $V[G]$ of V such that $V[G] \models 2^{\aleph_0} \geq \kappa$, $V[G]$ has the same cardinals as V, and $V[G]$ has the 2-localization property over V. In particular, it is consistent with an arbitrarily large size of the continuum that all cardinals in Cichoń's diagram are $\aleph_1$.*

PROOF: In V let P be the countable support product of κ copies of Sacks forcing. By CH, P has no antichains of size $\aleph_2$ and therefore does not collapse any cardinal above $\aleph_1$, as we mentioned before. By Lemma 62, P does not collapse $\aleph_1$. Let G be P-generic over V. Then $V[G]$ and V have the same cardinals and $V[G]$ has the Sacks property over V by Lemma 62. Now it follows from Corollary 32 that all the cardinals in Cichoń's diagram are $\aleph_1$. It remains to evaluate the size of the continuum in $V[G]$. As mentioned before, it is easily checked that the Sacks reals added by the different factors of P are pairwise different. It follows that $2^{\aleph_0}$ is at least κ in $V[G]$. □

In order to determine the exact value of $2^{\aleph_0}$ in $V[G]$ in this proof, one has to analyze the construction of names for subsets of ω. For simplicity, let us assume that V satisfies the Generalized Continuum Hypothesis (GCH) and suppose that κ is of uncountable cofinality in V. Now it can be shown that in V there is a set N of size κ such that every subset of ω in $V[G]$ has a name in N. This implies that in $V[G]$, $2^{\aleph_0}$ is exactly κ.

Countable support products of Sacks forcing have been used by Baumgartner [Bau85] to show the consistency of the total failure of Martin's Axiom: The topological version of Martin's Axiom can be stated as follows: No compact space without uncountable disjoint families of open sets is the union of $< 2^{\aleph_0}$ nowhere dense sets. The Baire Category Theorem implies that no compact space is the union of countably many nowhere dense sets. Martin's Axiom *fails totally* if every compact space without uncountable disjoint families of open sets is the union of $\aleph_1$ nowhere dense sets, but $2^{\aleph_0} > \aleph_1$. Baumgartner's article includes a nice introduction to countable support products of Sacks forcing.

7 Iteration of Sacks forcing

Side-by-side products of partial orders adding reals, such as the countable support products of Sacks forcing introduced in Section 6, have one major disadvantage when it comes to strong independence results: The reals added by the different factors of the product are independent over the ground model. That is, if P is the countable support product of κ copies of Sacks forcing, G is P-generic over the the ground model V, and for each $\alpha < \kappa$, x_α denotes the Sacks real added by the α-th factor of P, then for every set $A \in V$ with $A \subseteq \kappa$ we have

$$(2^\omega)^{V[(x_\alpha)_{\alpha\in A}]} \cap (2^\omega)^{V[(x_\alpha)_{\alpha\in\kappa\setminus A}]} = (2^\omega)^V.$$

Recall that officially we only defined $V[X]$ if X is a set of ordinals. But it should be clear how to code $(x_\alpha)_{\alpha\in A}$ and $(x_\alpha)_{\alpha\in\kappa\setminus A}$ by sets of ordinals.

In Section 6 we showed that forcing with countable support products of copies of Sacks forcing over models of CH produces models of set theory in which many cardinal characteristics of the continuum, such as the cardinals in Cichoń's diagram, are small (i.e., equal to $\aleph_1$), while the size of the continuum is big. However, there is one class of cardinal characteristics of the continuum whose members are still big in such models.

Let X be a set and $f : X \to X$. A point $(x, y) \in X^2$ is *covered* by f iff $f(x) = y$ or $f(y) = x$. A family $\mathcal{F}$ of functions from X to X covers $A \subseteq X^2$ iff every point in A is covered by some member of $\mathcal{F}$.

One might ask how many continuous functions are needed to cover $\mathbb{R}^2$. As it turns out, the number of continuous functions needed to cover $(2^\omega)^2$ is the the same as for $\mathbb{R}^2$ [GesGolKoj0*]. In the present context it is more convenient to consider 2^ω instead of $\mathbb{R}$.

