THE NONEXISTENCE OF UNIVERSAL METRIC FLOWS

STEFAN GESCHKE

ABSTRACT. We consider dynamical systems of the from (X, f) where X is a compact metric space and $f : X \to X$ is either a continuous map or a homeomorphism and provide a new proof that there is no universal metric dynamical system of this kind. The same is true for metric minimal dynamical systems and for metric abstract ω-limit sets, answering a question by Will Brian.

1. INTRODUCTION

We call a compact space X with a continuous map $f : X \to X$ an $\mathbb{N}$-*flow.* If f is a homeomorphism, then the pair (X, f) is a $\mathbb{Z}$-*flow.* X is the *phase space* of the flow (X, f). If (X, f) is a G-flow for $G = \mathbb{N}$ or $G = \mathbb{Z}$, then the action of G on X is given by the map

$$G \times X \to X; (n, x) \mapsto f^n(x).$$

The G-*orbit* of $x \in X$ is the set $\{f^n(x) : n \in G\}$.

Given two G-flows (X, f) and (Y, g) for $G \in \{\mathbb{N}, \mathbb{Z}\}$, a map $h : X \to Y$ is *equivariant* if $h \circ f = g \circ h$. Two G-flows are *isomorphic* if there is an equivariant homeomorphism of their phase spaces. A G-flow (Y, g) is a *factor* of a G-flow (X, f) if there is a continuous equivariant surjection $p : X \to Y$.

If $\mathcal{C}$ is a class of G-flows, then a G-flow $(X, f) \in \mathcal{C}$ is *universal* (for $\mathcal{C}$) if every $(Y, g) \in \mathcal{C}$ is a factor of (X, f).

It is clear that there are no universal objects in the class of all G-flows, simply because there are arbitrarily large phase spaces of G-flows. We investigate what happens if we restrict our attention to metric G-flows.

A G-flow (X, f) is *minimal* if X has no proper closed subsets that are G-flows with respect to the restriction of f. It is well known that there are universal minimal G-flows [7] and an active field of research to determine those Polish groups G for which the universal minimal flow is metrizable (see for example [4]). However, for $G = \mathbb{N}$ or $G = \mathbb{Z}$ the phase space of a universal minimal G-flow is homeomorphic to an infinite subspace of the Čech-Stone compactification of the integers and hence not metrizable. This brings up the question whether there is a universal minimal metric G-flow.

From results of Beleznay and Foreman [3] it follows that this is not the case if $G = \mathbb{Z}$. We sketch the argument[1] which is based on Furstenberg's structure theory of minimal *distal* flows [8] and refer the reader to [3] or [8] for the notion of distality.

Every minimal flow has a maximal distal factor, which is also minimal. If there were a universal metric minimal flow, its maximal distal factor would be a universal metric minimal distal flow. However, there are no universal metric minimal distal flows:

Date: June 4, 2018.

2000 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 37B05, 37B10.

Key words and phrases. universal metric dynamical system, minimal dynamical system.

The author was supported through the DAAD project 57156702.

[1]We thank the anonymous referee of a previous version of this article for pointing out this argument.

To every minimal distal flow one can assign an ordinal, the *Furstenberg rank* of the flow. The Furstenberg rank does not increase when taking factors. Hence the rank of a universal element in any class of minimal distal flows would have to be the maximal rank in that class. But Beleznay and Foreman showed that the ranks of metric minimal distal flows are precisely the countable ordinals. In particular, there is no such flow of maximal rank. It follows that there are no universal metric minimal distal flows and hence no universal metric minimal flows. We provide a much more elementary and completely different proof of this result.

A third class of flows that we look at is the class of *abstract ω-limit sets*.

Definition 1.1. Let $G \in \{\mathbb{N}, \mathbb{Z}\}$. For a G-flow (X, f) and $x \in X$ let

$$\omega(x) = \bigcap_{n \geq 0} \mathrm{cl}\{f^m(x) : m \geq n\}$$

be the *ω-limit set* of x.

A G-flow is an *abstract ω-limit set* if it is isomorphic to the ω-limit set of a point in some G-flow.

