# THE NONEXISTENCE OF UNIVERSAL METRIC FLOWS FOR COUNTABLE GROUPS

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ABSTRACT. Let $G$ be a countably infinite discrete group. If a $G$-flow $X$ has all minimal metric $G$-flows as a factor, then it is of weight at least $2^{\aleph_0}$. In particular, neither the class of minimal metric $G$-flows nor the class of metric $G$-flows contain a universal element.

## 1. INTRODUCTION

Let $G$ be a topological group. A *$G$-flow* is a compact space $X$ together with a continuous group action of $G$ on $X$. A $G$-flow $X$ is *minimal* if no nonempty closed subset of $X$ is closed under the group action. That every $G$-flow contains a minimal $G$-flow is a straight forward application of Zorn's lemma. If $G$ has the discrete topology, then $G$ acts continuously on its Čech-Stone compactification $\beta G$. From the universal property of $\beta G$ it follows that every minimal subflow $X$ of $\beta G$ is in fact a *universal* minimal $G$-flow in the sense that every minimal $G$-flow $Y$ is a factor of $X$, i.e., there is a continuous map from $X$ onto $Y$ that commutes with the group action. However, the universal minimal $G$-flows obtained in this way are of weight $2^{\aleph_0}$ and therefore not metrizable. Since universal minimal $G$-flows are unique up to homeomorphism, this argument shows that infinite discrete groups have large universal minimal flows.

This brings up the question of what happens if we restrict our attention to metric $G$-flows. If $G = \mathbb{Z}$, then there are no universal metric minimal $G$-flows, which follows from work of Beleznay and Foreman [2] (see [5] for a sketch of the argument). Motivated by a question by Brian [3], the author has previously provided a different argument for the non-existence of universal metric minimal $\mathbb{Z}$-flows [5]. The purpose of this note is to extend the methods from [5] using a result of Gao, Jackson, and Seward [4] to all discrete countable groups.

## 2. ZERO-DIMENSIONAL FLOWS AND BOOLEAN ALGEBRAS

Let us fix a discrete group $G$. The following lemma generalizes a result due to Anderson [1].

**Lemma 2.1.** *Let $X$ be a $G$-flow of weight $\kappa$. Then $X$ is a factor of a zero-dimensional $G$-flow of weight at most $|G| + \kappa$.*

*Proof.* We may assume that $X$ is infinite. Otherwise it is zero-dimensional and we have nothing to prove. Let $(U_\alpha, V_\alpha)_{\alpha<\kappa}$ be an enumeration of all pairs of nonempty, disjoint open subsets of $X$ from a fixed basis of the topology on $X$. Consider the space $2^{\kappa\times G}$ with the following continuous $G$-action:

For all $s \in 2^{\kappa\times G}$, $g, h \in G$, and $\alpha < \kappa$ let

$$(hs)(\alpha, g) = s(\alpha, h^{-1}g).$$

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Note that $G$ acts continuously on $X \times 2^{\kappa \times G}$ by the coordinate wise action.

Now let $Y \subseteq X \times 2^{\kappa \times G}$ consist of all pairs $(x, s)$ such that for all $\alpha < \kappa$ and all $g \in G$ we have

(1) if $x \in g[U_\alpha]$, then $s(\alpha, g) = 0$ and
(2) if $x \in g[V_\alpha]$, then $s(\alpha, g) = 1$.

We first observe that $Y$ is a closed subset of $X \times 2^{\kappa \times G}$. Suppose $(x, s) \in X \times 2^{\kappa \times G}$ is not in $Y$. Assume there are $\alpha$ and $g$ violating (1). So $x \in g[U_\alpha]$ and $s(\alpha, g) = 1$. Now the pairs $(x', s')$ with $x' \in g[U_\alpha]$ and $s'(\alpha, g) = 1$ form an open neighborhood of $(x, s)$ that is disjoint from $Y$. If there are $\alpha$ and $g$ violating (2), we obtain an open neighborhood of $(x, s)$ that is disjoint from $Y$ in the same way.