Let V be a model of GCH and suppose that $\kappa > \aleph_1$ is a cardinal of uncountable cofinality in V. Let P be the countable support product of κ copies of Sacks forcing and let G be P-generic over V. For $\alpha < \kappa$ let x_α denote the Sacks real added by the α-th factor of P, as above. Suppose that in $V[G]$, $\mathcal{F}$ is a family of size $< \kappa$ of continuous functions from 2^ω to 2^ω. Since every continuous function from 2^ω to 2^ω is in fact a closed subset of $(2^\omega)^2$, it can be coded (for example, using Borel codes) by a subset of ω. Now it can be shown that there is a set $A \in V$ such that A is of size $< \kappa$ in V and $\mathcal{F} \subseteq V[(x_\alpha)_{\alpha\in A}]$. Let $\beta, \gamma \in \kappa \setminus A$. Since $x_\gamma \notin V[(x_\alpha)_{\alpha\in A\cup\{\beta\}}]$ and $x_\beta \notin V[(x_\alpha)_{\alpha\in A\cup\{\gamma\}}]$, no function from $\mathcal{F}$ covers (x_β, x_γ). It follows that in $V[G]$, $\mathcal{F}$ does not cover $(2^\omega)^2$. Since $\mathcal{F}$ was an arbitrary family of size $< \kappa$ of continuous functions on 2^ω, we have shown that in $V[G]$, $(2^\omega)^2$ cannot be covered by less than κ continuous functions.

So, if we want to show that it is consistent that $(2^\omega)^2$ can be covered by less than $2^{\aleph_0}$ continuous functions, then we have to look for a different construction of models of set theory. We have to *iterate* Sacks forcing. The fundamental paper about iterated Sacks forcing is [BauLav79].

Definition 71 *Let P be a partial order and let $\dot{Q}$ be a P-name for a partial order. Then $P * \dot{Q}$ is the partial order consisting of all pairs $(p, \dot{q})$ such that $p \in P$ and $\dot{q}$ is a P-name for an element of $\dot{Q}$. $P * \dot{Q}$ is ordered as follows: For $(p_0, \dot{q}_0), (p_1, \dot{q}_1) \in P * \dot{Q}$ let $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$ iff $p_0 \leq p_1$ and $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$.*

There is an obvious embedding $e : P \to P * \dot{Q}$: Let $\dot{1}_Q$ be a P-name for the largest element of $\dot{Q}$. For every $p \in P$ let $e(p) := (p, \dot{1}_Q)$. It is easily checked that e is a complete embedding. In particular, if G is $P * \dot{Q}$-generic over the ground model V, then $G_0 := e^{-1}[G]$ is P-generic over V. In $V[G]$ let $G_1 := \{\dot{q}_{G_0} : \exists p \in P((p, \dot{q}) \in G)\}$. Then G_1 is $\dot{Q}_{G_0}$-generic over $V[G_0]$. On the other hand, if G_0 is P-generic over V and G_1 is $\dot{Q}_{G_0}$-generic over $V[G_0]$, then $\{(p, \dot{q}) \in P * \dot{Q} : p \in G_0 \wedge \dot{q}_{G_0} \in G_1\}$ is $P * \dot{Q}$-generic over V.

This shows that forcing with $P * \dot{Q}$ is the same as first adding a P-generic filter G_0 and then adding a $\dot{Q}_{G_0}$-generic filter over it. The point of iterated forcing is that we can add generics over each other using a single forcing notion in the ground model. The iteration becomes really relevant only when we want to add infinitely many generics over each other.

Definition 72 *Let δ be an ordinal. A* countable support iteration of length *δ is an object of the form $((P_\alpha)_{\alpha \leq \delta}, (\dot{Q}_\alpha)_{\alpha < \delta})$ with the following properties:*

1. *$(P_\alpha)_{\alpha \leq \delta}$ is a sequence of forcing notions and P_0 is the trivial forcing notion containing just one element,*
2. *$(\dot{Q}_\alpha)_{\alpha < \delta}$ is a sequence such that for each $\alpha < \delta$, $\dot{Q}_\alpha$ is a P_α-name for a forcing notion (more precisely, a name for a set together with a name for an ordering on that set),*
3. *for all $\alpha < \delta$, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, and*
4. *if $\beta \leq \delta$ is a limit ordinal, then P_β consist of all functions q with domain β such that*
   (a) *for all $\alpha < \beta$, $q(\alpha)$ is a P_α-name for a condition in $\dot{Q}_\alpha$ and $q \restriction \alpha \in P_\alpha$ and*
   (b) *the support* $\mathrm{supt}(q) := \{\alpha < \beta : 1_{P_\alpha} \not\Vdash q(\alpha) = 1_{\dot{Q}_\alpha}\}$ *of q is countable (where $1_{\dot{Q}_\alpha}$ is aP_α-name for the largest element of $\dot{Q}_\alpha$)*

   and P_β is ordered as follows: for $p, q \in P_\beta$ we have $p \leq q$ iff for all $\alpha < \beta$, $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \leq q(\alpha)$.

This definition deserves several remarks. First, it should be clear how one has to change the definition in order to obtain the definition of *finite support iteration*: Just replace "countably many" in 4b by "finitely many". While finite supports are suitable for iterating c.c.c. forcings since finite support iterations of c.c.c. forcings are again c.c.c., countable supports are appropriate for Axiom A forcings since with countable supports it is possible to extend the fusion technology used to deal with Axiom A forcings to the iteration.