The ω-limit set of a point in a G-flow (X, f) is a nonempty closed subset of the phase space X that is also a G-flow. It follows that every minimal flow is the ω-limit set of each of its points. Will Brian asked whether there are universal metric abstract ω-limit sets [6].

We show that for $G \in \{\mathbb{N}, \mathbb{Z}\}$ the classes of metric minimal G-flows, metric abstract ω-limit sets and metric G-flows do not have universal elements.

2. Algebraic flows

In [1], Anderson showed that for $G \in \{\mathbb{N}, \mathbb{Z}\}$ every metric G-flow is a factor of a G-flow whose phase space is the Cantor space $\{0, 1\}^{\mathbb{N}}$. Anderson also observed that every minimal G-flow with a metric phase space is a factor of a minimal G-flow on $\{0, 1\}^{\mathbb{N}}$.

An analog of this is true for abstract ω-limit sets.

Lemma 2.1. *Every metric abstract ω-limit set is a factor of a metric ω-limit set whose phase space is zero-dimensional.*

Proof. Let (X, f) be an abstract ω-limit set. Bowen's proof of his characterization of abstract ω-limit sets in [5] actually shows that (X, f) is isomorphic to the ω-limit set of a point y in a G-flow whose phase space is a subset of $X \times [0, 1]$. In particular, (X, f) is isomorphic to the ω-limit set of a point y in a metric G-flow (Y, g).

By Anderson's result mentioned above, the G-flow (Y, g) is a factor of a G-flow $(\{0, 1\}^{\mathbb{N}}, h)$. Let $p : \{0, 1\}^{\mathbb{N}} \to Y$ be a continuous surjection witnessing this fact and let $z \in p^{-1}(y)$. It is easily checked that p is a continuous equivariant map from $\omega(z)$ onto $\omega(y)$. Hence (X, f) is a factor of the ω-limit set of z. $\square$

This shows that if there are universal elements in the class of metric G-flows, minimal metric G-flows, or metric abstract ω-limit sets, then there are zero-dimensional ones.

Via Stone duality we can investigate G-flows with a zero-dimensional phase space by studying Boolean algebras and their endomorphisms, respectively automorphisms.

If (X, f) is a G-flow and X is zero-dimensional, then its *dual* is the Boolean algebra $\mathrm{Clop}(X)$ of clopen subsets of X together with the endomorphism

$$f^* : \mathrm{Clop}(X) \to \mathrm{Clop}(X); a \mapsto f^{-1}[a].$$

The endomorphism f^* is an automorphism of $\mathrm{Clop}(X)$ iff f is a homeomorphism.

Definition 2.2. Let A be a Boolean algebra and let f be an endomorphism of A. The pair (A, f) is a *Boolean algebraic* $\mathbb{N}$*-flow* (*Ba* $\mathbb{N}$*-flow*). If f is an automorphism of A, then (A, f) is a *Boolean algebraic* $\mathbb{Z}$*-flow* (*Ba* $\mathbb{Z}$*-flow*).

The natural structure preserving maps between Ba G-flows are the equivariant Boolean homomorphisms and we call two Ba G-flows *isomorphic* if there is an equivariant isomorphism between them.

If A is a Boolean algebra with an endomorphism f, then the space $\mathrm{Ult}(A)$ of ultrafilters of A is a compact zero-dimensional space and the Stone dual

$$f^* : \mathrm{Ult}(A) \to \mathrm{Ult}(A); p \mapsto f^{-1}(p)$$

is a continuous map. The map f^* is a homeomorphism iff f is an automorphism of A. The G-flow $(\mathrm{Ult}(A), f^*)$ is the *dual* of the Ba G-flow (A, f).

Taking the double dual of a zero-dimensional G-flow (X, f) yields an isomorphic G-flow.

Definition 2.3. Let $G \in \{\mathbb{N}, \mathbb{Z}\}$. If (A, f) is a Ba G-flow and $a \in A$, then by $\langle a \rangle_G$ we denote the smallest subalgebra B of A such that $a \in B$ and $(B, f \upharpoonright B)$ is a Ba G-flow. The Boolean algebra $\langle a \rangle_G$ is the subalgebra of A generated by the G-orbit of a.