$Y$ is zero-dimensional since any two of its elements can be separated by a clopen set. Namely, let $(x, s), (x', s') \in Y$ be distinct. If $s \neq s'$ there is a clopen subset $A$ of $2^{\kappa \times G}$ such that $s \in A$ and $s' \notin A$. Now $Y \cap (X \times A)$ is a clopen subset of $Y$ that contains $(x, s)$ but not $(x', s')$. If $x \neq x'$, then there is $\alpha$ such that $x \in U_\alpha$ and $x' \in V_\alpha$. By the definition of $Y$, this implies that $s \neq s'$ and hence $(x, s)$ and $(x', s')$ are separated by a clopen subset of $Y$.

Also, $Y$ is closed under the action of $G$. To see this, let $(x, s) \in Y$ and $h \in G$. We have to show $(hx, hs) \in Y$. Let $\alpha < \kappa$ and $g \in G$. Suppose $hx \in g[U_\alpha]$. Then $x \in h^{-1}g[U_\alpha]$. Since $(x, s) \in Y$, $0 = s(\alpha, h^{-1}g) = (hs)(\alpha, g)$. Hence $(hx, hs)$ satisfies condition (1) for $\alpha$ and $g$. The argument for condition (2) is the same.

It follows that $Y$ is a zero-dimensional $G$-flow of weight at most $|G| + \kappa$. Let $\pi_1 : X \times 2^{\kappa \times G} \to X$ be the projection onto the first coordinate and let $\varphi$ be its restriction to $Y$. It is clear that $\varphi$ is continuous, $G$-equivariant, and onto. This finishes the proof of the lemma. $\square$

Given a topological space $X$, let $\mathrm{Clop}(X)$ denote the Boolean algebra of clopen subsets of $X$. If $X$ is a $G$-flow, then $G$ acts on $\mathrm{Clop}(X)$ by automorphisms as follows:

We consider each $g \in G$ as a homeomorphism $g : X \to X$. Given $a \in \mathrm{Clop}\, X$, let $ga$ be the clopen set $g^{-1}[a]$.

If $G$ acts on a Boolean algebra $B$ by automorphisms, we call $B$ a $G$-Boolean algebra, or $G$-Ba for short. The natural structure preserving maps between $G$-Bas are the equivariant Boolean homomorphisms and we call two $G$-Bas *isomorphic* if there is an equivariant isomorphism between them.

As an example, consider the space $2^G$ with the *shift action*, i.e., for each $x \in 2^G$ and all $g, h \in G$ we let $(gx)(h) = x(g^{-1}h)$, similar to the construction in the proof of Lemma 2.1. The dual Boolean algebra $\mathrm{Clop}(X)$ is the free Boolean algebra over a set of generators that is indexed by the elements of $G$.

**Definition 2.2.** Let $\mathrm{Fr}(G)$ be the free Boolean algebra over the set $\{a_g : g \in G\}$ of generators, where we assume that for different group elements $g$ and $g'$ the generators $a_g$ and $a_{g'}$ are also different. $G$ acts on $\mathrm{Fr}(G)$ as follows:

For all $g, h \in G$ let $ha_g = a_{h^{-1}g}$. This defines the action of $G$ on the generators of $\mathrm{Fr}(G)$. This is enough since every permutation of the generators extends to a unique automorphism of $\mathrm{Fr}(G)$.

The $G$-Ba $\mathrm{Fr}(G)$ is isomorphic to $\mathrm{Clop}(2^G)$.