A countable support iteration $((P_\alpha)_{\alpha\leq\delta}, (\dot{Q}_\alpha)_{\alpha<\delta})$ is already determined by the $\dot{Q}_\alpha$. Therefore, slightly abusing notation, we will frequently call P_δ the *countable support iteration of the* $\dot{Q}_\alpha$, $\alpha < \delta$. For all $\alpha, \beta \leq \delta$ with $\alpha < \beta$ there is a natural embedding $e_{\alpha\beta} : P_\alpha \to P_\beta$ mapping every $p \in P_\alpha$ to p' where $p' \restriction \alpha = p$ and for all $\gamma \in [\alpha, \beta)$, $p'(\gamma) = 1_{\dot{Q}_\gamma}$. The $e_{\alpha\beta}$ are complete embeddings. Via $e_{\alpha\delta}$ we may consider P_α as a subset of P_δ.

Let G be a P_δ-generic filter over the ground model V. Then for every $\alpha < \delta$, $G_\alpha := G \cap P_\alpha$ is P_α-generic over V. Thus, we have an increasing chain

$$V \subseteq V[G_1] \subseteq \cdots \subseteq V[G_\alpha] \subseteq \cdots \subseteq V[G]$$

of models of set theory. For every $\alpha < \delta$, $V[G]$ is a generic extension of $V[G_\alpha]$, obtained by adding a Q-generic filter over $V[G_\alpha]$, where $Q \in V[G_\alpha]$ is the so-called *quotient of* P_δ *over* G_α. Q can be constructed in a natural way from $((P_\alpha)_{\alpha\leq\delta}, (\dot{Q}_\alpha)_{\alpha<\delta})$ and G_α and is a countable support iteration of length $\delta - \alpha$. (See [Gol93, Section 4] for more on quotient forcing.)

The *countable support iteration of Sacks forcing of length* δ is the forcing notion S_δ where $((S_\alpha)_{\alpha\leq\delta}, (\dot{Q}_\alpha)_{\alpha<\delta})$ is a countable support iteration such that for every $\alpha < \delta$, $\dot{Q}_\alpha$ is an S_α-name for Sacks forcing. *The Sacks model* is a model of set theory of the form $V[G]$ where G is S_{ω_2}-generic over V and V is a model of CH. The formulation “the Sacks model” is slightly misleading, since the model is not uniquely determined. However, S_{ω_2}-generic extensions of models of CH are sufficiently similar to each other to be considered the same most of the time.

If the ground model satisfies CH, then S_{ω_2} has no antichains of length $\aleph_2$ (see [BauLav79] or, for a more general result about proper forcing, [She98, Chapter III, Theorem 4.1]) and therefore does not collapse any cardinal $\geq \aleph_2$. Since countable support iterations of Sacks forcing have the 2-localization property, as we will show in a moment, they do not collapse $\aleph_1$. It follows that forcing with S_{ω_2} over models of CH does not collapse cardinals. Since S_{ω_2} has the 2-localization property, it follows from Corollary 32 that all the cardinals in Cichoń's diagram are $\aleph_1$ in the Sacks model.

If G is S_δ-generic over the ground model V and $\alpha \leq \delta$, then the quotient Q of S_δ over $G \cap S_\alpha$ is (equivalent to) the countable support iteration of Sacks forcing of length $\delta - \alpha$ in $V[G_\alpha]$. Moreover, $V[G_{\alpha+1}] = V[G_\alpha][x_\alpha]$ where x_α is the Sacks real added by the last factor of the iteration $S_{\alpha+1}$. Using this notation we see that for all $\beta \leq \delta$, $V[G_\beta] = V[(x_\alpha)_{\alpha<\beta}]$. As in the case of countable support products, an easy density argument shows that the x_α are pairwise different. Analyzing S_{ω_2}-names for subsets of ω one can see that the size of the continuum does not exceed $\aleph_2$ in the Sacks model. It follows that the size of the continuum is exactly $\aleph_2$ in the Sacks model.

We already mentioned that countable support iterations of Sacks forcing cannot be used to construct models of set theory where $2^{\aleph_0} > \aleph_2$. This is because countable support iterations of nontrivial forcing notions of some length with cofinality $\aleph_1$ collapse the size of the continuum to $\aleph_1$. (See [Gol93, Section 0] for more details.) Thus, if δ is an ordinal with cofinality $> \aleph_1$ and G is S_δ-generic over the ground model V, then for cofinally many $\alpha < \delta$, $V[G_\alpha]$ is a model of CH. It follows that $2^{\aleph_0}$ cannot be bigger than $\aleph_2$ in $V[G]$.