Given two Ba G-flows (A, f) and (B, g) and elements $a \in A$ and $b \in B$, we call the triples (A, f, a) and (B, g, b) *isomorphic* if there is an isomorphism between (A, f) and (B, g) that maps a to b.

Given a Ba G-flow (A, f) and $a \in A$, the *type* of a is the isomorphism type of the triple $(\langle a \rangle_G, f \upharpoonright \langle a \rangle_G, a)$.

If (A, f) is a Ba $\mathbb{N}$-flow and $I \subseteq A$ is an ideal that is closed under f, then f induces an endomorphism f/I of the quotient A/I. If (A, f) is a Ba $\mathbb{Z}$-flow and $I \subseteq A$ is an ideal that is closed under f and f^{-1}, then f induces an automorphism f/I of the quotient A/I. On the other hand, the kernel of an G-equivariant homomorphism from a Ba G-flow (A, f) to a Ba G-flow (B, g) is an ideal that is closed under f if $G = \mathbb{N}$ and closed under f and f^{-1} if $G = \mathbb{Z}$.

Definition 2.4. For $G \in \{\mathbb{N}, \mathbb{Z}\}$ let $\mathrm{Fr}(G)$ be the free Boolean algebra over the set $\{g_n : n \in G\}$ of generators. We assume that the g_n are pairwise distinct. Let $s_G : \mathrm{Fr}(G) \to \mathrm{Fr}(G)$ be the Boolean homomorphism extending the map $g_n \mapsto g_{n+1}$.

Clearly, $s_{\mathbb{Z}}$ is an automorphism of $\mathrm{Fr}(\mathbb{Z})$ and hence $(\mathrm{Fr}(\mathbb{Z}), s_{\mathbb{Z}})$ is a Ba $\mathbb{Z}$-flow. Also, $(\mathrm{Fr}(\mathbb{N}), s_{\mathbb{N}})$ is a Ba $\mathbb{N}$-flow.

Lemma 2.5. *Let (A, f) be a BA G-flow for $G = \mathbb{N}$ or $G = \mathbb{Z}$ and let $a \in A$. Then there is a unique Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ such that $\pi(g_0) = a$ and $\pi(s_G(b)) = f(\pi(b))$ for all $b \in \mathrm{Fr}(G)$.*

Proof. There is a unique Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ such that for all $n \in G$, $\pi(g_n) = f^n(a)$. It is clear that π is as desired.

On the other hand, every Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ with $\pi(g_0) = a$ and $\pi(s_G(b)) = f(\pi(b))$ for all $b \in \mathrm{Fr}(G)$ satisfies $\pi(g_n) = f^n(a)$ for all $n \in G$. □

Lemma 2.6. *Let (A, f) and (B, g) be Ba G-flows for $G = \mathbb{N}$ or $G = \mathbb{Z}$, $a \in A$, and $b \in B$. Suppose that $A = \langle a \rangle_G$ and $B = \langle b \rangle_G$. Let $\pi_A : \mathrm{Fr}(G) \to A$ and $\pi_B : \mathrm{Fr}(G) \to B$ be the unique equivariant homomorphims with $\pi_A(g_0) = a$ and $\pi_B(g_0) = b$. Then a and b have the same type iff the ideals $\pi_A^{-1}(0)$ and $\pi_B^{-1}(0)$ are identical.*

Proof. If $\pi_A^{-1}(0) = \pi_B^{-1}(0)$, then (A, f, a) and (B, f, b) are isomorphic since both triples are isomorphic to the quotient $\mathrm{Fr}(G)/\pi_A^{-1}(0)$ with the endomorphism induced by s_G and the distinguished element $g_0/\pi_A^{-1}(0)$. Note that if $G = \mathbb{Z}$, then the endomorphism induced by s_G on $\mathrm{Fr}(G)/\pi_A^{-1}(0)$ is actually an automorphism.