**Lemma 2.3.** *Let $A$ be a $G$-Ba and let $a \in A$. Then there is a unique $G$-equivariant Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ mapping $a_{1_G}$ to $a$.*

*Proof.* The homomorphism maps every generator $a_g$ to $g^{-1}a$. This maps extends to a unique Boolean homomorphism which is also $G$-equivariant. $\square$

**Lemma 2.4.** *Let $A$ and $B$ be $G$-Bas and let $a \in A$ and $b \in B$. Suppose that $A = \langle a \rangle_G$ and $B = \langle b \rangle_G$. Let $\pi_A : \mathrm{Fr}(G) \to A$ and $\pi_B : \mathrm{Fr}(G) \to B$ be the unique* *equivariant homomorphims with $\pi_A(a_{1_G}) = a$ and $\pi_B(a_{1_G}) = b$. Then $a$ and $b$ have the same type iff the ideals $\pi_A^{-1}(0)$ and $\pi_B^{-1}(0)$ are equal.*

*Proof.* If $\pi_A^{-1}(0) = \pi_B^{-1}(0)$, then $(A, a)$ and $(B, b)$ are isomorphic since both structures are isomorphic to the quotient $\mathrm{Fr}(G)/\pi_A^{-1}(0)$ with the distinguished element $a_{1_G}/\pi_A^{-1}(0)$.

If $a$ and $b$ are of the same type, then there is a $G$-equivariant isomorphism $\iota : A \to B$ such that $\iota(a) = b$. Now $\iota \circ \pi_A = \pi_B$ by the uniqueness of $\pi_B$. Since $\iota$ is an isomorphism, $\pi_B^{-1}(0) = \pi_A^{-1}(\iota^{-1}(0)) = \pi_A^{-1}(0)$. □

## 3. Conclusion

We use the following result by Gao, Jackson, and Seward [4, Theorem 7.4.9].

**Theorem 3.1.** *For every countably infinite group $G$ there are $2^{\aleph_0}$ pairwise disjoint minimal subflows of $2^G$.*

**Theorem 3.2.** *Let $G$ be a countably infinite group and let $X$ be a $G$-flow that has all metric minimal $G$-flows as factors. Then the weight of $X$ is at least $2^{\aleph_0}$.*

*Proof.* Let $\kappa$ be the weight of $X$. By Theorem 3.1, $\kappa$ is infinite. By Lemma 2.1, $X$ is a factor of a zero-dimensional $G$-flow $Y$ of weight $\kappa$. Now $\mathrm{Clop}(Y)$ is a $G$-Ba of size $\kappa$ and for all zero-dimensional metric minimal $G$-flows $Z$, $\mathrm{Clop}(Z)$ embeds into $\mathrm{Clop}(Y)$ by a $G$-equivariant embedding.

In particular, the $2^{\aleph_0}$ subflows of $2^G$ from Theorem 3.1 embed into $\mathrm{Clop}(Y)$. For each subflow $Z$ of $2^G$, let $h_Z : \mathrm{Fr}(G) \to \mathrm{Clop}(Z)$ be the Stone dual of the embedding of $Z$ into $2^G$. Clearly, if $Z$ and $Z'$ are two different subflows of $2^G$, then the kernels of $h_Z$ and $h_{Z'}$ are different ideals. Now it follows from Lemma 2.4 that $h_Z(a_{1_G})$ and $h_{Z'}(a_{1_G})$ are of a different type.

Since there are $2^{\aleph_0}$ different minimal subflows of $2^G$, there are $2^{\aleph_0}$ different types of elements of $G$-Bas of the form $\mathrm{Clop}(Z)$ where $Z$ is a metric minimal $G$-flow. Since all $G$-Bas of the form $\mathrm{Clop}(Z)$, $Z$ a metric minimal $G$-flow, embed into $\mathrm{Clop}(Y)$, the $G$-Ba $\mathrm{Clop}(Y)$ has elements of $2^{\aleph_0}$ different types. Hence $\mathrm{Clop}(Y)$ is of size at least $2^{\aleph_0}$. This shows $\kappa \geq 2^{\aleph_0}$. □

**Corollary 3.3.** *If $G$ is a countably infinite discrete group, then there is no metric $G$-flow that has all minimal metric $G$-flows as factors. In particular, there is now universal minimal metric $G$-flow.*

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