The Sacks model is generally known as the model of set theory in which all reasonable cardinal characteristics of the continuum (that can consistently be smaller than $2^{\aleph_0}$) are $\aleph_1$. Recently this has been turned into a provable statement by Zapletal [Zap0*]. For a large class of combinatorial cardinal characteristics of the continuum he showed that if there is a forcing extension of the ground model V in which the cardinal characteristic in question is smaller than $2^{\aleph_0}$, then this cardinal is $\aleph_1$ in $V[G]$ where G is $S_{(2^{\aleph_0})^+}$-generic over V. $V[G]$ actually is “the” Sacks model. Even if V is not a model of CH, forcing with S_{ω_1}, an initial part of $S_{(2^{\aleph_0})^+}$, collapses the $2^{\aleph_0}$ to $\aleph_1$. Now the $(2^{\aleph_0})^+$ of V is $\aleph_2$. The remaining part of the iteration is just forcing with S_{ω_2} over a model of

CH.

The proof of the following theorem illustrates the main techniques for proving statements about the Sacks model. The theorem together with Lemma 74 summarizes a sizeable portion of what is known about this model of set theory.

Theorem 73 *In the Sacks model, $(2^\omega)^2$ can be covered by $\aleph_1$ continuous functions from 2^ω to 2^ω, while $2^{\aleph_0} = \aleph_2$.*

This theorem was explicitly shown in [HarSte02]. However, implicitly it already follows from Miller's [Mil83] about mapping sets of reals onto the reals and from the work of Groszek [Gro81] showing that the constructible degrees of reals in a S_{ω_2}-generic extension of L are wellordered of ordertype ω_2. Constructible degrees of reals are defined as follows: For $x, y \in 2^\omega$ let $x \leq y$ iff $x \in L[y]$. $L[y]$ can be constructed without forcing. (See [Jec78, Section 15].) A constructible degree is an equivalence class with respect to the relation $\leq \cap \geq$. The order $\leq$ induces an order on the constructible degrees, the *order of constructible degrees*. Stronger versions of Theorem 73 can be found in [CiePaw0*], [GesKojKubSch02], and [Ste99].

There are more consistency results about the structure of constructible degrees of reals in models of set theory. Let us mention two examples.

Groszek [Gro94] showed that it is consistent to have a reversed copy of ω_1 as an initial segment of the order of constructible degrees. Kanovei and Zapletal [KanZap98] showed that there is a generic extension $L[G]$ of L in which there are a strictly increasing sequence $(a_n)_{n\in\omega}$ of constructible degrees and a constructible degree b such that the following holds: For two constructible degrees c, d let $\langle c, d\rangle$ denote the constructible degree which is the least upper bound of c and d. Then the sequence $(\langle a_n, b\rangle)_{n\in\omega}$ is strictly increasing and if $x \in L[G]$ is not an element of any intermediate model of the form $L[a_0, \ldots, a_n, b]$, then $L[G] = L[x]$.

We go back to the proof of Theorem 73 . Since the number of ground model reals is $\aleph_1$ in the Sacks model, the theorem follows immediately from the next lemma, which gives some additional information.

Lemma 74 *a) Let δ be an ordinal and suppose that G be S_δ-generic over the ground model V. For $\alpha < \delta$ let $G_\alpha := G \cap S_\alpha$. Then in $V[G]$ the following holds: Let $x \in 2^\omega$ and let $\alpha \leq \delta$ be the first ordinal such that $x \in V[G_\alpha]$. Then for all $y \in 2^\omega \cap V[G_\alpha]$ there is a continuous function $f : 2^\omega \to 2^\omega$ coded in V such that $f(x) = y$. In particular, $(2^\omega)^2$ is covered by the continuous functions coded in the ground model.*

b) For every ordinal δ, S_δ has the 2-localization property and therefore has the Sacks property.

PROOF: Our proof of a) will be more or less the same as the proof of a similar statement in [GesGolKoj0*], which talks about slightly different forcing notions. In order to find the required continuous functions in the ground model, we first observe that for 2^ω an extension theorem similar to the Tietze-Uryson Theorem holds.

Claim 75 *Whenever A is a closed subset of 2^ω and $f : A \to 2^\omega$ is continuous, then f can be extended to a continuous function $\overline{f} : 2^\omega \to 2^\omega$.*

This can be seen as follows: First of all, it suffices to extend every continuous $g : A \to 2$ to a continuous function $\overline{g} : 2^\omega \to 2$ since f is built from countably many such functions. If $g : A \to 2$ is continuous, then $g^{-1}(0)$ and $g^{-1}(1)$ are disjoint closed subsets of 2^ω. Since the topology on 2^ω is generated by clopen sets and since $g^{-1}(0)$ and $g^{-1}(1)$ are in fact compact, there is a clopen set $B \subseteq 2^\omega$ such that $g^{-1}(1) \subseteq B$ and $g^{-1}(0) \subseteq 2^\omega \setminus B$. Let $\overline{g} : 2^\omega \to 2$ be the characteristic function of B. Since B is clopen, $\overline{g}$ is continuous. By the choice of B, $\overline{g}$ extends g. This finishes the proof of Claim 75.