If a and b are of the same type, then there is an equivariant isomorphism $\iota : A \to B$ such that $\iota(a) = b$. Now $\iota \circ \pi_A = \pi_B$. Since ι is an isomorphism, $\pi_B^{-1}(0) = \pi_A^{-1}(\iota^{-1}(0)) = \pi_A^{-1}(0)$. $\square$

3. Symbolic dynamics

Definition 3.1. Let $G = \mathbb{N}$ or $G = \mathbb{Z}$. On the space $\{0,1\}^G$ we consider the *Bernoulli shift* $S_G : \{0,1\}^G \to \{0,1\}^G$ which is defined by letting $S_G(x) : G \to \{0,1\}$ be the map satisfying $S_G(x)(n) = x(n+1)$ for all $n \in G$. Clearly, $S_{\mathbb{Z}} : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ is a homeomorphism and $S_{\mathbb{N}} : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is a continuous map.

Note that the shift S_G on $\{0,1\}^G$ is (isomorphic to) the Stone dual of the shift s_G on $\mathrm{Fr}(G)$.

Our theorem on the nonexistence of universal metric flows will follow from the fact that $(\{0,1\}^G, S_G)$ has many minimal *subshifts*, i.e., closed subsets that are minimal G-flows with respect to the restriction of S_G. One way of constructing continuum many minimal subshifts is to consider *Sturmian subshifts.* All the facts about Sturmian subshifts that we use can be found in [9].

Definition 3.2. Let $G = \mathbb{N}$ or $G = \mathbb{Z}$. A *Sturmian word* is a word $x \in \{0,1\}^G$ such that there are two real numbers, the *slope* α and the *intercept* ρ, with $\alpha \in [0,1)$ irrational such that for all $i \in G$ we have

$$x(i) = 1 \quad \Leftrightarrow \quad (\rho + i \cdot \alpha) \mod 1 \in [0, \alpha).$$

Another way of phrasing this is that a Sturmian word is the characteristic function of the set of return times of ρ to the interval $[0,\alpha)$ under the rotation action by α.

In the context of $\mathbb{N}$-flows, we consider Sturmian words in $\{0,1\}^{\mathbb{N}}$ and when we talk about $\mathbb{Z}$-flows, we consider Sturmian words in $\{0,1\}^{\mathbb{Z}}$.

It is well known that the orbit closure $C_x = \mathrm{cl}\{s_G^n(x) : n \in G\}$ of a Sturmian word with the restriction of the shift is a minimal G-flow (see the discussion in [9] following Lemma 7.1). If $x \in \{0,1\}^{\mathbb{Z}}$ is a Sturmian word of slope α, then for all y in the orbit closure of x the limit

$$\lim_{n\to\infty} \frac{|y^{-1}(1) \cap \{-n, \dots, n\}|}{2n+1}$$

exists and equals α [9, Lemma 7.1]. In other words, α is the *Banach density* of the set with the characteristic function y.

Similarly, if $x \in \{0,1\}^{\mathbb{N}}$ is a Sturmian word of slope α, then for all y in the orbit closure of x the limit

$$\lim_{n\to\infty} \frac{|y^{-1}(1) \cap \{0, \dots, n-1\}|}{n}$$

exists and equals α.

It follows that for different irrational numbers $\alpha, \beta \in [0,1)$, Sturmian words of slope α and β have different (even disjoint) orbit closures. We call the orbit closure of a Sturmian word together with the restriction of S_G a *Sturmian subshift.* A Sturmian subshift is a G-flow.

Given a Sturmian subshift $(X, S_G \restriction X)$, we denote the common slope of all Sturmian words that generate X by $\alpha(X)$.

Lemma 3.3. *Let $(X, S_G \restriction X)$ be a Sturmian subshift and let $p : \mathrm{Fr}(G) \to \mathrm{Clop}(X)$ be the homomorphism dual to the embedding of X into $\{0,1\}^G$. Then $\langle p(g_0)\rangle_G = \mathrm{Clop}(X)$ and the type of $p(g_0)$ determines $\alpha(X)$.*

Proof. Since X is a subspace of $\{0,1\}^G$, p is onto. Since $\mathrm{Fr}(G) = \langle g_0 \rangle_G$, $\langle p(g_0) \rangle_G = \mathrm{Clop}(X)$. By Lemma 2.6, the type of $p(g_0)$ determines the kernel $p^{-1}(0)$. But by standard Stone duality, the ideals of $\mathrm{Fr}(G)$ are in 1-1 correspondence to the subspaces of $\{0,1\}^G$. It follows that the type of $p(g_0)$ determines the subspace X of $\{0,1\}^G$ and hence the slope $\alpha(X)$. □

Definition 3.4. In the context of Lemma 3.3 we call $p(g_0)$ the *generator* of $\mathrm{Clop}(X)$ and denote it by g_X.