This extension property shows that it actually suffices to show that for x and y as in a) there are a closed set $A \subseteq 2^\omega$ and a continuous function $f : A \to 2^\omega$ such that f (and therefore A) is coded in V and $f(x) = y$. Since we intend to work in V, we have to choose suitable names for x and y first. It is worth mentioning that no new reals are added at limit stages of forcing iterations of uncountable cofinality. That is, if α is minimal with $x \in V[G_\alpha]$, then α is of countable cofinality. (See for example [BarJud95, Lemma 1.5.7].)

Claim 76 *Let $\dot{x}$ be an S_α-name such that there is no $\beta < \alpha$ with $\dot{x}_{G_\alpha} \in V[G_\beta]$. Then there is an S_α-name $\dot{z}$ such that $\dot{x}_{G_\alpha} = \dot{z}_{G_\alpha}$ and for all $\beta < \alpha$ and all S_α-generic filters H over V, $\dot{z}_H \notin V[H_\beta]$ where $H_\beta := H \cap S_\beta$.*

We first show that there is $p \in S_\alpha$ such that for no S_α-generic filter H over V with $p \in H$ there is $\beta < \alpha$ such that $x \in V[H_\beta]$. Note that every S_β-name, $\beta < \alpha$, corresponds canonically to some S_α-name for the same object. We may therefore identify every S_β-name with the corresponding S_α-name. Now consider the sets

$$D_0 := \{p \in S_\alpha : \text{For no } S_\alpha\text{-generic filter } H \text{ over } V \\ \text{with } p \in H \text{ there is } \beta < \alpha \text{ such that } \dot{x}_H \in V[H_\beta]\}$$

and

$$D_1 := \{p \in S_\alpha : \text{There are } \beta < \alpha \\ \text{and an } S_\beta\text{-name } \dot{z} \text{ such that } p \Vdash \dot{x} = \dot{z}\}.$$

$D_0 \cup D_1$ is dense in S_α: Suppose that $p \in S_\alpha$ is not in D_0. Then there are an S_α-generic filter H, $\beta < \alpha$, and an S_β-name $\dot{z}$ such that $p \in H$ and $\dot{x}_H = \dot{z}_{H_\beta}$. Since $\dot{x}_H = \dot{z}_{H_\beta}$ there is some $q \in H$ which forces $\dot{x} = \dot{z}$. Since H is a filter, we may assume $q \leq p$. Clearly, $q \in D_1$. This shows the density of $D_0 \cup D_1$. By the genericity of G, there is $p \in G \cap (D_0 \cup D_1)$. If $p \in D_1$, then α is not minimal with $x \in V[G_\alpha]$. This shows $p \in D_1$. Therefore p is as required.

Using the Maximality Principle it is now easy to construct an S_α-name $\dot{z}$ for an element of 2^ω such that $p \Vdash \dot{z} = \dot{x}$ and for no S_α-generic filter H over V there is $\beta < \alpha$ such that $\dot{z}_H \in V[H_\beta]$. (Choose an S_α-name $\dot{y}$ for an element of 2^ω which is not added before stage α. Let $\dot{z}$ be such that $p \Vdash \dot{z} = \dot{x}$ and $q \Vdash \dot{z} = \dot{y}$ for all $q \in S_\alpha$ that are incompatible with p.) This finishes the proof of Claim 76.

Using Claim 76 we can find an S_α-name $\dot{x}$ for an element of 2^ω such that $x = \dot{x}_{G_\alpha}$ and for all S_α-generic filters H and all $\beta < \alpha$, $\dot{x}_H \notin V[H_\beta]$. In other words, $\dot{x}$ is a name for a real not added before stage α of the iteration.

For every $p \in S_\alpha$ we will construct $q \leq p$ such that for $A := \mathrm{supt}(p)$ the following condition $(*)_{q,A,\dot{x}}$ holds:

$(*)_{q,A,\dot{x}}$ Let $T_q(\dot{x})$ be the tree of q-possibilities for $\dot{x}$. Then in V we have a homeomorphism $h : [T_q(\dot{x})] \to (2^\omega)^A$ such that if H is S_α-generic over V with $q \in H$, then h maps $\dot{x}_H$ to a sequence $(z_\gamma)_{\gamma\in A} \in (2^\omega)^S$ such that for all $\gamma \in A$, z_γ is the image of the γ'th generic real under the natural homeomorphism from $[q(\gamma)_H]$ to 2^ω.

So in a weak sense we can reconstruct the restriction of the sequence of generic reals to $\mathrm{supt}(q)$ from $\dot{x}_H$ using a ground model function. We will see soon that we can really reconstruct the sequence of generic reals below α from $\dot{x}_H$.

It is not difficult to see

Claim 77 *If $(*)_{q,A,\dot{x}}$ holds for some $q \in S_\alpha$ and a countable set $A \subseteq \alpha$, then $(*)_{r,A,\dot{x}}$ holds for every $r \leq q$.*

Now let $\beta \leq \alpha$ be such that β is minimal with $y \in V[G_\beta]$. Applying Claim 76 we find an S_β-name $\dot{y}$ for an element of 2^ω such that $\dot{y}_{G_\beta} = y$ and for no S_β-generic filter H over V there is $\gamma < \beta$ such that $\dot{y}_H \in V[H_\gamma]$. Note that we may assume that $\beta > 0$ since otherwise $y \in V$ and thus there is a constant function in V mapping x to y.

Let $p \in S_\alpha$ and suppose we find $q \in S_\beta$ such that $q \leq p \restriction \beta$ and $(*)_{q,\mathrm{supt}(q),\dot{x}}$ holds. Suppose we can then find $r \in S_\alpha$ such that $r \leq q{}^\frown p \restriction [\beta, \alpha)$ and $(*)_{r,\mathrm{supt}(r),\dot{x}}$ holds. Let $h : [T_r(\dot{x})] \to (2^\omega)^{\mathrm{supt}(r)}$ be the homeomorphism (in the ground model) guaranteed by $(*)_{r,\mathrm{supt}(r),\dot{x}}$. Let $g : [T_r(\dot{y})] \to (2^\omega)^{\mathrm{supt}(q)}$ be the homeomorphism guaranteed by $(*)_{r,\mathrm{supt}(q),\dot{y}}$, which holds by Claim 77.

Finally let $\pi : (2^\omega)^{\mathrm{supt}(r)} \to (2^\omega)^{\mathrm{supt}(q)}$ be the natural projection and put $f := g^{-1} \circ \pi \circ h$. f is only defined on a closed subset of 2^ω, but by Claim 75 we can extend it to a continuous function $\overline{f} : 2^\omega \to 2^\omega$. Clearly $r \Vdash f(\dot{x}) = \dot{y}$. This finishes the proof of Lemma 74 provided we know

Lemma 78 *Let α be an ordinal and $\dot{x}$ an S_α-name for an element of 2^ω which is not added in a proper initial stage of the iteration. Then for every $p \in S_\alpha$ there is $q \leq p$ such that $(*)_{q,\mathrm{supt}(q),\dot{x}}$ holds.*

PROOF: The proof of this lemma is closely related to the proof of the 2-localization property of countable support products of Sacks forcing in Section 6. We have to define the notion of a fusion sequence for iterations of Sacks forcing. Let F be a finite subset of α and $\eta : F \to \omega$. For $p, q \in S_\alpha$ let $p \leq_{F,\eta} q$ iff for all $\gamma \in F$, $p \restriction \gamma \Vdash p(\gamma) \leq_{\eta(\gamma)} q(\gamma)$. A sequence $(p_n)_{n\in\omega}$ is a *fusion sequence* if there are an increasing sequence $(F_n)_{n\in\omega}$ of finite subsets of α and a sequence $(\eta_n)_{n\in\omega}$ such that for all $n \in \omega$, $\eta_n : F_n \to \omega$, for all $i \in F_n$ we have $\eta_n(i) \leq \eta_{n+1}(i)$, $p_{n+1} \leq_{F_n,\eta_n} p_n$, and for all $\gamma \in \mathrm{supt}(p_n)$ there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geq n$. The *fusion* p_ω of a fusion sequence $(p_n)_{n\in\omega}$ in S_α is defined as follows: Let $\gamma < \alpha$ and suppose we have already defined $p_\omega \restriction \gamma$. Let $p_\omega(\gamma)$ be an S_γ-name for a condition in S such that $p_\omega \restriction \gamma$ forces $p_\omega(\gamma)$ to be the intersection of the $p_n(\gamma)$, $n \in \omega$.

We also use a notion of faithfulness. For F and η as above, $p \in S_\alpha$, and $\sigma \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$ let $p * \sigma \in S_\alpha$ be such that for all $\gamma \in F$,

$$p * \sigma \restriction \gamma \Vdash p * \sigma(\gamma) = p(\gamma) * \sigma(\gamma)$$

and for $\gamma \in \alpha \setminus F$, $p * \sigma(\gamma) = p(\gamma)$. A condition $p \in S_\alpha$ is (F, η)*-faithful* if for all $\sigma, \tau \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$ with $\sigma \neq \tau$, $\dot{x}_{p*\sigma}$ and $\dot{x}_{p*\tau}$, the longest initial segments of $\dot{x}$ decided by $p * \sigma$, respectively by $p * \tau$, are incomparable (with respect to $\subseteq$). (Compare this to the corresponding definitions in Section 6.)