Theorem 3.5. *Let $G = \mathbb{N}$ or $G = \mathbb{Z}$. Then there is no metric G-flow that has all Sturmian subshifts as factors.*

Proof. Suppose there is a metric G-flow (X, f) such that every Sturmian subshift is a factor of (X, f). By Anderson's result mentioned above, we may assume that X is zero-dimensional. Let (A, f^*) be the Stone dual of (X, f). Then A is a countable Boolean algebra.

If a Sturmian subshift $(Y, S_G \restriction Y)$ is a factor of (X, f), then there is an equivariant embedding of $\mathrm{Clop}(Y)$ into A. In particular, A has an element whose type is the same as the type of the generator g_Y of $\mathrm{Clop}(Y)$.

Since there are uncountably many slopes of Sturmian words, by Lemma 3.3 there are uncountably many different types of generators of algebras of the form $\mathrm{Clop}(Y)$ where $(Y, S_G \restriction Y)$ is a Sturmian subshift. But since A is countable, its elements realize only countably many different types. A contradiction. □

Corollary 3.6. *Let $G = \mathbb{N}$ or $G = \mathbb{Z}$. The following classes of G-flows contain no universal elements:*

(1) *Metric G-flows*
(2) *Metric minimal G-flows*
(3) *Metric abstract ω-limit sets*

Proof. The corollary follows from the previous theorem together with the fact that all Sturmian subshifts are contained in each of the three classes of G-flows. □

The proof of Theorem 3.5 shows that no zero-dimensional G-flow of weight less that $2^{\aleph_0}$ has all Sturmian subshifts as factors. It follows that the weight of a universal minimal G-flow is at least $2^{\aleph_0}$. In fact, using a result due to Balcar and Błaszczyk [2], Turek [10] showed that the universal minimal flow of an arbitrary discrete, countable abelian group is the Gleason space of the Cantor cube $\{0,1\}^{2^{\aleph_0}}$.

References

[1] R.D. Anderson, *On raising flows and mappings*, Bull. Amer. Math. Soc., Volume 69, Number 2 (1963), 259–264

[2] B. Balcar and A. Błaszczyk, *On minimal dynamical systems on Boolean algebras*, Comment. Math. Univ. Carolinae 31 (1) (1990), 7–11

[3] F. Beleznay, M. Foreman, *The Collection of Distal Flows is not Borel*, Amer. J. Math. 117 (1) (1995), 203–239

[4] I. Ben Yaacov, J. Melleray, T. Tsankov, *Metrizable universal minimal flows of Polish groups have a comeagre orbit*, Geometric and Functional Analysis, Volume 27, Issue 1 (2017), 67–77

[5] R. Bowen, *ω-Limit Sets for Axiom A Diffeomorphism*, Journal of Differential Equations **18** (1975), 333–339

[6] W. Brian, *Is there a universal ω-limit set?*, mathoverflow.net/questions/209634

[7] R. Ellis, **Lectures on Topological Dynamics**, W.A. Benjamin, Inc., 1969. MR 42:2463

[8] H. Furstenberg, *The structure of distal flows*, Amer J. Math. 83 (1963), 477–515

[9] M. Morse and G.A. Hedlund, *Symbolic Dynamics II. Sturmian Trajectories*, American Journal of Mathematics, Vol. 62, No. 1 (1940), pp. 1–42

[10] S. Turek, *A note on universal minimal dynamical systems*, Comment Math. Univ. Carolinae 32 (4) (1991), 781–783

[11] E. van Douwen, *The maximal totally bounded group topology on G and the biggest minimal G-space, for Abelian groups G*, Topology and its Appl. 34 (1990), 69–91

Department of Mathematics, University of Hamburg, Bundesstr. 55, 20146 Hamburg, Germany

E-mail address: stefan.geschke@uni-hamburg.de