The following claim is the iteration version of Claim 65.

Claim 79 *Let F and η be as before and suppose that $q \in S_\alpha$ is (F, η)-faithful.*

a) Let $\beta \in \alpha \setminus F$ and let $F' := F \cup \{\beta\}$ and $\eta' := \eta \cup \{(\beta, 0)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F', η')-faithful.

b) Let $\beta \in F$ and let $\eta' := \eta \restriction F \setminus \{\beta\} \cup \{(\beta, \eta(\beta) + 1)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F, η')-faithful.

a) follows immediately from the definitions. For b) let $\delta := \max F$ and let $\{\sigma_0, \ldots, \sigma_m\}$ be an enumeration of $\prod_{\gamma\in F} 2^{\eta(\gamma)}$. We define a $\leq_{F,\eta}$-decreasing sequence $(q_j)_{j\leq m}$ in S_α along with names $q_{\sigma,0}$ and $q_{\sigma,1}$, $\sigma \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$, for conditions.

Let $j \in \{1, \ldots m\}$ and assume that q_{j-1} has been constructed already. Since $\dot{x}$ is not added in a proper initial stage of the iteration, there are $q_{\sigma_j,0}$ and $q_{\sigma_j,1}$ such that for all $i \in 2$

$$q_{j-1} * \sigma_j \restriction \delta \Vdash q_{\sigma_j,i} \leq (q(\delta) * (\sigma_j(\delta) ^\frown i)) ^\frown q \restriction (\delta, \alpha)$$

and

$$q_{j-1} * \sigma_j \restriction \delta \Vdash x_{q_{\sigma_j},0} \text{and } x_{q_{\sigma_j},1} \text{ are incomparable.}$$

Let $q_j \leq_{F,\eta} q_{j-1}$ be such that $q_j * \sigma \restriction \delta$ decides $x_{q_{\sigma_j},0}$ and $x_{q_{\sigma_j},1}$. This finishes the inductive construction of the q_j.

Now let $r \leq_{F,\eta} q_m$ be such that $r \restriction \delta = q_m \restriction \delta$ and for all $\sigma \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$ and all coordinatewise extensions $\tau \in \prod_{\gamma\in F} 2^{\eta'(\gamma)}$ of σ,

$$r * \tau \restriction \delta \Vdash r * \tau \restriction [\delta, \alpha) = q_{\sigma,\tau(\eta(\beta))}.$$

It is easy to check that r works for Claim 79.

To conclude the proof of Lemma 78, let $p \in S_\alpha$. Using some bookkeeping and parts a) and b) of Claim 79 we construct a sequence $(p_n)_{n\in\omega}$ and a sequence $(F_n, \eta_n)_{n\in\omega}$ witnessing that $(p_n)_{n\in\omega}$ is a fusion sequence such that $p = p_0$ and for all $n \in \omega$, p_n is (F_n, η_n)-faithful.

Let q be the fusion of the sequence $(p_n)_{n\in\omega}$. We have to check that $(*)_{q,\mathrm{supt}(q),\dot{x}}$ holds. Let $a \in [T_q(\dot{x})]$ and $n \in \omega$. Now $q \leq_{F_n,\eta_n} p_n$ and p_n is (F_n, η_n)-faithful. It follows that there is exactly one $\sigma_{a,n} \in \prod_{\gamma\in F_n} 2^{\eta_n(\gamma)}$ such that $\dot{x}_{q*\sigma_{a,n}} \subseteq a$. Let $h(a) := (\bigcup_{n\in\omega} \sigma_{a,n}(\gamma))_{\gamma\in\mathrm{supt}(q)}$. Since for all $\gamma \in \mathrm{supt}(q)$ and all $m \in \omega$ there is some $n \in \omega$ such that $\gamma \in F_n$ and $\eta_n(\gamma) \geq m$, $h(a) \in (2^\omega)^{\mathrm{supt}(q)}$. It is easily checked that $h : [T_q(\dot{x})] \to (2^\omega)^{\mathrm{supt}(q)}$ is a homeomorphism witnessing $(*)_{q,\mathrm{supt}(q),\dot{x}}$. □

Now we turn the proof of part b) of Lemma 74. Let G be S_δ-generic over the ground model V. By Claim 76, for $x \in \omega^\omega$ there is $\alpha \leq \delta$ and an S_α-name for an element of ω^ω such that $x = \dot{x}_{G_\alpha}$ and for all S_α-generic filters H over V there is no $\beta < \alpha$ such that $\dot{x}_H \in V[H_\beta]$.

Using the proof of Lemma 78 for the name $\dot{x}$ for an element of ω^ω instead of 2^ω, for every $p \in S_\alpha$ we obtain a condition $q \leq p$ such that $(*)_{q,\mathrm{supt}(q),\dot{x}}$ is satisfied. This is because the proof of Lemma 78 does not depend on the fact that $\dot{x}$ is a name for an element of 2^ω. It works for names for elements of ω^ω as well. From the construction of q it follows that $T_q(\dot{x})$ is binary. Clearly, q forces $\dot{x}$ to be a branch through $T_q(\dot{x})$.

By the genericity of G_α, there is $q \in G_\alpha$ such that $T_q(\dot{x})$ is binary. This shows the 2-localization property of S_δ. □

Lemma 74 shows how S can collapse cardinals: Let G be S_{ω_2} generic over the ground model V. Suppose that V satisfies CH. Then in $V[G]$ the size of the continuum is $\aleph_2$. Let H be S-generic over $V[G]$. Then $V[G][H]$ is obtained by forcing with an iteration of Sacks forcing of length $\omega_2 + 1$ over V. For $\gamma < (\omega_2)^V + 1$ let x_γ denote the generic real added at stage γ. Here $(\omega_2)^V$ denotes the ordinal that is ω_2 in V. By Lemma 74, for every $\gamma < (\omega_2)^V$ there is a continuous function $f_\gamma : 2^\omega \to 2^\omega$ such that $f_\gamma \in V$ and $f_\gamma(x_{(\omega_2)^V}) = x_\gamma$. Since there are only $\aleph_1$ continuous function from 2^ω to 2^ω in V, this implies $(\omega_2)^V \leq \aleph_1$ in $V[G][H]$. This shows that adding a Sacks real over the Sacks model collapses $\aleph_2$.

At the end of this section let us mention another description of S_δ, which is related more closely to the representation of Sacks forcing as the partial order of uncountable Borel subsets of $\mathbb{R}$.

Definition 710 *Let δ be an ordinal. A set $B \subseteq \mathbb{R}^\delta$ is* `countable`-perfect *iff*

1. *for every ordinal $\beta < \delta$ and every sequence $s \in B \restriction \beta := \{u \restriction \beta : u \in B\}$ the set $\{r \in \mathbb{R} : s^\frown r \in B \restriction \beta + 1\}$ is uncountable and*
2. *for every increasing sequence $(\beta_n)_{n\in\omega}$ of ordinals below δ and every increasing sequence (with respect to inclusion) $(s_n)_{n\in\omega}$ such that for all $n \in \omega$, $s_n \in B \restriction \beta_n$ it is the case that $\bigcup_{n\in\omega} s_n \in B \restriction \bigcup_{n\in\omega} \beta_n$.*

It can be shown that the partial order consisting of all `countable`-perfect Borel subsets of $\mathbb{R}^\delta$ is forcing equivalent to S_δ (see [Zap0*, 3.1]). This way of representing iterations of Sacks forcing was used by Kanovei [Kan99] to iterate Sacks forcing along non-wellfounded linear orders. He showed that whenever I is a linearly ordered index set, it is possible to add a sequence $(a_i)_{i\in I}$ of reals to the ground model V such that for each $j \in I$, a_j is a Sacks real over $V[(a_i)_{i<j}]$.

Using the representation of S_δ in terms of `countable`-perfect Borel sets, we can give the formulation of a set theoretic axiom that axiomatizes the properties of the Sacks model very well.

Definition 711 *Consider the following game between player I and player II lasting ω_1 rounds. At round $\beta \in \omega_1$ player I plays an ordinal $\alpha_\beta < \omega_1$, a* `countable`*-perfect Borel set $B_\beta \subseteq \mathbb{R}^{\alpha_\beta}$, and a Borel function $f_\beta : B_\beta \to \mathbb{R}$. Then player II responds by a* `countable`*-perfect Borel set $C_\beta \subseteq B_\beta$. Player I wins iff $\bigcup_{\beta<\omega_1} f_\beta[C_\beta] = \mathbb{R}$. The* Covering Property Axiom (CPA) *is the statement "CH fails and player II has no winning strategy in the above game".*

CPA was invented by Ciesielski and Pawlikowski [CiePaw0*]. They showed that CPA holds in the Sacks model and implies many statements that were previously known to be true in the Sacks model. In fact, many of these statements already follow from weaker versions of CPA which are also defined in [CiePaw0*] and which seem to be much easier to apply. In particular, these weaker version of CPA avoid some of the technicalities and notational inconveniences usually associated with countable support iterations.

Recently, Zapletal [Zap0*] proved that if V satisfies CPA, then for a large class of cardinal characteristics of the reals the following holds: If there is a generic extension $V[G]$ of V where the cardinal characteristic in question is $< 2^{\aleph_0}$, then this cardinal characteristic is $< 2^{\aleph_0}$ in V. This shows that CPA indeed axiomatizes the Sacks model very well. The formulation of CPA is sufficiently general to be easily adapted for other models of set theory as well. (See [CiePaw0*] and [Zap0*] for this.)

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